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ON SOME PROPERTIES OF ORTHOGONAL POLYNOMIALS OVER AN AREA IN DOMAINS OF THE COMPLEX PLANE. III

ПРО ДЕЯКІ ВЛАСТИВОСТІ ОРТОГОНАЛЬНИХ НА ПЛОЩИНІ ПОЛІНОМІВ В ОБЛАСТЯХ КОМПЛЕКСНОЇ ПЛОЩИНИ. III

We study orthogonal polynomials of higher orders in domains with weight under a condition that boundary and weight functions possess singularities and do not satisfy interference conditions.

Вивчаються ортогональні поліноми вищих порядків в областях з вагою за умови, що крайові та вагові функції мають сингулярності і не задовольняють умови накладання.

1. Introduction and definitions. This work is the continuation of research of author started in [1, 2].

Let G be a finite region, with $0 \in G$, bounded by a Jordan curve $L := \partial G$, let σ be the two-dimensional Lebesgue measure on G, and let $h(z) \in L^1(G, d\sigma)$ be a positive weight function in G. The polynomials $\{K_n(z)\}$, $\deg K_n = n$, $n = 0, 1, 2, \dots$, satisfying the condition

$$\iint\limits_G h(z) K_n(z) \overline{K_m(z)}\, d\sigma_z \;=\; \delta_{n,m}$$

are called orthogonal polynomials for the pair (G, h). They are determined uniquely if the coefficient of the term of the highest degree is positive.

Let $\{z_i\}$, $i = \overline{1, m}$, be a fixed system of points on L and let the weight function $h(z)$ be defined as follows:

$$h(z) \;=\; h_0(z) \prod_{i=1}^{m} |z - z_i|^{\gamma_i}, \tag{1.1}$$

where $\gamma_i > -2$ and $h_0(z)$ is satisfying the condition

$$\exists\, c_0 > 0 \quad \forall z \in G \quad h_0(z) \,\geq\, c_0 \,>\, 0.$$

Let us some definitions.

Throughout this paper, $c, c_1, c_2, \dots$ are positive and $\varepsilon, \varepsilon_1, \varepsilon_2, \dots$ sufficiently small positive constants (in general, different in different relations), which depend on G in general.

For $\delta > 0$ and $z \in G$, let us set: $B(z, \delta) := \{\zeta : |\zeta - z| < \delta\}$, $B := B(0, 1)$, $\Delta(z, \delta) := \operatorname{ext} \overline{B(z, \delta)} = \{\zeta : |\zeta - z| > \delta\}$, $\Delta := \operatorname{ext} B$, $\Omega := \operatorname{ext} G$, $\Omega(z, \delta) := \Omega \cap B(z,$

$\delta)$, $\omega = \varphi(z)$ $(\omega = \Phi(z))$ is the conformal mapping of $G(\Omega)$ onto the $B(\Delta)$ normalized by $\varphi(0)=0$, $\varphi'(0)>0$ $(\Phi(\infty)=\infty$, $\Phi'(\infty)>0)$, $\psi := \varphi^{-1}$ $(\Psi := \Phi^{-1})$.

Definition 1. *A bounded Jordan region G is called a k-quasidisk, $0 \le k < 1$, if any conformal mapping ψ can be extended to a K-quasiconformal, $K = \dfrac{1+k}{1-k}$, homeomorphism of the plane $\overline{C}$ on the $\overline{C}$. In this case, the curve $L := \partial G$ is called a K-quasicircle. The region G (curve L) is called a quasidisk (quasicircle), if it is k-quasidick (k-quasicircle) with some $0 \le k < 1$.*

Definition 2. *We say that $G \in Q_\alpha$, $0 < \alpha \le 1$, if*

1) *L is a quasicircle;*

2) $\Phi \in \mathrm{Lip}\,\alpha$, $z \in \overline{\Omega}$.

Definition 3. *We say that $G \in Q_{\alpha,\beta_1,\ldots,\beta_m}$, $0 < \beta_i \le \alpha \le 1$, $i = \overline{1,m}$, if*

i) for every sequence of pairwise disjoint circles $\{D(z_i,\delta_i)\}_{i=1}^m$ restriction of the function Φ on $\Omega(z_i,\delta_i)$ belongs to $\mathrm{Lip}\,\beta_i$ and restriction

$$\Phi \Big| \Omega \setminus \bigcup_{i=1}^{m} \Omega(z_i,\delta_i) \in \mathrm{Lip}\,\alpha;$$

ii) there exists a sequence of pairwise disjoint circles $\{D(z_i,\delta_i^*)\}_{i=1}^m$ such that $\forall\, i = \overline{1,m}$ and $\forall\, \xi,\ z \in \Omega(z_i,\delta_i^*)$, $z \ne z_i \ne \xi$, the following estimate is true:

$$|\Phi(z) - \Phi(\xi)| \le k_i(z,\xi)\,|z-\xi|^\alpha,$$

where

$$k_i(z,\xi) = c_i \max\left(|\xi - z_i|^{\beta_i-\alpha}; |z-z_i|^{\beta_i-\alpha}\right)$$

and c_i do not depend on z and ξ.

Assume that the system of points $\{z_i\}$, $i = \overline{1,m}$, mentioned in (1.1) and Definition 3 is identically ordered on L. In [1], we showed that if the interference condition

$$1 + \frac{\gamma_i}{2} = \frac{\beta_i}{\alpha} \tag{1.2}$$

is satisfied for any singular point $\{z_i\}$, $i = \overline{1,m}$, of the weight functions and boundary contour, then the order of the height of polynomials $K_n(z)$ in $\overline{G}$ acts itself identically neither weight $h(z)$ and boundary contour L have not got singularity nor they have got singularity. In [2], we studied the order of the height of polynomials $K_n(z)$ on boundary points of the region, when

$$1 + \frac{\gamma_i}{2} > \frac{\beta_i}{\alpha}. \tag{1.3}$$

In the present paper, we investigate the case where

$$1 + \frac{\gamma_i}{2} < \frac{\beta_i}{\alpha} \tag{1.4}$$

for any singular points $\{z_i\}$, $i = \overline{1,m}$.

2. Main results.

Theorem 1. *Suppose that $G \in Q_{\alpha,\beta_1}$, $0 < \beta_1 \le \alpha \le 1$, and $h(z)$ is defined by (1.1). If*

$$1 + \frac{\gamma_1}{2} < \frac{\beta_1}{\alpha}, \tag{2.1}$$

then for every $z \in \overline{G}$ *and each* $n = 1, 2, \ldots$

$$|K_n(z)| \leq c_1 n^{s_1} + c_2 |z - z_1|^{\sigma_1} n^{1/\alpha}, \tag{2.2}$$

where

$$s_1 = \frac{2+\gamma_1}{2\beta_1}, \quad \sigma_1 = \frac{\beta_1}{\alpha} - \frac{2+\gamma_1}{2}. \tag{2.3}$$

Since $\alpha \geq \beta_1$, (2.1) is satisfied when $-2 < \gamma_1 < 0$. This and (2.2) enable us to see that the order of the height of K_n at point z_1 and points $z \in L$, $z \neq z_1$, where $h(z) \to \infty$ and curve L doesn't have singularity, acts itself identically. Thus, the conditions (2.1) we will call algebraic pole conditions of the order $\lambda_1 = 1 - \frac{\alpha}{\beta_1}\left(1 + \frac{\gamma_1}{2}\right)$.

This theorem can be extended to the case where L and $h(z)$ have a lot of singular points. For example, in the case of two singular points, we can write

$$|K_n(z)| \leq c_1 |z - z_1|^{\sigma_1} n^{s_2} + c_2 |z - z_2|^{\sigma_2} n^{s_1} + c_3 |z - z_1|^{\sigma_1} |z - z_2|^{\sigma_2} n^{1/\alpha},$$
$$z \in \overline{G}, \tag{2.4}$$

where s_i, σ_i, $i = 1, 2$, are defined as it is in (2.3), respectively.

Theorem 1 is also correct if the curve L has at point z_1 algebraic pole and at points $\{z_k\}$, $k \geq 2$, singularities which satisfy the interference conditions (1.2).

3. Some auxiliary results. In the following, we shall use the notations "$a \prec b$" and "$a \asymp b$" equivalent to $a \leq b$ and $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2, respectively.

Let G be a quasidisk. Then there exists a quasiconformal reflection $y(\cdot)$ across L such that $y(G) = \Omega$, $y(\Omega) = G$ and $y(\cdot)$ fixes the points of L. The quasiconformal reflection $y(\cdot)$ is such that it satisfies the following condition [3, 4, p.26]:

$$|y(\zeta) - z| \asymp |\zeta - z|, \quad z \in L, \quad \varepsilon < |\zeta| < \frac{1}{\varepsilon},$$
$$|y_{\bar\zeta}| \asymp |y_\zeta| \asymp 1, \quad \varepsilon < |\zeta| < \frac{1}{\varepsilon}, \tag{3.1}$$
$$|y_{\bar\zeta}| \asymp |y(\zeta)|^2, \quad |\zeta| < \varepsilon, \quad |y_{\bar\zeta}| \asymp |\zeta|^{-2}, \quad |\zeta| > \frac{1}{\varepsilon}.$$

For $t > 0$, let $L_t := \{z : |\varphi(z)| = t$, if $t < 1$, $|\Phi(z)| = t$, if $t > 1\}$, $L_1 := L$, $G_t := \operatorname{int} L_t$, $\Omega_t := \operatorname{ext} L_t$, and for $t > 1$, let $L^* := y(L_t)$, $G^* := \operatorname{int} L^*$, $\Omega^* := \operatorname{ext} L^*$; let $w = \Phi_R(z)$ be the conformal mapping of Ω^* onto the Δ normalized by $\Phi_R(\infty) = \infty$, $\Phi_R'(\infty) > 0$; $\Psi_R := \Phi_R^{-1}$; $L_t^* := \{z : |\Phi_R(z)| = t\}$, $G_t^* := \operatorname{int} L_t^*$, $\Omega_t^* := \operatorname{ext} L_t^*$; $d(z, L) = \operatorname{dist}(z, L)$.

According to [4], for all $z \in L^*$ and $t \in L$ such that $|z - t| = d(z, L_R)$ we have

$$d(z, L) \asymp d(t, L_R) \asymp d(z, L_R). \tag{3.2}$$

Lemma 1 [5]. *Let* G *be a quasidisk, let* $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \prec d(z_1, L_{r_0})\}$, *and let* $\omega_j = \Phi(z_j)$, $j = 1, 2, 3$. *Then:*

a) *the statements* $|z_1 - z_2| \prec |z_1 - z_3|$ *and* $|w_1 - w_2| \prec |w_1 - w_3|$ *are equivalent. So,* $|z_1 - z_2| \asymp |z_1 - z_3|$ *and* $|w_1 - w_2| \asymp |w_1 - w_3|$;

b) *if* $|z_1 - z_2| \prec |z_1 - z_3|$, *then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{\varepsilon} \prec \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \prec \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c}$$

where $0 < r_0 < 1$ *is a constant depending on* G *and* k.

Let $A_p(h, G)$, $p > 0$, denote the class of functions f which are analytic in G and satisfy the condition

$$\|f\|_{A_p} := \|f\|_{A_p(h, G)} := \left(\iint_G h(z)|f(z)|^p d\sigma_z\right)^{1/p} < \infty.$$

Lemma 2. *Let* $p > 0$, *let* f *be an analytic function in* $|z| > 1$ *and have at* $z = \infty$ *a pole of degree at most* n, $n \geq 1$. *Then for all* R_1, R_2, $1 < R_1 < R_2$,

$$\|f\|_{A_p(R_1 < |z| < R_2)} \leq \left(\frac{R_2 - R_1}{R_1 - 1}\right)^{1/p} R_2^{n+1/p} \|f\|_{A_p(1 < |z| < R_1)}.$$

Proof. According to Riesz theorem [6, p. 443], for any ρ, $R_1 \leq \rho < R_2$, and s, $1 < s \leq R_1$, we can write

$$\int_{|z|=\rho} \left|\frac{f(z)}{z^{n+1/p}}\right|^p |dz| \leq \int_{|z|=R_1} \left|\frac{f(z)}{z^{n+1/p}}\right|^p |dz|, \tag{3.3}$$

$$\int_{|z|=R_1} \left|\frac{f(z)}{z^{n+1/p}}\right|^p |dz| \leq \int_{|z|=s} \left|\frac{f(z)}{z^{n+1/p}}\right|^p |dz|, \tag{3.4}$$

respectively. After integrating (3.3) over ρ from R_1 to R_2 and (3.4) over s from 1 to R_1, we get

$$\iint_{R_1 < |z| < R_2} |f(z)|^p d\sigma_z \leq \frac{R_2^{np+2} - R_1^{np+2}}{R_1^{np+2} - 1} \iint_{1 < |z| < R_1} |f(z)|^p d\sigma_z. \tag{3.5}$$

Let us set

$$S := \frac{R_2^{np+2} - R_1^{np+2}}{R_1^{np+2} - 1}. \tag{3.6}$$

By applying the Lagrange theorem to the numerator and denominator of the fraction, we obtain

$$S = \frac{(np+2) r_2^{np+1} (R_2 - R_1)}{(np+2) r_1^{np+1} (R_1 - 1)}$$

for some r_1, $1 < r_1 < R_1$, and r_2, $R_1 < r_2 < R_2$. Then

$$S < \frac{R_2 - R_1}{R_1 - 1} R_2^{np+1}. \tag{3.7}$$

Taking into account (3.6) and using (3.7) and (3.5), we complete the proof.

Lemma 3. *Let* G *be a quasidisk, let* $P_n(z)$, $\deg P_n \leq n$, $n = 1, 2, \ldots$, *be arbitrary polynomial, and let weight function* $h(z)$ *satisfy the condition* (1.1). *Then, for any* $R_1 > 1$, $p > 0$, *and* $n = 1, 2, \ldots$,

$$\|P_n\|_{A_p(h,G_{1+c(R-1)})} \leq c_1 R^{n+1/p} \|f\|_{A_p(h,G)}, \tag{3.8}$$

where c, c_1 *are independent of* n *and* R.

Proof. We present the proof of (3.8) under several headings. First of all, it is easy to convince ourselves that to prove (3.8), it suffices to show that the estimate:

$$\|P_n\|_{A_p(h,G_R\setminus G)} \prec [1+c(R-1)]^{n+1/p} \|P_n\|_{A_p(h,G\setminus G^*)} \tag{3.9}$$

is true for some $c > 0$.

Now, we consider the two numbers ρ_1, ρ_2, $\rho_1 < \rho_2$, such that

$$G^*_{\rho_1} \subseteq G, \tag{3.10}$$

$$G_R \subset G^*_{\rho_2}, \tag{3.11}$$

and show that one can choose the numbers ρ_1, ρ_2 that satisfy the following conditions:

$$\rho_1 - 1 \asymp R - 1, \tag{3.12}$$

$$\rho_2 - 1 \asymp R - 1. \tag{3.13}$$

In fact, assume that ρ_1, ρ_2 are arbitrary numbers satisfying (3.10) and (3.11), $z \in L^*$, $\tilde{z} = y(z)$. We define the points $z_1 \in L^*_{\rho_1}$, $z_2 \in L$, and $z_3 \in L^*_{\rho_2}$ as $d\left(z, L^*_{\rho_1}\right) = |z - z_1|$, $d(z, L) = |z - z_2|$, and $d\left(z, L^*_{\rho_2}\right) = |z - z_3|$, respectively. According to (3.2), there exist c_3, c_4 that are independent of z and R such that

$$c_3 d(z_2, L_R) \leq d(z, L) \leq c_4 d(z_2, L_R). \tag{3.14}$$

Since L^* is a quasicircle, applying Lemma 1 to functions Φ_R, we obtain

$$\left|\frac{z-z_2}{z-z_1}\right| \geq c_5 \left|\frac{\Phi_R(z)-\Phi_R(z_2)}{\Phi_R(z)-\Phi_R(z_1)}\right|^{\varepsilon_1} \geq c_6 \left(\frac{|\Phi_R(z)-\Phi(z_2)|}{\rho_1 - 1}\right)^{\varepsilon_1},$$

whence

$$|z - z_1| \leq c_6^{-1} \left(\frac{\rho_1 - 1}{|\Phi_R(z)-\Phi_R(z_2)|}\right)^{\varepsilon_1} |z - z_2|. \tag{3.15}$$

In view of the D-property of the mapping $y_R(z)$ [7, p. 18], we have

$$|z - z_2| \geq c_3 d(z_2, L_R) \geq c_7 |\tilde{z} - z_2|$$

and, by Lemma 1, we get

$$|\Phi_R(z) - \Phi_R(z_2)| \geq c_8 |\Phi_R(\tilde{z}) - \Phi_R(z_2)| \geq c_8 (R - 1).$$

Then, from (3.15) we obtain

$$|z - z_1| \leq c_6^{-1} \left(\frac{\rho_1 - 1}{c_8 (R-1)}\right)^{\varepsilon_1} |z - z_2|.$$

So, we can take

$$\rho_1 = 1 + c_9 (R - 1) \tag{3.16}$$

with $c_9 = c_8 c_6^{-\varepsilon_1}/2$, which also leads to (3.10) and (3.12).

We now define ρ_2. By applying Lemma 1 to Φ_R, we get

$$\left|\frac{z-\tilde{z}}{z-z_3}\right| \leq c_{10}\left|\frac{\Phi_R(z)-\Phi_R(\tilde{z})}{\Phi_R(z)-\Phi_R(z_3)}\right|^c,$$

whence

$$|z-z_3| \geq c_{11}\left(\frac{\rho_2-1}{|\Phi_R(z)-\Phi_R(\tilde{z})|}\right)^c|z-\tilde{z}|. \tag{3.17}$$

Since $|\Phi_R(z)-\Phi_R(z_2)| \leq c_{12}|\Phi_R(\tilde{z})-\Phi_R(z_2)|$, we have

$$|\Phi_R(z)-\Phi_R(\tilde{z})| \leq |\Phi_R(z)-\Phi_R(z_2)| + |\Phi_R(\tilde{z})-\Phi_R(z_2)| \leq$$
$$\leq (c_{12}+1)|\Phi_R(\tilde{z})-\Phi_R(z_2)| \leq c_{13}(R-1),$$

and (3.17) implies that

$$|z-z_3| \geq c_{11}\left(\frac{\rho_2-1}{c_{13}(R-1)}\right)^c|z-\tilde{z}|.$$

Choosing

$$\rho_2 = 1 + c_{14}(R-1) \tag{3.18}$$

with $c_{14} = c_8\cdot c_6^{-\varepsilon_1} + c_{13}\cdot c_{11}^{c^{-1}}$, we see that (3.11) and (3.13) are satisfied.

Now, let us make a proof of (3.9). Let us include the Blashke finctions with respect to the singular points of the weight functions $h(z)$:

$$B_R(z) = \prod_{i=1}^{m} B_R^i(z) := \prod_{i=1}^{m}\frac{\Phi_R(z)-\Phi_R(z_i)}{1-\overline{\Phi_R(z_i)}\Phi_R(z)}, \quad z\in\Omega^*. \tag{3.19}$$

It is easy to see that $B_R(z_i)=0$ and $|B_R(z)|\equiv 1$ at $z\in L^*$.

For $p>0$ and $R>1$, let us set

$$f_R(w) := h_0(\Psi_R(w))\prod_{i=1}^{m}\left[\frac{\Psi_R(w)-\Psi_R(w_i)}{wB_R^i(\Psi_R(w))}\right]^{\gamma_i/p} P_n(\Psi_R(w))[\Psi_R'(w)]^{2/p}, \quad w=\Phi_R(z).$$

The function f_R is analytic in Δ and has pole of degree at most n on $z=\infty$. Then, according to Lemma 2, we have

$$\|f_R\|_{A_p(\rho_1<|w|<\rho_2)} \leq \left(\frac{\rho_2-\rho_1}{\rho_1-1}\right)^c \rho_2^{n+1/p}\|f_R\|_{A_p(1<|w|<\rho_1)}$$

or

$$\iint_{G_R\setminus G} h_0(z)\prod_{i=1}^{m}\left|\frac{z-z_i}{\Phi_R(z)B_R^i(z)}\right|^{\gamma_i}|P_n(z)|^p d\sigma_z \leq$$

$$\leq \iint_{G^*_{\rho_2}\setminus G^*_{\rho_1}} h_0(z)\prod_{i=1}^{m}\left|\frac{z-z_i}{\Phi_R(z)B_R^i(z)}\right|^{\gamma_i}|P_n(z)|^p d\sigma_z \leq$$

$$\le \frac{\rho_2-\rho_1}{\rho_1-1}\rho_2^{pn+1}\iint\limits_{G^*_{\rho_2}\setminus G^*} h_0(z)\prod_{i=1}^{m}\left|\frac{z-z_i}{\Phi_R(z)B_R^i(z)}\right|^{\gamma_i}|P_n(z)|^p d\sigma_z \le$$

$$\le \frac{\rho_2-\rho_1}{\rho_1-1}\rho_2^{pn+1}\iint\limits_{G\setminus G^*} h_0(z)\prod_{i=1}^{m}\left|\frac{z-z_i}{\Phi_R(z)B_R^i(z)}\right|^{\gamma_i}|P_n(z)|^p d\sigma_z.$$

From (3.16) and (3.18) we get

$$\iint\limits_{G_R\setminus G} h_0(z)|P_n(z)|^p d\sigma_z \prec$$

$$\prec \prod_{i=1}^{m}\left[\frac{\max_{z\in\overline{G_R\setminus G}}\left|\Phi_R(z)B_R^i(z)\right|}{\max_{z\in\overline{G\setminus G^*}}\left|\Phi_R(z)B_R^i(z)\right|}\right]^{\gamma_i}\rho_2^{pn+1}\iint\limits_{G\setminus G^*} h(z)|P_n(z)|^p d\sigma_z. \tag{3.20}$$

Since

$$\left|\Phi_R(z)B_R^i(z)\right| = \left|\Phi_R(z)\frac{\Phi_R(z)-\Phi_R(z_i)}{\left(\overline{\Phi_R(z_i)}\right)^{-1}-\Phi_R(z)}\frac{1}{\overline{\Phi_R(z_i)}}\right| =$$

$$= \left|\frac{\Phi_R(z)}{\overline{\Phi_R(z_i)}}\right|\left|\frac{\Phi_R(z)-\Phi_R(z_i)}{\Phi_R(z_i)-\Phi_R(z)}\right| = \left|\frac{\Phi_R(z)}{\overline{\Phi_R(z_i)}}\right|,$$

from (3.20) we obtain

$$\iint\limits_{G_R\setminus G} h(z)|P_n(z)|^p d\sigma_z \prec$$

$$\prec \prod_{i=1}^{m}\left[\frac{\max_{z\in\overline{G_R\setminus G}}|\Phi_R(z)|}{\max_{z\in\overline{G\setminus G^*}}|\Phi_R(z)|}\right]^{\gamma_i}\rho_2^{pn+1}\iint\limits_{G\setminus G^*} h(z)|P_n(z)|^p d\sigma_z \prec$$

$$\prec \rho_2^{pn+1}\iint\limits_{G\setminus G^*} h(z)|P_n(z)|^p d\sigma_z.$$

Since ρ_2 and R are symmetric, the proof is completed.

4. Case of arbitrary polynomials. Let $P_n(z)$ be arbitrary polynomial of degree at most n and let $M_{n,p} := \|P_n\|_{A_p(h,G)}$.

Theorem 2. *Suppose that* $p > 1$, $G \in Q_{\alpha,\beta_1}$, $0 < \beta_1 \le \alpha \le 1$, *and* $h(z)$ *is defined by* (1.1). *If*

$$1 + \frac{\gamma_1}{2} < \frac{\beta_1}{\alpha},$$

then for every $z \in \overline{G}$ *and each* $n = 1, 2, \ldots$,

$$|P_n(z)| \le \left(c_1 n^{s_1} + c_2|z-z_1|^{\sigma^1} n^{2/(p\alpha)}\right)M_{n,p}, \tag{4.1}$$

where

$$s^1 = \frac{(2+\gamma_1)}{p\beta_1}, \quad \sigma^1 = \frac{2\beta_1}{p\alpha} - \frac{2+\gamma_1}{p}. \tag{4.2}$$

Proof. Since L is a quasicircle, we have that any L_R, $n = 1 + cn^{-1}$ is also a quasicircle. Therefore, we can construct a reflection y_R, $y_R(0) = \infty$ across L_R such that it satisfies the conditions (3.1) described for $y_R(\zeta)$. For this $y_R(\zeta)$, we can write for $P_n(z)$ the following integral representations [3, p. 105]:

$$P_n(z) = -\frac{1}{\pi}\int_{G_R} \frac{P_n(\zeta) y_{R,\bar{\zeta}}(\zeta)}{(y_R(\zeta)-z)^2} d\sigma_\zeta, \quad z \in G_R. \tag{4.3}$$

For $\varepsilon > 0$, by setting $U_\varepsilon(z) := \{\zeta : |\zeta - z| < \varepsilon\}$ and without loss of generality, we may take $U_\varepsilon := U_\varepsilon(0) \subset G^*$. For $z_1 \in L$ we have

$$|P_n(z_1)| \le \frac{1}{\pi}\left\{\iint_{U_\varepsilon} + \iint_{G_R\setminus U_\varepsilon}\right\} \frac{|P_n(\zeta)||y_{R,\bar{\zeta}}(\zeta)|}{|y_R(\zeta)-z_1|^2} d\sigma_\zeta =: J_1 + J_2. \tag{4.4}$$

To estimate the integral J_1, we multiply the numerator and denominator of integrant by $h^{1/p}(\zeta)$, and applying the Hölder inequality, we get

$$J_1 \prec \left\{\iint_{U_\varepsilon} h(\zeta)|P_n(\zeta)|^p d\sigma_\zeta\right\}^{1/p} \left\{\iint_{U_\varepsilon} \frac{|y_{R,\bar{\zeta}}(\zeta)|^q}{h^{q-1}(\zeta)|y_{R,\zeta}(\zeta)-z_1|^{2q}} d\sigma_\zeta\right\}, \quad \frac{1}{p}+\frac{1}{q} = 1.$$

The first multiplier is smaller than $\pi^{-1}M_{n,p}$. According to (3.1), $|y_{R,\bar{\zeta}}| \asymp |y_R(\zeta)|^2$ for all $\zeta \in U_\varepsilon$ and, by virtue of $|\zeta - z_1| \ge \varepsilon$, we have $|y_R(\zeta) - z| \asymp |y_R(\zeta)|$ for $z \in L$ and $\zeta \in U_\varepsilon$. This relations imply

$$J_1 \prec M_{n,p}. \tag{4.5}$$

If $\mathcal{L}_{y_R} := |y_{R,\zeta}|^2 - |y_{R,\bar{\zeta}}|^2$ is a Jacobian of the reflection $y_R(\zeta)$, we can obtain

$$|\mathcal{L}_{y_R}| \succ |y_{R,\bar{\zeta}}|^2 \tag{4.6}$$

as it is in [1]. Then, for J_2, we get

$$J_2 \prec M_{n,p}\left\{\iint_{G_R\setminus U_\varepsilon} \frac{d\sigma_\zeta}{|\zeta - z_1|^{\gamma(q-1)}|y_R(\zeta)-z_1|^{2q}}\right\}^{1/q} \prec$$

$$\prec M_{n,p}\left\{\iint_{y_R(G_R\setminus U_\varepsilon)} \frac{d\sigma_\zeta}{|y_R(\zeta)-z_1|^{\gamma(q-1)}|\zeta - z_1|^{2q}}\right\}^{1/q} \tag{4.7}$$

from (3.1), (4.6), and Lemma 3.

First of all, we establish that

$$|\zeta - z_1| \prec |y_R(\zeta) - z| \tag{4.8}$$

for all $\zeta \in G_R \setminus U_\varepsilon$ and $z_1 \in L$.

In fact, let $|z_1 - t| = d(z_1, L_R)$, $t \in L_R$. According to (3.1),

$$c_1|\zeta - z| \le |y_R(\zeta) - z| \le c_2|\zeta - z| \tag{4.9}$$

for all $\zeta \in G_R \setminus U_\varepsilon$ and $z \in L_R$, whence

$$|\zeta - z_1| \le |\zeta - t| + |y_R(\zeta) - t| + |y_R(\zeta) - z_1| \le$$

$$\le (c_1^{-1}+1)|y_R(\zeta) - t| + |y_R(\zeta) - z_1| \prec |y_R(\zeta) - z_1|.$$

If $\gamma_1 \le 0$, after changing the variable $\zeta = y_R(\zeta)$ and using (4.8), (4.6), and (3.1), from (4.7) we have

$$J_2 \prec M_{n,p}\left\{\iint\limits_{G_R \setminus U_\varepsilon} \frac{d\sigma_\zeta}{|y_R(\zeta) - z_1|^{\gamma_1(q-1)+2q}}\right\}^{1/q} \prec$$

$$\prec M_{n,p}\left\{\iint\limits_{y_R(G_R \setminus U_\varepsilon)} \frac{d\sigma_\zeta}{|\zeta - z_1|^{\gamma_1(q-1)+2q}}\right\}^{1/q} \prec M_{n,p}\, d^{-\frac{2+\gamma_1}{p}}(z_1, L_R). \qquad (4.10)$$

If $\gamma_1 > 0$, by changing the variable $\zeta = y_R(\zeta)$ and applying (4.8), (4.6), and (3.1), we obtain

$$J_2 \prec M_{n,p}\left\{\iint\limits_{y_R(G_R \setminus U_\varepsilon)} \frac{d\sigma_\zeta}{|\zeta - z_1|^{\gamma_1(q-1)+2q}}\right\}^{1/q} \prec M_{n,p}\, d^{-\frac{2+\gamma_1}{p}}(z_1, L_R). \qquad (4.11)$$

From (4.5), (4.7), (4.10), and (4.11) we obtain

$$|P_n(z_1)| \prec M_{n,p}\, d^{-\frac{2+\gamma_1}{p}}(z_1, L_R).$$

Since $G \in Q_{\alpha,\beta_1}$, we have

$$|P_n(z_1)| \prec M_{n,p}\, n^{\frac{(2+\gamma_1)}{p\beta_1}}. \qquad (4.12)$$

Now, by using the integral representation (4.3), we obtain

$$\left|\frac{P_n(z) - P_n(z_1)}{(z - z_1)^{\sigma_1}}\right| \le \frac{1}{\pi}\iint\limits_{G_R} \frac{|P_n(\zeta)||y_{R,\bar\zeta}(\zeta)||z - z_1|^{1-\sigma^1}}{|y_R(\zeta) - z||y_R(\zeta) - z_1|^2}\, d\sigma_\zeta +$$

$$+ \frac{1}{\pi}\iint\limits_{G_R} \frac{|P_n(\zeta)||y_{R,\bar\zeta}(\zeta)||z - z_1|^{1-\sigma^1}}{|y_R(\zeta) - z_1||y_R(\zeta) - z|^2}\, d\sigma_\zeta =: A(z; z_1) + B(z; z_1). \qquad (4.13)$$

The definitions of the integrals $A(z; z_1)$ and $B(z; z_1)$ enable us to see that they are symmetric with respect to the points z and z_1. Thus, we estimate integrals $A(z; z_1)$ and $B(z; z_1)$ parallel. To estimate the integral $A(z; z_1)$ $(B(z; z_1))$, we multiply the numerator and denominator of integrant by $|\zeta - z_1|^{\gamma_1/p}$ and, after applying the Hölder inequality, from Lemma 3 we get

$$A(z; z_1) \prec M_{n,p}\left\{\left(\iint\limits_{U_\varepsilon} + \iint\limits_{G_R \setminus U_\varepsilon}\right) \frac{|y_{R,\bar\zeta}(\zeta)||z - z_1|^{q(1-\sigma^1)}\, d\sigma_\zeta}{|\zeta - z_1|^{\gamma_1(q-1)}|y_R(\zeta) - z|^q |y_R(\zeta) - z_1|^{2q}}\right\}^{1/q} =:$$

$$=: M_{n,p}\{A_1(z; z_1) + A_1(z; z_1)\}^{1/q}, \qquad \frac{1}{p} + \frac{1}{q} = 1, \qquad (4.14)$$

$$B(z; z_1) \prec M_{n,p}\left\{\left(\iint\limits_{U_\varepsilon} + \iint\limits_{G_R \setminus U_\varepsilon}\right) \frac{|y_{R,\bar\zeta}(\zeta)||z - z_1|^{q(1-\sigma^1)}\, d\sigma_\zeta}{|\zeta - z_1|^{\gamma_1(q-1)}|y_R(\zeta) - z|^{2q} |y_R(\zeta) - z_1|^{q}}\right\}^{1/q} \prec$$

$$\prec M_{n,p}\{B_1(z; z_1) + B_2(z; z_1)\}^{1/q}.$$

According to (3.1), $\left|y_{R,\overline{\zeta}}(\zeta)\right| \asymp |y_R(\zeta)|^2$ for all $\zeta \in U_\varepsilon$ and, in view of $|\zeta - z_1| \asymp 1$, we have $|y_R(\zeta)| \asymp |y_R(\zeta) - z| \asymp |y_R(\zeta) - z_1|$ for $z,\ z_1 \in L$ and $\zeta \in \in U_\varepsilon$. These relations imply

$$A_1(z; z_1) \prec 1 \qquad (B_1(z; z_1) \prec 1). \tag{4.15}$$

For the estimations of $A_2(z; z_1)$ $(B_2(z; z_1))$, we consider the different situations of the points z and z_1 on L. Let us set

$$F_1 := y_R(G_R \setminus U_\varepsilon) = E_0 := E_1 \cup E_2,$$

$$F_{11} := \left\{\zeta \in F_1 : |\zeta - z_1| \le \frac{1}{2}|z - z_1|\right\}, \qquad F_{11}^c := \left\{\zeta \in F_1 : |\zeta - z_1| \succ \frac{1}{2}|z - z_1|\right\},$$

$$F_{12} := \left\{\zeta \in F_1 : |\zeta - z| \le \frac{1}{2}|z - z_1|\right\}, \qquad F_{12}^c := \left\{\zeta \in F_1 : |\zeta - z| \succ \frac{1}{2}|z - z_1|\right\},$$

$$E_1 := \left\{y_R(G_R \setminus U_\varepsilon) \cap U_\delta(z_1)\right\}, \qquad E_2 := \left\{y_R(G_R \setminus U_\varepsilon) \setminus U_\delta(z_1)\right\}, \qquad 0 < \delta < \delta_0(G),$$

$$E_{01} := \left\{\zeta \in E_0 : |\zeta - z_1| \ge |\zeta - z|\right\}, \qquad E_{02} := \left\{\zeta \in E_0 : |\zeta - z_1| < |\zeta - z|\right\}.$$

a) Let $|z - z_1| \ge \delta > 0$.

Taking into account (3.1) (for the y_R) and (4.8), we have

$$A_2(z; z_1) \prec \iint\limits_{F_1} \frac{d\sigma_\zeta}{|y_R(\zeta) - z_1|^{\gamma_1(q-1)}|\zeta - z|^q|\zeta - z_1|^{2q}} \le$$

$$\le \left(\iint\limits_{F_{11}} + \iint\limits_{F_{12}} + \iint\limits_{F_{11}^c} + \iint\limits_{F_{12}^c}\right) \frac{d\sigma_\zeta}{|y_R(\zeta) - z_1|^{\gamma_1(q-1)}|\zeta - z|^q|\zeta - z_1|^{2q}}.$$

According to $|z - z_1| \ge \big||z - z_1| - |\zeta - z_1|\big| \ge \frac{1}{2}|z - z_1|$ for $\zeta \in F_{11}$ and $|\zeta - - z_1| \ge \frac{1}{2}|z - z_1|$ for $\zeta \in F_{12}$, we obtain

$$\left(\iint\limits_{F_{11}} + \iint\limits_{F_{12}}\right) \frac{d\sigma_\zeta}{|y_R(\zeta) - z_1|^{\gamma_1(q-1)}|\zeta - z|^q|\zeta - z_1|^{2q}} \prec$$

$$\prec \iint\limits_{F_{11}} \frac{d\sigma_\zeta}{|\zeta - z_1|^{\gamma_1(q-1)+2q}} + \iint\limits_{F_{12}} \frac{d\sigma_\zeta}{|\zeta - z|^q} \prec n^{\frac{(\gamma_1+2)(q-1)}{\beta_1}} + n^{\frac{q-2}{\alpha}},$$

$$\iint\limits_{F_{11}^c} + \iint\limits_{F_{12}^c} \frac{d\sigma_\zeta}{|y_R(\zeta) - z_1|^{\gamma_1(q-1)}|\zeta - z|^q|\zeta - z_1|^{2q}} \prec 1,$$

and

$$A_2(z; z_1) \prec n^{\frac{(\gamma_1+2)(q-1)}{\beta_1}} + n^{\frac{q-2}{\alpha}}. \tag{4.16}$$

Analogously,

$$B_2(z;z_1) \prec \iint_{F_1} \frac{d\sigma_\zeta}{|y_R(\zeta)-z_1|^{\gamma_1(q-1)}|\zeta-z|^{2q}|\zeta-z_1|^q} \le$$

$$\le \left(\iint_{F_{11}} + \iint_{F_{12}} + \iint_{F_{11}^c} + \iint_{F_{12}^c}\right) \frac{d\sigma_\zeta}{|y_R(\zeta)-z_1|^{\gamma_1(q-1)}|\zeta-z|^{2q}|\zeta-z_1|^q}.$$

Since

$$\left(\iint_{F_{11}} + \iint_{F_{12}}\right) \frac{d\sigma_\zeta}{|y_R(\zeta)-z_1|^{\gamma_1(q-1)}|\zeta-z|^{2q}|\zeta-z_1|^q} \prec$$

$$\prec \iint_{F_{11}} \frac{d\sigma_\zeta}{|\zeta-z_1|^{\gamma_1(q-1)+q}} + \iint_{F_{12}} \frac{d\sigma_\zeta}{|\zeta-z|^{2q}} \prec n^{\frac{\gamma_1(q-1)+q-2}{\beta_1}} + n^{\frac{2(q-2)}{\alpha}},$$

and

$$\iint_{F_{11}^c} + \iint_{F_{12}^c} \frac{d\sigma_\zeta}{|y_R(\zeta)-z_1|^{\gamma_1(q-1)}|\zeta-z|^{2q}|\zeta-z_1|^q} \prec 1,$$

we have

$$B_2(z;z_1) \prec n^{\frac{\gamma_1(q-1)+q-2}{\beta_1}} + n^{\frac{2(q-1)}{\alpha}}. \tag{4.17}$$

From (4.14), (4.15), (4.16), and (4.17) we get

$$A(z;z_1) \prec M_{n,p} n^{2/(\alpha p)}, \quad B(z;z_1) \prec M_{n,p} n^{2/(\alpha p)}. \tag{4.18}$$

b) Let $\delta > |z-z_1| \ge d(z_1, L_R)$.

Taking into account that $|z-z_1|^\varepsilon \le c(\varepsilon)(|\zeta-z|^\varepsilon + |\zeta-z_1|^\varepsilon)$ is satisfied for all $\varepsilon > 0$, we have

$$A_2(z;z_1) \prec \iint_{F_1} \frac{d\sigma_\zeta}{|\zeta-z_1|^{\gamma_1(q-1)+2q}|\zeta-z|^{q\sigma^1}} +$$

$$+ \iint_{F_1} \frac{d\sigma_\zeta}{|\zeta-z_1|^{\gamma_1(q-1)+2q-(1-\sigma^1)q}|\zeta-z|^q} \le$$

$$\le \iint_{E_{01}} \frac{d\sigma_\zeta}{|\zeta-z|^{\gamma_1(q-1)+2q+q\sigma^1}} + \iint_{E_{01}} \frac{d\sigma_\zeta}{|\zeta-z|^{\gamma_1(q-1)+2q-(1-\sigma^1)q+q}} +$$

$$+ \iint_{E_{02}} \frac{d\sigma_\zeta}{|\zeta-z_1|^{\gamma_1(q-1)+2q+q\sigma^1}} + \iint_{E_{02}} \frac{d\sigma_\zeta}{|\zeta-z_1|^{\gamma_1(q-1)+2q-(1-\sigma^1)q+q}} \prec$$

$$\prec n^{\frac{(\gamma_1+2)(q-1)+q\sigma^1}{\beta_1}}. \tag{4.19}$$

Completely, we see that

$$B_2(z;z_1) \prec n^{\frac{(\gamma_1+2)(q-1)+q\sigma^1}{\beta_1}}. \tag{4.20}$$

Therefore, in this case, by substituting (4.19) and (4.20) in (4.14) and using (4.15) and (4.2), we get

$$A(z;z_1) \prec M_{n,p} n^{\frac{\gamma_1+2+\sigma^1}{\beta_1}} \prec M_{n,p} n^{2/(p\alpha)} \tag{4.21}$$

and, respectively,

$$B(z;z_1) \prec M_{n,p} n^{\frac{\gamma_1+2+\sigma^1}{\beta_1}} \prec M_{n,p} n^{2/(p\alpha)}. \tag{4.22}$$

c) Let $|z-z_1| \leq d(z_1, L_R)$.

From (4.14) we have

$$A_2(z;z_1) \prec \iint\limits_{G_R \setminus U_\varepsilon} \frac{d^{q(1-\sigma^1)}(z_1, L_R) d\sigma_\zeta}{d^{\gamma_1(q-1)+3q}(z_1, L_R)} \prec n^{\frac{(\gamma_1+2)(q-1)+q\sigma^1}{\beta_1}} \tag{4.23}$$

and, respectively,

$$B_2(z;z_1) \prec n^{\frac{(\gamma_1+2)(q-1)+q\sigma^1}{\beta_1}}. \tag{4.24}$$

By substituting (4.23) and (4.24) in (4.14) and using (4.15) and (4.2), we obtain

$$A(z;z_1) \prec M_{n,p} n^{2/(p\alpha)}, \quad B(z;z_1) \prec M_{n,p} n^{2/(p\alpha)}. \tag{4.25}$$

So, from (4.25), (4.12) and (4.13) we obtain the proof of (4.1).

***Proof of Theorem* 1.** Since $M_{n,2} \equiv 1$ for $K_n(z)$, we get the proof of Theorem 1 from Theorem 2.

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