Український математичний вісник
Том 15 **(2018)**, № 2, 251 – 263

Approximate controllability of the wave equation with mixed boundary conditions

Leonid Pestov, Dmytro Strelnikov

(*Presented by V. O. Derkach*)

Abstract. We consider the initial boundary value problem for acoustic equation in time space cylinder $\Omega \times (0, 2T)$ with unknown variable speed of sound, zero initial data, and mixed boundary conditions. We assume that (Neumann) controls are located at some part $\Sigma \times [0, T]$, $\Sigma \subset \partial\Omega$ of lateral surface of the cylinder $\Omega \times (0, T)$. The domain of observation is $\Sigma \times [0, 2T]$ and the pressure at another part $(\partial\Omega \backslash \Sigma) \times [0, 2T]$) is assumed to be zero for any control. We prove approximate boundary controllability for functions from subspace $V \subset H^1(\Omega)$ which traces have vanished on Σ provided that the observation time is $2T$ more than two acoustical radii of the domain Ω. We give an explicit procedure for solving Boundary Control Problem (BCP) for smooth harmonic functions from V (i.e. we are looking for a boundary control f which generates a wave u^f such that $u^f(., T)$ approximates any prescribed harmonic function from V). Moreover using Friedrichs-Poincare inequality we obtain conditional estimate for this BCP. Notice that for solving BCP for these harmonic functions we do not need the knowledge of the speed of sound.

2010 MSC. Primary 35R30; Secondary 35M33, 46E35.

Key words and phrases. Acoustical tomography, inverse problem, wave equation, Boundary Control method, generalized Friedrichs-Poincare inequality.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with a smooth boundary $\partial\Omega$. Let $\Sigma \subset \partial\Omega$ be an open set with a smooth boundary. The problem,

Received 29.03.2018
This work was supported by the Volkswagen Foundation project "Modeling, Analysis and Approximation Theory toward Applications in Tomography and Inverse Problem". The authors would like also to thank M. Belishev, V. Derkach and T. Fastovska for useful discussions and valuable remarks.

ISSN 1810 – 3200.

which we refer to as a *forward one* is an initial boundary value problem for the wave equation with boundary conditions

$$\begin{aligned} \rho u_{tt} - \Delta u &= 0 \quad \text{in} \quad \Omega \times (0,T), && (1.1)\\ u_\nu|_{\Sigma\times[0,T]} &= f, && (1.2)\\ u|_{(\partial\Omega\setminus\Sigma)\times[0,T]} &= 0, && (1.3)\\ u|_{t=0} &= u_t|_{t=0} = 0 \quad \text{in} \quad \overline{\Omega}. && (1.4) \end{aligned}$$

so $f = 0$ outside of $\Sigma \times [0,T]$. Here $\rho(x) = 1/c^2(x)$ is a smooth positive function ($c(x)$ is the speed of sound) and ν is the outward normal vector to the boundary $\partial\Omega$, u_ν is the normal derivative. We call function f Neumann *boundary control.* Introduce *the control space*

$$\mathcal{F}^T = L_2(\Sigma \times (0,T))$$

and the set of smooth controls

$$\mathcal{M}^T = C_0^\infty(\Sigma \times (0,T)).$$

Let u^f be the solution to the forward problem (*a wave*). Notice, that smooth controls generate classical waves. Due to the finiteness of the wave propagation speed one has

$$\operatorname{supp} u^f(\cdot, s) \subset \Omega^s, \quad s > 0,$$

where

$$\Omega^s = \left\{x \in \overline{\Omega} \,:\, \operatorname{dist}_\rho(x,\Sigma) \le s\right\}, \quad s \ge 0$$

and the distance being understood in the sense of the Riemannian metric $\sqrt{\rho(x)}\,|dx|$. The subdomain Ω^s is the part of Ω filled with waves at the moment $t = s$. In particular, for $T > T^* := \sup\limits_{x\in\Sigma} \operatorname{dist}_\rho(x,\partial\Omega)$, the relation $\Omega^T = \overline{\Omega}$ holds. With the system (1.1)–(1.4) one associates the *response operator* R^T, which acts by the rule

$$R^T f = u^f|_{\Sigma\times[0,T]}.$$

The inverse problem consists of determining function ρ in $\overline{\Omega}$ via the response operator R^{2T} provided $T > T^*$. One of the natural ways to solve this problem is the Boundary Control method (BC-method, Belishev, 1986, see e.g. [2–4] and works cited there, and the version of the BC-method proposed in [12,13]). We do not give a BC-solution of the inverse problem in this paper. We study the boudary controllability problem of the dynamical system (1.1)–(1.4) only (see *Remark* 5.3 at the end of the paper where we shortly comment on the inverse problem).

The boundary controllability is the principal question in the BC-method. For scalar hyperbolic equations like (1.1), the system (1.1)–(1.4) turns out to be *approximately controllable*. To formulate this property, observe that the *set of final states* $\mathcal{U}^T := \{u^f(\cdot,T)\colon f \in \mathcal{M}^T\}$ is contained in $L_2(\Omega^T)$. Then the approximate boundary controllability means that this set is dense in $L_2(\Omega^T)$. However the set of final states $\mathcal{U}^T$ is not dense in the Sobolev space $H^1(\Omega^T)$. The main result of this paper (Theorem 4.3) states that the closure of $\mathcal{U}^T$ in $H^1(\Omega^T)$ coincides with the subspace $V = \{u \in H^1_\rho(\Omega) |\, u_{\partial\Omega\setminus\Sigma} = 0\}$ of functions $u \in H^1_\rho(\Omega)$ vanishing on $\partial\Omega \setminus \Sigma$. The definition of the real Hilbert space $H^1_\rho(\Omega)$ will be given in Section 4.

In contrast to the works cited above, we use measurements (waves) at the same part of the boundary as controls. It corresponds to the scheme of using the boundary triple technique in [2]. The boundary triple used in the present paper is associated with the Zaremba Laplacian with mixed boundary conditions studied in [9]. In order to give an H^1 estimate for the difference $\varphi - u^f(\cdot,T)$, where φ in V is a harmonic function, we use an analogue of the Friedrichs inequality with an estimate from a part of the boundary.

2. Bilinear forms

Here we introduce one of the main tools of the BC method — the symmetric energy forms defined on the set of smooth controls. In what follows we fix $T > T^*$ so that $\Omega^T = \Omega$. We define two symmetric bilinear forms on $\mathcal{M}^T \times \mathcal{M}^T$

$$[f,g]_1 := \int_\Omega \rho(x) u^f(x,T) u^g(x,T)\, dx, \tag{2.1}$$

$$[f,g]_2 := \int_\Omega \Big(\nabla u^f(x,T), \nabla u^g(x,T)\Big)\, dx. \tag{2.2}$$

Both forms $[\,\cdot,\cdot\,]_1$ and $[\,\cdot,\cdot\,]_2$ are explicitly determined by the inverse data, i.e. the response operator R^{2T}. We derive these formulas here. For a function u, which depends on time and, possibly, other variables, denote

$$u_\pm(\cdot,t) = \frac{u(\cdot,t) \pm u(\cdot,2T-t)}{2},$$

$$(Iu)(\cdot,t) = \int_0^t u(\cdot,s)\, ds, \quad t \in [0,2T].$$

Proposition 2.1. *For any controls $f, g \in \mathcal{M}^T$ the equalities*

$$\int_{\Omega} \rho(x) u^g(x,T) u^f(x,T)\, dx = \int_{\Sigma\times[0,T]} \left[\left(R^{2T} g\right)_+ If - g_+ I R^{2T} f \right] dt\, d\sigma \tag{2.3}$$

$$\int_{\Omega} \left(\nabla u^g, \nabla u^f\right)(x,T)\, dx = \int_{\Sigma\times[0,T]} \left[f \frac{\partial}{\partial t}(Rg)_+ + g_+ \frac{\partial}{\partial t}(Rf) \right] dt\, d\sigma, \tag{2.4}$$

are valid, where $d\sigma$ is the standard measure on the boundary $\partial\Omega$.

Proof. For any smooth solution v to the wave equation (1.1), the equality

$$\rho \left(v u_t^f - u^f v_t \right)_t = \operatorname{div}\left(v \nabla u^f - u^f \nabla v \right)$$

holds. Clearly functions $u_{\pm}^g$ satisfy the wave equation. Substituting u_+^g for v and integrating over $\Omega \times [0,T]$, we obtain (note, that $u_+^g(\cdot, T) = u^g(\cdot, T)$, $(u_+^g)_t(\cdot, T) = 0$)

$$\begin{aligned} \int_{\Omega} \rho(x) u^g(x,T) u_t^f(x,T)\, dx \quad &= \quad \int_{\partial\Omega\times[0,T]} \left(u_+^g \frac{\partial u^f}{\partial \nu} - u^f \frac{\partial u_+^g}{\partial \nu} \right) dt\, d\sigma \\ &\overset{(1.3)}{=} \int_{\Sigma\times[0,T]} \left(u_+^g \frac{\partial u^f}{\partial \nu} - u^f \frac{\partial u_+^g}{\partial \nu} \right) dt\, d\sigma \\ &\overset{(1.2)}{=} \int_{\Sigma\times[0,T]} \left(u_+^g f - u^f g_+ \right) dt\, d\sigma. \end{aligned}$$

Taking into account that $u^{f_t} = u_t^f$ and $u^{If} = Iu^f$, and denoting $f_t = \widetilde{f}$ we arrive at

$$\int_{\Omega} \rho(x) u^g(x,T) u^{\widetilde{f}}(x,T)\, dx = \int_{\Sigma\times[0,T]} \left(u_+^g I\widetilde{f} - I u^{\widetilde{f}} g_+ \right) dt\, d\sigma. \tag{2.5}$$

Since

$$(R^{2T} f)(x,t) = (R^T f)(x,t) \quad \text{for all} \quad x \in \partial\Omega,\ t \in (0,T) \tag{2.6}$$

then

$$\begin{aligned} u_+^g(x,t) &= \frac{1}{2} \left\{ \left(R^T g\right)(x,t) + \left(R^{2T} g\right)(x, 2T - t) \right\} \\ &= \left(R^{2T} g\right)_+ (x,t) \quad \text{for} \quad x \in \partial\Omega,\ t \in (0,T), \end{aligned} \tag{2.7}$$

$$u^{\widetilde{f}}(x,t) = (R^T \widetilde{f})(x,t) = (R^{2T} \widetilde{f})(x,t) \quad \text{for} \quad x \in \partial\Omega,\ t \in (0,T). \tag{2.8}$$

By substituting (2.7) and (2.8) into (2.5) and replacing $\widetilde{f}$ by f one obtains (2.3).

Consider the form $[\,\cdot,\cdot\,]_2$ in (2.2). For any smooth solution v to the wave equation the equality

$$\Big[\rho v_t u_t^f + (\nabla v, \nabla u^f)\Big]_t = \operatorname{div}\Big(v_t \nabla u^f + u_t^f \nabla v\Big)$$

holds. Substituting u_+^g for v and integrating over $\Omega \times [0,T]$, we get (using (1.2)) the equality

$$\int_\Omega \Big(\nabla u^g, \nabla u^f\Big)(x,T)\,dx = \int_{\Sigma\times[0,T]} \Big((u_+^g)_t f + u_t^f g_+\Big)\,dt\,d\sigma,$$

which coincides with (2.4). □

Remark 2.2. In the case when Σ is the whole boundary $\partial\Omega$ the formula (2.3) coincides with the formula of Blagoveshchenskii presented in [1]. The formula (2.4) in the case when $\Sigma = \partial\Omega$ was obtained in [13].

3. Boundary triple and Zaremba operator

Different versions of dynamical systems with boundary controls are related (see [2]) to different choices of boundary triples for the operator in the space domain, see definitions in [6,8]. Here we introduce the boundary triple corresponding to the system (1.1)–(1.4).

Let the minimal operator $-\Delta_{min}$ (resp. the maximal operator $-\Delta_{max}$) be defined as the closure in $L_2(\Omega)$ of the operator $-\Delta$ restricted to $C_0^\infty(\Omega)$ (resp. $C^\infty(\overline{\Omega})$). It is known (see, for instance, [5, Theorem 4.8]) that $-\Delta_{\max} = (-\Delta_{\min})^*$ and

$$\operatorname{dom}(-\Delta_{\min}) = H_0^2(\Omega), \quad \operatorname{dom}(-\Delta_{\max}) = \left\{u \in L^2(\Omega) \,|\, \Delta u \in L^2(\Omega)\right\},$$

where Δ is understood in the sense of distributions. Let γ_D and γ_N be the Dirichlet and the Neumann traces

$$\gamma_D\colon\ u \mapsto u|_{\partial\Omega}\,, \quad \gamma_N\colon\ u \mapsto u_\nu|_{\partial\Omega}\,, \quad f \in H^2(\Omega). \tag{3.1}$$

It is known, e.g. from Lions and Magenes [11] that γ_D and γ_N, defined originally on $H^2(\Omega)$ admit continuations to surjective operators

$$\gamma_D\colon \operatorname{dom}(-\Delta_{\max}) \to H^{-1/2}(\partial\Omega), \quad \gamma_N\colon \operatorname{dom}(-\Delta_{\max}) \to H^{-3/2}(\partial\Omega).$$

Dirichlet $-\Delta_D$ and Neumann $-\Delta_N$ realizations of $-\Delta$, defined as restrictions of the operator $-\Delta_{\max}$ to the domains

$$\operatorname{dom}(-\Delta_D) = \left\{u \in H^2(\Omega)\colon \gamma_D u = 0\right\},$$

$$\operatorname{dom}(-\Delta_N) = \left\{u \in H^2(\Omega) \colon \gamma_N u = 0\right\},$$

are selfadjoint extensions of the operator $-\Delta_{min}$. One more selfadjoint realization of $-\Delta$, so-called Zaremba extension $-\Delta_\Sigma$ of $-\Delta_{min}$, is defined as the restriction of $-\Delta_{\max}$ to the set

$$\operatorname{dom}(-\Delta_\Sigma) = \left\{u \in \operatorname{dom}(-\Delta_{\max}) \colon \gamma_D u|_{\partial\Omega\setminus\Sigma} = 0, \gamma_N u|_\Sigma = 0\right\}, \quad (3.2)$$

see [9]. Its domain is not contained in $H^{3/2}(\Omega)$, however for every $\epsilon > 0$ the following inclusion holds $\operatorname{dom}(-\Delta_\Sigma) \subset H^{3/2-\epsilon}(\Omega)$. Notice, that the operator $-\Delta_\Sigma$ in $L^2(\Omega)$ has a discrete spectrum.

Let $H^{3/2}_\Delta := \{u \in H^{3/2}(\Omega) \colon \Delta u \in L_2(\Omega)\}$. According to [11]

$$\gamma_D(H^{3/2}_\Delta(\Omega)) = H^1(\partial\Omega), \quad \gamma_N(H^{3/2}_\Delta(\Omega)) = L^2(\partial\Omega)$$

and for all $u, v \in H^{3/2}_\Delta$ the following Green formula holds

$$\int_\Omega (u\Delta v - v\Delta u)\, dx = \int_{\partial\Omega} (\gamma_D u \gamma_N v - \gamma_N u \gamma_D v)\, d\sigma. \quad (3.3)$$

Let us define the subspace D_* of $H^{3/2}_\Delta(\Omega)$ by

$$D_* := \left\{u \in H^{3/2}_\Delta(\Omega) \colon (\gamma_D u)|_{\partial\Omega\setminus\Sigma} = 0\right\}, \quad (3.4)$$

and let the operators $\gamma^\Sigma_N, \gamma^\Sigma_D$ be defined as restrictions of the operators $u \mapsto \gamma_N u|_\Sigma$, $u \mapsto \gamma_D u|_\Sigma$ to the domain D_*.

$$\gamma^\Sigma_N u := \gamma_N u|_\Sigma, \quad \gamma^\Sigma_D u := \gamma_D u|_\Sigma, \quad (u \in D_*). \quad (3.5)$$

The triple $\{L_2(\Sigma), \gamma^\Sigma_N, \gamma^\Sigma_D\}$ is a boundary triple for the operator $-\Delta_{\max}$ in the sense of [6]. As was shown in [7], the operator $-\Delta_{0,\Sigma}$ defined as the restriction of $-\Delta_{\max}$ to the domain

$$\operatorname{dom}(-\Delta_{0,\Sigma}) = \left\{u \in H^{3/2}_\Delta(\Omega) \colon (\gamma_D u)|_{\partial\Omega\setminus\Sigma} = (\gamma_N u)|_\Sigma = 0\right\} \quad (3.6)$$

is essentially selfadjoint in $L^2(\Omega)$. Namely, the closure of $-\Delta_{0,\Sigma}$ coincides with the Zaremba operator $-\Delta_\Sigma$.

Remark 3.1. By the terminology used in [7, Definition 1.8] the triple $\{L_2(\Sigma), \gamma^\Sigma_N, \gamma^\Sigma_D\}$ is called an ES-generalized boundary triple for $-\Delta_{\max}$, with "ES" meaning that the operator $-\Delta_{0,\Sigma}$ in (3.6) is essentially selfadjoint.

The Green formula corresponding to this boundary triple takes the form

$$\int_\Omega (u\Delta v - v\Delta u)\, dx = \int_\Sigma (\gamma_D u \gamma_N v - \gamma_N u \gamma_D v)\, d\sigma \quad (3.7)$$

with $u, v \in D_*$. This formula will be used in the next section.

4. Approximate controllability

In order to apply the general results of [11] to the system (1.1)–(1.4) let us consider the Hilbert space $H(\Omega) \coloneqq L^2_\rho(\Omega)$ with the norm

$$\|u\|_{H(\Omega)} \coloneqq \left(\int_\Omega \left\{\rho(x)|u(x)|^2\right\} dx\right)^{1/2}. \tag{4.1}$$

If there are $\rho_1, \rho_2 > 0$ such that ρ satisfies the inequalitues

$$\rho_1 \le \rho(x) \le \rho_2 \quad (x \in \Omega) \tag{4.2}$$

then the norm (4.1) is equivalent to the standard norm in $L^2(\Omega)$. Similarly, the Sobolev space $H^1_\rho(\Omega)$ is defined as the standard Sobolev space $H^1(\Omega)$ endowed with the inner product

$$(u, v)_{H^1_\rho(\Omega)} \coloneqq \int_\Omega \{\rho uv + (\nabla u, \nabla v)\}\, dx. \tag{4.3}$$

Let V be a subspace of $H^1_\rho(\Omega)$ specified by the equality

$$V = \{u \in H^1_\rho(\Omega) \colon u_{\partial\Omega\setminus\Sigma} = 0\}. \tag{4.4}$$

Next we recall (see [10, Lemma 2.3]) the following Friedrichs type inequality:

$$k_1 \int_\Omega |u|^2\, dx \le \int_\Omega |\nabla u|^2\, dx + \int_{\Sigma_0} |u|^2\, d\sigma, \tag{4.5}$$

which is valid for every open set $\Sigma_0 \subset \partial\Omega$ with $\sigma(\Sigma_0) > 0$, for some constant $k_1 > 0$ and for all $u \in H^1_\rho(\Omega)$.

Proposition 4.1. *Let Σ_0 be an open subset of $\partial\Omega$ with $\sigma(\Sigma_0) > 0$, and let $W^1_{2,2}(\Omega, \Sigma_0)$ be the completion of the set of functions $C^\infty(\Omega) \cap C(\overline{\Omega})$ with respect to the norm*

$$\|u\|_{W^1_{2,2}(\Omega,\Sigma_0)} = \left(\int_\Omega |\nabla u|^2\, dx\right)^{1/2} + \left(\int_{\Sigma_0} |u|^2\, d\sigma\right)^{1/2}, \quad u \in H^1(\Omega), \tag{4.6}$$

and let ρ satisfy the inequalities (4.2)*. Then:*

(1) $W^1_{2,2}(\Omega, \Sigma_0) = H^1_\rho(\Omega)$ and there exist constants $C_1, C_2 > 0$ such that

$$C_1\|u\|_{W^1_{2,2}(\Omega,\Sigma)} \le \|u\|_{H^1(\Omega)} \le C_2\|u\|_{W^1_{2,2}(\Omega,\Sigma)} \tag{4.7}$$

for all $u \in H^1_\rho(\Omega)$.

(2) If Σ is an open subset of $\partial\Omega$ with $\sigma(\partial\Omega \setminus \overline{\Sigma}) > 0$, and ρ satisfies (4.2) *then the norm $\|\cdot\|_{H^1_\rho(\Omega)}$ on V is equivalent to the norm $\left(\int_\Omega |\nabla u|^2\, dx\right)^{1/2}$.*

Proof. (1) The proof of (1) follows from the Friedrichs type inequality (4.5) and from the estimate (see [11])

$$\|u\|_{L_2(\partial\Omega)} \le k_2 \|u\|_{H^1(\Omega)} \quad (u \in H^1(\Omega)),$$

which is valid for some $k_2 > 0$.

(2) The statement (2) is immediate from (1) applied for $\Sigma_0 = \partial\Omega \setminus \overline{\Sigma}$, since then the second integral in (4.6) vanishes for all $u \in V$. □

Proposition 4.2. *Let Σ be an open subset of $\partial\Omega$ with $\sigma(\partial\Omega \setminus \overline{\Sigma}) > 0$, let ρ satisfies the inequalities* (4.2)*, let the form $a(u,v)$ be defined on V by*

$$a(u,v) := \int_\Omega (\nabla u, \nabla v) dx, \quad u, v \in V.$$

and let the operator A in $L^2_\rho(\Omega)$ be given by $A := \rho^{-1}(-\Delta_\Sigma)$. Then:

(1) The operator A is selfadjoint and nonnegative in $H = L^2_\rho(\Omega)$. The spectrum of A is discrete.

(2) The form $a(u,v)$ admits the representation

$$a(u,v) = (Au, v)_H, \quad u, v \in \mathrm{dom}(-\Delta_\Sigma).$$

Proof. The statement (1) follows from the properties of the Zaremba operator $-\Delta_\Sigma$ mentioned in Section 3. The discretness of the spectrum of A is implied by the Courant minimax principle and the corresponding statement for the operator $-\Delta_\Sigma$ in $L^2(\Omega)$. The nonnegativity of A is postponed until the next paragraph.

Substituting in the 1-st Green formula

$$-(\Delta u, v)_{L^2(\Omega)} = \int_\Omega (\nabla u(x), \nabla v)\, dx - \int_{\partial\Omega} u(x) \overline{v_\nu(x)}\, d\sigma,$$

$u, v \in \mathrm{dom}(-\Delta_\Sigma)$ one obtains

$$a(u,v) = -(\Delta u, v)_{L^2(\Omega)} = -(\rho^{-1}\Delta u, v)_{L^2_\rho(\Omega)} = (Au, v)_H.$$

This formula proves (2) and the nonnegativity of A. □

Recall that the set of reachable states of the wave at the instant of time $t = T$ is defined by the equality

$$\mathcal{U}^T := \left\{ u^f(\cdot, T) \colon f \in \mathcal{M} \right\}. \tag{4.8}$$

It is clear, that $\mathcal{U}^T \subset V$.

Theorem 4.3. *Let Σ be an open subset of $\partial\Omega$ with $\sigma(\partial\Omega \setminus \overline{\Sigma}) > 0$ and let ρ satisfy the inequalities* (4.2)*. Then the variety $\mathcal{U}^T$ is dense in V.*

Proof. Let $\varphi \in V \ominus \mathcal{U}^T$, i.e. for every control $f \in \mathcal{M}^T$ the following equality holds

$$\left(u^f(\cdot, T), \varphi \right) = 0. \tag{4.9}$$

Let us show that $\varphi = 0$. By [11, Theorem 3.8.1] the following system

$$\rho(x) v_{tt} - \Delta v = 0 \qquad \text{in } \Omega \times (0, T), \tag{4.10}$$

$$v|_{t=T} = 0, \quad v_t|_{t=T} = \varphi \qquad \text{in } \Omega, \tag{4.11}$$

$$v|_{\partial\Omega\setminus\Sigma} = v_\nu|_\Sigma = 0 \tag{4.12}$$

has a weak solution $v \in L^2(0, T; V)$ such that $v_t \in L^2(0, T; H)$. The latter means, that

$$a(v, u) + \int_\Omega \rho u v_{tt} \, dx = 0 \tag{4.13}$$

Let the wave $u^f(x, t)$ be the solution of the problem (1.1)–(1.4), and let v be the solution of the system (4.10)–(4.12). Then by the 1-st Green formula

$$a(v, u) - \int_\Sigma v u_\nu d\sigma + \int_\Omega \rho u_{tt} v \, dx = 0.$$

Subtracting the last equation from (4.13) and substituting $u^f_\nu|_{\Sigma\times[0,T]}$ by f one obtains

$$\int_\Omega \rho(u^f v_t - v u^f_t)_t \; dx = -\int_\Sigma u^f_\nu v \, d\sigma = -\int_\Sigma f v \, d\sigma. \tag{4.14}$$

Integrating this identity on $[0, T]$ yields

$$\int_\Omega \rho u^f(x, T) \varphi(x) \; dx = -\int_{\Sigma\times[0,T]} f v \, d\sigma \, dt. \tag{4.15}$$

Let now w be the weak solution of the system (see [11, Theorem 3.8.1])

$$\rho(x) w_{tt} - \Delta w = 0 \qquad \text{in } \Omega \times (0, T), \tag{4.16}$$

$$w|_{t=T} = \varphi, \quad w_t|_{t=T} = 0 \qquad \text{in } \Omega, \tag{4.17}$$

$$w|_{\partial\Omega\setminus\Sigma} = w_\nu|_\Sigma = 0, \tag{4.18}$$

such that $w \in L^2(0,T;V)$ such that $w_t \in L^2(0,T;H)$. Notice that

$$\operatorname{div}\left(w_t \nabla u^f + u_t^f \nabla w\right) = \left(\rho u_t^f w_t + (\nabla u^f, \nabla w)\right)_t \tag{4.19}$$

and by the Gauss–Ostrogradskii formula

$$\int_\Omega \left(\rho u_t^f w_t + (\nabla u^f, \nabla w)\right)_t \, dx = \int_{\partial\Omega} (w_t u_\nu + u_t w_\nu)\, d\sigma. \tag{4.20}$$

Integrating this identity on $t \in [0,T]$ one obtains

$$\int_\Omega \left(\nabla u^f, \nabla\varphi\right)(x,T)\, dx = \int_{\Sigma\times[0,T]} f w_t \, d\sigma\, dt. \tag{4.21}$$

Combining the equalities (4.15) and (4.21) and taking into account (4.9) one obtains

$$0 = (u^f, \varphi)_{H^1_\rho(\Omega)} = -\int_{\Sigma\times[0,T]} f(v - w_t)\, d\sigma\, dt. \tag{4.22}$$

Since $\mathcal{M}^T$ is dense in $L_2(\Sigma\times[0,T])$ one obtains from (4.22)

$$(v - w_t)|_{\Sigma\times[0,T]} = 0. \tag{4.23}$$

By virtue of (4.12), (4.18) one obtains from (4.23)

$$(v - w_t)|_{\partial\Omega\times[0,T]} = (v - w_t)_\nu|_{\Sigma\times[0,T]} = 0. \tag{4.24}$$

Therefore, the function $u = v - w_t$ is a solution of the system

$$\rho(x) u_{tt} - \Delta u = 0 \qquad \text{in } \Omega\times(0,T), \tag{4.25}$$

$$u(x,T) = 0 \quad \text{in} \quad \Omega, \tag{4.26}$$

$$u|_{\partial\Omega\times[0,T]} = u_\nu|_{\Sigma\times[0,T]} = 0. \tag{4.27}$$

Let us consider the odd extension of u to $\Omega\times[T,2T]$:

$$\widetilde{u}(\cdot,t) := \begin{cases} u(\cdot,t), & t \in [0,T), \\ -u(\cdot,2T-t), & t\in[T,2T]. \end{cases} \tag{4.28}$$

Then the function $\widetilde{u}$ satisfies the system

$$\rho\widetilde{u}_{tt} - \Delta\widetilde{u} = 0, \quad \text{in } \Omega\times(0,2T), \tag{4.29}$$

$$\widetilde{u}|_{\partial\Omega\times[0,2T]} = \widetilde{u}_\nu|_{\Sigma\times[0,2T]} = 0. \tag{4.30}$$

The reasonings of [3] show that by the Holmgren–John–Tataru theorem [14] $\widetilde{u} = 0$ in the space-time domain

$$\{(x,t) \in \Omega \times (0,2T)\colon\ \mathrm{dist}_\rho(x,\Sigma) < t < 2T - \mathrm{dist}_\rho(x,\Sigma)\}. \tag{4.31}$$

Therefore, $v = w_t$ in the domain $\{(x,t) \in \Omega \times (0,T)\colon\ \mathrm{dist}_\rho(x,\Sigma) < t \le T\}$ and hence

$$w(\cdot,t) = \varphi - \int_t^T v(\cdot,s)\,ds. \tag{4.32}$$

By the last equality one gets $\varphi_\nu|_\Sigma = 0$, $\varphi|_{\partial\Omega\setminus\Sigma} = 0$ and

$$\begin{aligned} 0 &= \rho w_{tt} - \Delta w = -\Delta\varphi - \int_t^T [\rho v_{ss} - \Delta v](\cdot,s)ds + \rho v_t|_{t=T} \\ &= \rho\varphi - \Delta\varphi = \rho(I + A)\varphi \quad \text{in } \Omega. \end{aligned} \tag{4.33}$$

Since $A > 0$ this eigenvalue problem has no non-trivial solutions, therefore $\varphi = 0$. □

5. Estimates for the error of approximation

Now we are going to estimate the error $\|u^f(\cdot,T) - \varphi\|_{L_2(\Sigma)}$ of the approximation of a harmonic function φ by the wave $u^f(\cdot,T)$ ($f \in \mathcal{M}^T$) in terms of the response operator.

For every $\varphi \in H^1_\rho(\Omega)$ let us set

$$\Phi(f) := \int_\Omega \left|\nabla u^f(\cdot,T) - \nabla\varphi\right|^2 dx. \tag{5.1}$$

Proposition 5.1. *Let $\varphi \in V$ be a harmonic function in Ω and let $f \in \mathcal{M}^T$. Then the functional $\Phi(f)$ takes the form*

$$\Phi(f) = [f,f]_2 - 2\int_\Sigma (R^{2T} f)(\cdot,T)\varphi_\nu\,d\sigma + \int_\Sigma \varphi\varphi_\nu\,d\sigma. \tag{5.2}$$

Proof. Indeed, it follows from (5.1) that

$$\Phi(f) = \int_\Omega |\nabla u^f(\cdot,T)|^2 dx - 2\int_\Omega (\nabla u^f(\cdot,T), \nabla\varphi)\,dx + \int_\Omega |\nabla\varphi|^2 dx.$$

In view of the 1-st Green formula

$$\int_\Omega (\nabla u^f(x,T), \nabla\varphi)\,dx = \int_{\partial\Omega} u^f(x,T)\varphi_\nu(x)\,d\sigma,$$

$$\int_\Omega |\nabla\varphi|^2 dx = \int_{\partial\Omega} \varphi(x)\varphi_\nu(x)\,d\sigma$$

and the assumptions $u^f|_{\Omega\setminus\Sigma} = \varphi|_{\Omega\setminus\Sigma} = 0$ this implies (5.2). □

Proposition 5.2. *Let Σ be an open subset of $\partial\Omega$ with $\sigma(\partial\Omega \setminus \overline{\Sigma}) > 0$, let ρ satisfy the inequalities* (4.2) *and let $\varepsilon > 0$. Then:*

(1) For every harmonic function $\varphi \in V$ there exists a control $f \in \mathcal{M}^T$ such that
$$\Phi(f) \leq \varepsilon^2. \tag{5.3}$$

(2) There exists a constant $C > 0$ such that if (5.3) *holds for some $\varphi \in V$ and $f \in \mathcal{M}^T$ then*
$$\left\| u^f(\cdot, T) - \varphi(\cdot) \right\|_{H^1_\rho(\Omega)} \leq C\varepsilon. \tag{5.4}$$

Proof. (1) Since $\mathcal{U}^T$ is dense in $V(\subset H^1_\rho(\Omega))$ then for every $\varepsilon > 0$ there is $f \in \mathcal{M}^T$ such that
$$\left\| (u^f)(\cdot, T) - \varphi \right\|^2_{L_2(\Omega)} + \left\| (\nabla u^f)(\cdot, T) - \nabla\varphi \right\|^2_{L_2(\Omega)} \leq \varepsilon^2. \tag{5.5}$$

In view of (5.1) this yields (5.3).

(2) It follows from (4.3), the Friedrichs type inequality (4.6) and (5.3) that
$$\begin{aligned} \left\| u^f(\cdot, T) - \varphi(\cdot) \right\|^2_{H^1_\rho(\Omega)} &\leq \max_{x\in\Omega} \rho(x)^{1/2} \left\| (u^f)(\cdot, T) - \varphi \right\|^2_{L_2(\Omega)} \\ &\quad + \left\| (\nabla u^f)(\cdot, T) - \nabla\varphi \right\|^2_{L_2(\Omega)} \\ &\leq \left(1 + \frac{\rho_2}{k_1}\right) \left\| (\nabla u^f)(\cdot, T) - \nabla\varphi \right\|^2_{L_2(\Omega)} \\ &\leq \left(1 + \frac{\rho_2}{k_1}\right) \varepsilon^2. \end{aligned}$$

This proves (5.4). □

Remark 5.3. In the case when Σ coincides with $\partial\Omega$ one has the equality $V = H^1(\Omega)$ and hence the system is approximately controllable in $H^1(\Omega)$. As is known the set of products of harmonic functions $\varphi, \psi \in H^1(\Omega)$ is dense in $L^2(\Omega)$ and this allows to solve the inverse problem for the speed of sound ρ. In the case when $\Sigma \neq \partial\Omega$ and $m(\partial\Omega \setminus \Sigma) > 0$ we do not know how big the set $\{\varphi\psi \colon \varphi, \psi \in V\}$ is. If it is dense in $L^2(\Omega)$ then one can use the strategy of [13] in order to give the procedure of reconstruction of ρ.

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Contact information

Leonid Pestov Immanuel Kant Baltic Federal University,
Kaliningrad, Russia
E-Mail: `lpestov@kantiana.ru`

Dmytro Strelnikov Vasyl' Stus Donetsk National University,
Vinnytsya, Ukraine
E-Mail: `d.strelnikov@donnu.edu.ua`