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V.I. Ryazanov [1, 2], S.V. Volkov [3]

[1] Institute of Applied Mathematics and Mechanics of the NAS of Ukraine, Slov'yansk
[2] Bogdan Khmelnytsky National University of Cherkasy
[3] Donetsk National Technical University, Pokrovsk
E-mail: vl.ryazanov1@gmail.com, serhii.volkov@donntu.edu.ua

Mappings with finite length distortion and Riemann surfaces

Presented by Corresponding Member of the NAS of Ukraine V.Ya. Gutlyanskiĭ

We prove a series of criteria in terms of dilatations for the continuous and homeomorphic extension of the mappings with finite length distortion between domains on Riemann surfaces to the boundary. The criterion for the continuous extension of the inverse mapping to the boundary is turned out to be a very simple condition on the integrability of the dilatations in the first power. Moreover, the domain of the mapping is assumed to be locally connected on the boundary and its range has a weakly flat boundary. The criteria for the continuous extension of the direct mappings to the boundary have a much more refined nature. One of such criteria is the existence of a majorant for the dilation in the class of functions with finite mean oscillation, i.e., having a finite mean deviation from its mean value over infinitesimal disks centered at the corresponding boundary point. A stronger (but simpler) one is that the mean value of the dilatation over infinitesimal disks centered at the corresponding boundary point is finite. The domain is again assumed to be locally connected on the boundary and its range has a strongly accessible boundary. We give also many other criteria for the continuous extension of the direct mappings to the boundary. As consequences, the corresponding criteria for a homeomorphic extension of mappings with finite length distortion to the closures of domains are obtained.

Keywords: *Riemann surfaces, boundary behavior, mappings with finite length distortion, strongly accessible and weakly flat boundaries.*

1. Introduction. The present paper is a natural continuation of our previous papers [1-3], where the reader can find the corresponding historic comments and a discussion of many definitions and relevant results. The given papers were devoted to the theory of the boundary behavior of mappings with finite distortion by Iwaniec.

Here, we will develop the theory of the boundary behavior of the so-called mappings with finite length distortion first introduced in [4] for $\mathbb{R}^n$, $n \geqslant 2$, see also Chapter 8 in [5]. As was shown in [6], such mappings, generally speaking, are not mappings with finite distortion by Iwaniec, because their first partial derivatives can be not locally integrable. At the same time, this class is a natural generalization of the well-known classes of bi-Lipschitz mappings, as well as isometries and quasi-isometries.

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2. Definitions and preliminary remarks. We assume that all mappings under consideration are continuous. The previous definitions can be found in [1-3]. Here, we restrict ourselves in the main by new conceptions.

Let us start from the main definitions in [4] adopted to the case of domains D in the complex plane $\mathbb{C}$, see also Chapter 8 in [5]. It is said that a mapping $f: D \to \mathbb{C}$ is of *finite metric distortion*, written $f \in$ **FMD**, if f has the (N)-property by Luzin with respect to the area in $\mathbb{C}$ and

$$0 < l(z, f) \leqslant L(z, f) < \infty \text{ a.e.} \tag{1}$$

where

$$l(z, f) := \liminf_{\zeta \to z,\, \zeta \in D} \frac{|f(\zeta) - f(z)|}{|\zeta - z|}, \quad L(z, f) := \limsup_{\zeta \to z,\, \zeta \in D} \frac{|f(\zeta) - f(z)|}{|\zeta - z|}. \tag{2}$$

Now, we say that a mapping $f: D \to \mathbb{C}$ has (L)*-property*, if, for a.e. path γ in D, the path $\tilde{\gamma} = f \circ \gamma$ is locally rectifiable, and $f|_{\gamma}$ has the (N)-property by Luzin with respect to the length measure. Recall that a path γ in D is a mapping $y: \Delta \to D$, where Δ is an interval in $\mathbb{R}$. Moreover, it is said that a property holds for almost every (a.e.) path of a family, if the property fails only for its subfamily of paths of conformal modulus zero, see the definition of the conformal modulus on Riemann surfaces in [1–3].

We say also that a homeomorphism f between domains D and D^* in $\mathbb{C}$ is of *finite length distortion*, written $f \in$ **FLD**, if $f \in$ FMD and, moreover, f and f^{-1} have (L)-property. A special case is *bi-Lipschitz* homeomorphisms for which the quantities in (1) are uniformly in the domain D separated from zero, as well as from infinity. Thus, homeomorphisms of finite length distortion are a far reaching generalization of isometries and quasiisometries.

Remark 1. By Theorem 6.10 in [4] or Theorem 8.6 in [5], a homeomorphism $f \in$ FLD between domains D and D^* in $\mathbb{C}$ satisfies the inequality

$$M(f\Gamma) \leqslant \int_D Q(z) \cdot \rho^2(z)\, dm(z) \tag{3}$$

with $Q = K_f$ for any family Γ of paths γ in D and $\rho \in \mathrm{a}dm\Gamma$, see [1—3] for definitions of the dilatation K_f, the conformal modulus M of families of paths, and admissible functions $\rho: D \to [0, \infty]$.

Homeomorphisms f between domains D and D^* in the complex plane $\mathbb{C}$ satisfying conditions of the type (3) are called *Q-homeomorphisms*, see [7], and also Chapters 4—6 in [5]. Correspondingly to Remark 1, such homeomorphisms form a wider class of mappings than homeomorphisms with finite length distortion.

Let us pass to the corresponding definitions on Riemann surfaces. So, let f be a homeomorphism between domains D and D^* on Riemann surfaces $\mathbb{S}$ and $\mathbb{S}^*$. First of all, we say that f is a mapping with *finite length distortion*, written $f \in$ **FLD**, if f is so in charts of $\mathbb{S}$ and $\mathbb{S}^*$. In view of properties of conformal mappings, namely, the (N)-properties by Luzin with respect to area, as well as to length, and invariance of local rectifiable paths, see e.g. Theorem 5.6 in [8], the definition is independent of the choice of charts. We also say that f is a *local Q-homeomorphism* for a measurable function $Q: \mathbb{S} \to (0, \infty)$, if (3) holds for any family Γ of paths γ in D laying inside an arbitrary prescribed chart U of the Riemann surface $\mathbb{S}$.

Remark 2. As is known, if a function $\rho: V \to [0,\infty]$ is admissible for a family A of paths α in an open set V of the complex plane $\mathbb{C}$, then the function $\rho^*(\zeta) = \rho(\varphi^{-1}(\zeta))/|\varphi'(\varphi^{-1}(\zeta))|$ is admissible for the family $B := \varphi A$ of paths $\beta := \varphi \circ \alpha$ under every conformal mapping $\varphi: V \to \mathbb{C}$, see again Theorem 5.6 in [8]. Thus, the right-hand side in inequality (3) is a conformal invariant, because the Jacobian of $\varphi(z)$ is equal to $|\varphi'(z)|^2$.

Proposition 1. *Every homeomorphism f with finite length distortion between domains D and D^* on Riemann surfaces $\mathbb{S}$ and $\mathbb{S}^*$, correspondingly, is a local Q-homeomorphism with $Q = K_f$.*

Here and below, we assume that K_f is extended by zero outside of D.

Proof. Let $g: U \to \mathbb{C}$ be a chart of the Riemann surface $\mathbb{S}$. Since the space $\mathbb{S}$ is separable, the open set $D \cap U$ consists of a countable collection of its components U_k every of which is homeomorphic to the plane domain $V_k := g(U_k)$. Thus, every domain $U_k^* := f(U_k)$ is also homeomorphic to the plane domains V_k and, consequently, by the general Koebe principle, see Section II.3 in [9], U_k^* is a chart of the Riemann surface $\mathbb{S}^*$.

Note also that the path family Γ is split into a countable collection of mutually disjoint path families Γ_k lying in the domains U_k. Hence, the path family $\Gamma^* := f\Gamma$ is split also into a countable collection of mutually disjoint path families $\Gamma_k^* := f\Gamma_k$ lying in the domains U_k^*, i.e., in the corresponding charts of the Riemann surface $\mathbb{S}^*$. Thus, by Remark 1 in [1] and by Remarks 1 and 2 of the present paper, we obtain the desired conclusion.

3. The main lemma. Recall that the factor $\mathbb{D}/G$ of the unit disk $\mathbb{D}$ with a discrete group G of fractional mappings of $\mathbb{D}$ onto itself without fixed points is a Riemann surface with charts from the natural (locally homeomorphic) projection $\pi: \mathbb{D} \to \mathbb{D}/G$, see Theorem 6.2.1 in [10].

Lemma 1. *Let G be a discrete group of fractional maps of $\mathbb{D}$ onto itself with no fixed points, and $f: D \to D^*$ be a homeomorphism of finite length distortion between domains D and D^* on Riemann surfaces $\mathbb{S} := \mathbb{D}/G$ and $\mathbb{S}^*$, $p_0 \in \overline{D}$.*

Then there is $\varepsilon(p_0)$ such that the natural projection $\pi: \mathbb{D} \to \mathbb{D}/G$ is injective on a hyperbolic disk $B_0 := \{z \in \mathbb{D} : h(z, z_0) < \varepsilon(p_0)\}$, where $z_0 \in \pi^{-1}(p_0)$, and

$$M(f(\Gamma)) \leqslant \int_D K_f(p)\xi^2(p)\,dh(p) \tag{4}$$

for families Γ of paths in $D \cap \pi(B_0)$ and measurable functions $\xi: D \to [0,\infty]$, such that

$$\int_\gamma \xi(p)\,ds_h(p) \geqslant 1 \quad \forall \gamma \in \Gamma. \tag{5}$$

Remark 3. By the Klein–Poincaré theorem on the uniformization, see II.3 in [9], an arbitrary Riemann space $\mathbb{S}$ is conformally equivalent to the unit disk $\mathbb{D}$ factored by a discrete group G of fractional mappings of $\mathbb{D}$ onto itself without fixed points, excepting the simplest cases of $\mathbb{S}$ that are conformally equivalent to $\overline{\mathbb{C}}$, $\mathbb{C}$, a ring, or a torus.

In the case of a torus, $\mathbb{S}$ is conformally equivalent to a factor $\mathbb{C}/G$ with respect to a group G of shifts in $\mathbb{C}$ with 2 generators $z \to z + \omega_1$ and $z \to z + \omega_2$, where ω_1 and $\omega_2 \in \mathbb{C} \setminus \{0\}$ and $\operatorname{Im} \omega_1 / \omega_2 > 0$. In this case, a fundamental domain F is a parallelogram whose sides are parallel to ω_1 and ω_2, and gluing its opposite sides just gives a torus. The metric and the area on the surface $\mathbb{C}/G$ in small coincide with the Euclidean ones, because the Euclidean metric and area are invariant under shifts.

By the scheme of the proof below relations (4) are also valid for all the given special cases with the Euclidean metric and area instead of hyperbolic ones. Later on, for the universality, we keep the same notations in these cases, too.

Proof. By Section 2 in either [1] or [3], we may identify $\mathbb{D}/G$ with a fundamental set F in $\mathbb{D}$ for G with the metric d defined by (2.10) in [1] that contains a fundamental Poincaré polygon P_{z_0} for G centered at $z_0 \in \pi^{-1}(p_0)$. Let us choose $\varepsilon(p_0) > 0$ such that $d(z_0, z) = h(z_0, z)$ for $d(z_0, z) \leqslant \varepsilon(p_0)$ and $\varepsilon(p_0) < \delta_0 := \min\left[\inf\limits_{\varsigma\in\partial P_{z_0}} d(z_0, \zeta), \sup\limits_{z\in D} d(z_0, z)\right]$.

Since $ds_h(z) = 2|dz|/(1-|z|^2)$, we see that, for every ξ satisfying (5), $\int\limits_\gamma \eta(z)|dz| \geqslant 1 \ \ \forall\gamma\in\Gamma$, where $\eta(z) := \dfrac{2\xi(z)}{1-|z|^2}$, i.e., the function η is admissible for the family Γ of paths γ in $D\cap\pi(B_0)$. Moreover, since $dh(z) = 4dxdy/(1-|z|^2)^2$, $z = x+iy$, we obtain that

$$\int\limits_D K_f(z)\xi^2(z)dh(z) = \int\limits_D K_f(z)\eta^2(z)dm(z), \tag{6}$$

where $dm(z) := dxdy$ corresponds to the Lebesgue area in the plane $\mathbb{C}$. Thus, the conclusion of Lemma 1 follows from Proposition 1.

Remark 4. In other words, the statement of Lemma 1 means that every homeomorphism f of finite length distortion between domains on Riemann surfaces is a local K_f-homeomorphism with respect to the hyperbolic metric and the hyperbolic area. Note also that Riemann surfaces are locally the so-called Ahlfors 2-regular spaces with the given metric and measure h, see Theorem 7.2.2 in [10]. Hence, we may apply results in [11] on the boundary behavior of Q-homeomorphisms in metric spaces with measures to homeomorphisms with finite length distortion between domains on Riemann surfaces. It makes possible us, in comparison with [12], to formulate new results in terms of the metric and measure h but not in terms of local coordinates on arbitrary Riemann surfaces, see [1] or [3] and the end of Remark 3 on notations.

4. On the extending of the inverse mapping to the boundary. By contrast with the direct mappings, see the next section, we have the following simple criterion for the inverse mappings, see definitions in [1-3].

Theorem 1. *Let $\mathbb{S}$ and $\mathbb{S}^*$ be Riemann surfaces, D and D^* be domains in $\overline{\mathbb{S}}$ and $\overline{\mathbb{S}^*}$, correspondingly, $\partial D \subset \mathbb{S}$ and $\partial D^* \subset \mathbb{S}^*$, D be locally connected on its boundary, and let ∂D^* be weakly flat. Suppose that $f: D \to D^*$ is a homeomorphism of finite length distortion with $K_f \in L^1_{\mathrm{loc}}$. Then the mapping $g = f^{-1}: D^* \to D$ can be extended by continuity to a mapping $\overline{g}: \overline{D^*} \to \overline{D}$.*

Proof. By the Uryson theorem, $\overline{\mathbb{S}}$ is a metrizable space. Hence, the compactness of $\overline{\mathbb{S}}$ is equivalent to its sequential compactness, and the closure $\overline{D}$ is a compact subset of $\mathbb{S}$. Thus, the conclusion of Theorem 1 is true by Theorem 6.1 in [11] and by Lemma 1 and Remarks 3–4.

5. On the extending of the direct mappings to the boundary. As was already established in the plane, no degree of integrability of Q leads to the extension of direct mappings of Q-homeomorphisms to the boundary, see Proposition 6.3 in [5]. The corresponding criterion for FLD given below is much more refined. Namely, by Lemma 5.1 in [11], as well as Lemma 1 and Remarks 3-4 above, we obtain the following, see definitions in [1-3].

Lemma 2. *Let $\mathbb{S}$ and $\mathbb{S}^*$ be Riemann surfaces, D and D^* be domains in $\overline{\mathbb{S}}$ and $\overline{\mathbb{S}^*}$, $\partial D \subset \mathbb{S}$, $\partial D^* \subset \mathbb{S}^*$, D be locally connected at a point $p_0 \in \partial D$. Suppose that $f : D \to D^*$ is a homeomorphism of finite length distortion, ∂D^* is strongly accessible at least at one point in the cluster set $C(p_0, f)$, and*

$$\int\limits_{\varepsilon < h(p,\, p_0) < \varepsilon_0} K_f(p) \cdot \psi^2_{p_0,\varepsilon}(h(p, p_0))\, dh(p) = o(I^2_{p_0,\varepsilon_0}(\varepsilon)) \;\; as \;\; \varepsilon \to 0 \tag{7}$$

for some $\varepsilon_0 > 0$, where $\psi_{p_0,\varepsilon}(t)$ is a family of nonnegative measurable functions on $(0, \infty)$ such that

$$0 < I_{p_0,\varepsilon_0}(\varepsilon) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{p_0,\varepsilon}(t)\, dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0)\,. \tag{8}$$

Then f is extended by continuity to the point p_0, and $f(p_0) \in \partial D^*$.

Note that conditions (7), (8) imply that $I_{p_0,\varepsilon_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$, and ε_0 can be chosen arbitrarily small with keeping (7), (8).

Lemma 2 makes it possible to derive a series of criteria on the continuous extension of mappings with finite length distortion to the boundary, for instance:

Theorem 2. *Let $\mathbb{S}$, $\mathbb{S}^*$ be Riemann surfaces, D, D^* be domains on $\overline{\mathbb{S}}$, $\overline{\mathbb{S}^*}$, $\partial D \subset \mathbb{S}$, $\partial D^* \subset \mathbb{S}^*$, D be locally connected on ∂D, ∂D^* be strongly accessible. Suppose that $f : D \to D^*$ is a homeomorphism in* FLD *and, for all $p_0 \in \partial D$,*

$$\int\limits_0^{\varepsilon(p_0)} \frac{dr}{\| K_f \| (p_0, r)} = \infty\,, \quad \| K_f \| (p_0, r) := \int\limits_{h(p,\, p_0) = r} K_f(p)\, ds_h(p)\,. \tag{9}$$

Then the mapping f is extended by continuity to $\overline{D}$ and $f(\partial D) = \partial D^*$.

Proof. Indeed, setting $\psi_{p_0}(t) = 1 / \| K_f \| (p_0, t)$ for all $t \in (0, \varepsilon_0)$, $\varepsilon_0 := \varepsilon(p_0)$, and $\psi_{p_0}(t) = 1$ for all $t \in (\varepsilon_0, \infty)$, we obtain from condition (9) that

$$\int\limits_{\varepsilon < h(p,\, p_0) < \varepsilon_0} K_f(p) \cdot \psi^2_{p_0}(h(p, p_0))\, dh(p) = I_{p_0,\varepsilon_0}(\varepsilon) = o(I^2_{p_0,\varepsilon_0}(\varepsilon)) \quad as \quad \varepsilon \to 0\,,$$

where, in view of the condition $K_f(p) \in [1, \infty)$ a.e. in D, $0 < I_{p_0,\varepsilon_0}(\varepsilon) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{p_0}(t)\, dt < \infty$.

Thus, the first conclusions of Theorem 2 follow from Lemma 2. The second conclusion of Theorem 2 follows, for instance, from Proposition 2.5 in [11], see also Proposition 13.5 in [5].

Corollary 1. *In particular, the conclusion of Theorem 2 holds, if*

$$K_f(p) = O\left(\log \frac{1}{h(p, p_0)} \right) \; as \;\; p \to p_0 \quad \forall p_0 \in \partial D \tag{10}$$

or, more generally,

$$k_{p_0}(\varepsilon) = O\left(\log \frac{1}{\varepsilon} \right) \; as \;\; \varepsilon \to 0 \quad \forall p_0 \in \partial D\,, \tag{11}$$

where $k_{p_0}(\varepsilon)$ *is the mean value of the function* K_f *over the circle* $h(p, p_0) = \varepsilon$.

By Theorem 3.1 in [13], we have the following consequence of Theorem 2.

Theorem 3. *Under hypotheses of Theorem 2, suppose, instead of* (9), *that*

$$\int_U \Phi(K_f(p))\,dh(p) < \infty \tag{12}$$

in a neighborhood U *of* ∂D, *where* $\Phi : \overline{R}^+ \to \overline{R}^+$ *is a nondecreasing convex function with the condition*

$$\int_\delta^\infty \frac{d\tau}{\tau\Phi^{-1}(\tau)} = \infty, \quad \delta > \Phi(0). \tag{13}$$

Then the mapping f *is extended by continuity to* $\overline{D}$ *and* $f(\partial D) = \partial D^*$.

Remark 5. Note that, by Theorem 5.1 and Remark 5.1 in [14], condition (13) is not only sufficient, but also necessary for the continuous extension of all mappings f of finite length distortion with integral restrictions of the form (12) to the boundary. Note also that, by Theorem 2.1 in [13], condition (13) is equivalent to a series of other conditions, and the most interesting of them is

$$\int_\Delta^\infty \log\Phi(t)\frac{dt}{t^2} = +\infty \quad \text{for some} \quad \Delta > 0. \tag{14}$$

Corollary 2. *In particular, the conclusion of Theorem 3 holds, if, for some* $\alpha > 0$,

$$\int_U e^{\alpha K_f(p)}\,dh(p) < \infty. \tag{15}$$

The next statement holds by Remarks 3—4 and Lemma 2 with $\psi(t) = 1/t$.

Theorem 4. *Under the hypotheses of Theorem 2, if, instead of* (9),

$$\int_{\varepsilon < h(p, p_0) < \varepsilon_0} K_f(p)\frac{dh(p)}{h(p, p_0)^2} = o\left(\left[\log\frac{1}{\varepsilon}\right]^2\right) \ \text{as}\ \varepsilon \to 0 \quad \forall p_0 \in \partial D, \tag{16}$$

then the mapping f *is extended by continuity to* $\overline{D}$ *and* $f(\partial D) = \partial D^*$.

Following [11], see also Section 13.4 in [5], we say that a function $\varphi : \mathbb{S} \to R$ has *finite mean oscillation* at a point $p_0 \in \mathbb{S}$, written $\varphi \in FMO(p_0)$, if

$$\limsup_{\varepsilon \to 0} \frac{1}{|B(p_0, \varepsilon)|}\int_{B(p_0, \varepsilon)} |\varphi(p) - \tilde{\varphi}_\varepsilon|\,dh(p) < \infty, \tag{17}$$

where $\tilde{\varphi}_\varepsilon$ is the mean value of φ over the disk $B(p_0, \varepsilon) = \{p \in \mathbb{S} : h(p, p_0) < \varepsilon\}$.

By Remarks 3-4 and Lemma 2 with the choice $\psi_{p_0, \varepsilon}(t) \equiv 1/t\log\frac{1}{t}$, in view of Lemma 4.1 and Remark 4.1 in [11], see also Lemma 13.2 and Remark 13.3 in [5], we come to the next

Theorem 5. *Under the hypotheses of Theorem 2, if, instead of* (9),

$$K_f(p) \leqslant Q(p) \in FMO(p_0) \quad \forall p_0 \in \partial D, \; for\ some\ Q : \mathbb{S} \to R^+. \tag{18}$$

Then the mapping f *is extended by continuity to* $\overline{D}$ *and* $f(\partial D) = \partial D^*$.

By Corollary 4.1 in [11], see also Corollary 13.3 in [5], we have the following proposition.

Corollary 3. *In particular, the conclusion of Theorem 5 holds, if*

$$\limsup_{\varepsilon \to 0} \frac{1}{|B(p_0, \varepsilon)|} \int_{B(p_0, \varepsilon)} K_f(p)\, dh(p) < \infty \quad \forall\ p_0 \in \partial D. \tag{19}$$

Remark 6. Note that Lemma 2 allows the pointwise analysis. Note also that, combining the above results on the continuous extension with Theorem 1, we come to the corresponding results on the homeomorphic extension of mappings with finite length distortion to the boundary. However, we do not formulate them in the explicit form here because of the restrictions on the volume of the paper.

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В.І. Рязанов [1,2], *С.В. Волков* [3]

[1] Інститут прикладної математики і механіки НАН України, Слов'янськ
[2] Черкаський національний університет ім. Богдана Хмельницького
[3] Донецький національний технічний університет, Покровськ
E-mail: vl.ryazanov1@gmail.com, serhii.volkov@donntu.edu.ua

ВІДОБРАЖЕННЯ ЗІ СКІНЧЕННИМ СПОТВОРЕННЯМ ДОВЖИНИ ТА РІМАНОВІ ПОВЕРХНІ

У термінах дилатацій доведено ряд критеріїв для неперервного та гомеоморфного продовження на границю відображень зі скінченним спотворенням довжини між областями на ріманових поверхнях. Критерієм для неперервного продовження обернених відображень на границю є дуже проста умова про інтегрованість дилатації в першому степені. При цьому область визначення відображення передбачається локально зв'язною на границі, а область значень — зі слабо плоскою границею. Критерії для неперервного продовження на границю прямих відображень мають набагато тоншу природу. Один із критеріїв полягає в існуванні мажоранти дилатації в класі функцій зі скінченним середнім коливанням, тобто таких, що мають кінцеве середнє відхилення від свого середнього значення над інфінітезимальними (нескінченно малими) колами з центром у відповідній граничній точці. Більш жорстка, але більш проста вимога полягає в тому, що середнє значення дилатації над інфінітезимальними колами з центром у відповідній граничній точці скінченне. Область визначення знову передбачається локально зв'язною на границі, а область значень — із сильно досяжною границею. Також наведено багато інших критеріїв неперервного продовження на границю прямих відображень. Як наслідки отримуємо відповідні критерії для гомеоморфного продовження на границю областей відображень зі скінченним спотворенням довжини.

Ключові слова: *ріманові поверхні, відображення зі скінченним спотворенням довжини, сильно досяжні й слабо плоскі границі.*

В.И. Рязанов [1,2], *С.В. Волков* [3]

[1] Институт прикладной математики и механики НАН Украины, Славянск
[2] Черкасский национальный университет им. Богдана Хмельницкого
[3] Донецкий национальный технический университет, Покровск
E-mail: vl.ryazanov1@gmail.com, serhii.volkov@donntu.edu.ua

ОТОБРАЖЕНИЯ С КОНЕЧНЫМ ИСКАЖЕНИЕМ ДЛИНЫ И РИМАНОВЫ ПОВЕРХНОСТИ

В терминах дилатаций доказан ряд критериев для непрерывного и гомеоморфного продолжения на границу отображений с конечным искажением длины между областями на римановых поверхностях. Критерием для непрерывного продолжения обратных отображений на границу оказывается очень простое условие об интегрируемости дилатаций в первой степени. При этом область определения отображения предполагается локально связной на границе, а область значений — со слабо плоской границей. Критерии для непрерывного продолжения на границу прямых отображений имеют гораздо более тонкую природу. Один из критериев состоит в существовании мажоранты дилатации в классе функций с конечным средним колебанием, т. е. имеющих конечное среднее отклонение от своего среднего значения над инфинитезимальными (бесконечно малыми) кругами с центром в соответствующей граничной точке. Более сильное, но более простое требование состоит в том, что среднее значение дилатации над инфинитезимальными кругами с центром в соответствующей граничной точке конечно. Область определения снова предполагается локально связной на границе, а область значений — с сильно достижимой границей. Также приведены многие другие критерии непрерывного продолжения на границу прямых отображений. В качестве следствий получаются соответствующие критерии для гомеоморфного продолжения в замыкание областей отображений с конечным искажением длины.

Ключевые слова: *римановы поверхности, отображения с конечным искажением длины, сильно достижимые и слабо плоские границы.*