On some problems arising from the application. *

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We consider some mathematical problems, arising from the study of the continuous media.

It is characteristic that these evolutionary problems are described by the equations, which do not belong to the systems of the Cauchy-Kovalevskaja class.

I. It is well known that rotating currents of the liquid and the gas are widely applied to the technology (see, for instance, [1]-[8]). In the paper [1] it was showed that one could get an axialy summetric velocity field with an linear dependence of the axial component of the velocity from the axial coordinate by the change of the sizes of the of the rotational camera. However, I would like to note that nobody observed this phenomenon in the large series of experiments. And its arising become possible for the special sizes of the rotational camera. If we use cylindrical coordinates (r, φ, z) and suppose that the radial u and the tangential v components of the velocity vector depend only on r and t, and the axial component is as follows

$$w = w_1(r,t) + z w_2(r,t)$$

we can reduce the boundary value problem for the Navier-Stokes equation to the following problem

$$\frac{\partial^2 F}{\partial t \partial x} = 4x \frac{\partial^3 F}{\partial x^3} + 2F \frac{\partial F}{\partial x} + 4 \frac{\partial^2 F}{\partial x^2} - 2 \left(\frac{\partial F}{\partial x} \right) + p_1(t). \tag{1}$$

$$F\Big|_{t=0} = F_0(x), \quad F\Big|_{x=1} = \varphi_0(t), \quad \frac{\partial F}{\partial x}\Big|_{x=1} = \varphi_1(t). \tag{2}$$

$$\frac{\partial \Theta}{\partial t} = 4x \frac{\partial^2 \Theta}{\partial x^2} - 2F \frac{\partial \Theta}{\partial x} \tag{3}$$

$$\Theta\Big|_{t=0} = \Theta_0(x), \qquad \Theta\Big|_{x=1} = \psi(t) \tag{4}$$

$$\frac{\partial w_1}{\partial t} = 4x \frac{\partial^2 w_1}{\partial x^2} + 2F \frac{\partial w_1}{\partial x} - w_1 w_2 + 4 \frac{\partial w_1}{\partial x} - p_2(t). \tag{5}$$

$$w_1\Big|_{t=0} = w_1^0(x), \quad w_1\Big|_{x=1} = n(t), \quad p_2 = \frac{\partial \mathcal{P}}{\partial z}\Big|_{z=0}$$

$$w_2(t,x) = \frac{\partial F}{\partial x}, \quad w_2\Big|_{t=0} = F_0'(x), \quad w_2\Big|_{x=1} = \varphi_1(t) \tag{6}$$

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$$\frac{\partial \mathcal{P}}{\partial x} \equiv -2\frac{\partial^2 F}{\partial x^2} + \frac{1}{2x}\frac{\partial F}{\partial t} + \frac{F^2+\Theta^2}{4x} - \frac{F\frac{\partial F}{\partial x}}{x} \tag{7}$$

$$u \cdot r = -F, \quad v \cdot r = \Theta, \quad x = r^2, \quad \mathcal{P} = p_1(t)z^2 + p_2(t) + p(t,r) \tag{8}$$

$$U = \Big(u_1, u_2, w\Big), \;\; u = \sqrt{u_1^2+u_2^2}\cos\Big(\tilde{\Theta}-\varphi\Big), \;\; v = \sqrt{u_1^2+u_2^2}\sin\Big(\tilde{\Theta}-\varphi\Big)$$

$$\Big(u_1, u_2,\Big) = \sqrt{u_1^2+u_2^2}\Big(\cos\tilde{\Theta}, \sin\tilde{\Theta}\Big)\Big(r,\varphi,z\Big)$$

Thus, if we set the initial velocity vector such that its divergence is equal to zero ($w_2(0,x) = \frac{\partial F_0}{\partial x}$), the all three components of the velocity on the lateral partitions for $x = \alpha$, and the pressure gradient on the entrance ($p_2 = \frac{\partial \mathcal{P}}{\partial z}|_{z=0}$), are only depended from r and t, respectivily, we can get the problem (1)-(8). In the paper of O. Provorova, V. Kislyh and the author of the talk the stationary solutions of this problem were studied. V. Belonosov and me established some existence and uniqueness theorems for the nonstationary problem for the small time. N. Keilman made the careful analysis of stationary solutions. The first Lyapunov method was justified for distinct functional spaces by V. Belonosov and the author of the talk. These results were used by V. Kislyh and myself for the definition of the effective viscosity coefficients by the experience. These investigations were useful for the preparation of the rotational camera, which are used in the bibliography. It seems to be perspective to use such apparatus in the weishtlessness.

II. Below we discuss some equations, incountered in small water theory (Bussineski linear equation [9]), on the theory of electromagnatic waves in the long lines [10], in the plasma theory [11], in the dynamics of the stratified fluid.

I would not like to discuss the physical sence of the coefficients of these equations, which are as follows

$$a\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(b\frac{\partial u}{\partial x}\right) - \frac{\partial^2}{\partial x \partial t}\left(c\frac{\partial^2 u}{\partial x \partial t}\right) = q \tag{9}$$

My interest to such problems is connected with the fact, that the equations of such type are simple, and at the same time, very important model of the class of the equations, which cannot be solved with respect to t.

The qualitative analysis of the corresponding problems leads to the necessary investigation of operators, which have some continuous spectrum. Let us consider the following problems:

$$\frac{\partial^2}{\partial t^2}\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}\left(p(x\frac{\partial u}{\partial x})\right) = 0 \tag{10}$$

$$u\Big|_{x=0} = u\Big|_{x=1} = 0 \tag{11}$$

or

$$u_x\Big|_{x=0} = u\Big|_{x=1} = 0 \tag{12}$$

and initial data

$$u\Big|_{t=0} = u_0(x), \qquad u_t\Big|_{t=0} = u_1(x). \tag{13}$$

It is especially simple to represent a solution of the problem (10), (12), (13). I suppose that $p(x)$ is a smooth enough and a strictly monotone function. Then, if $p(x) = q^2(x) > 0$ we have

$$u_x(x,t) = A(x)\sin qt + B(x)\cos qt \tag{14}$$

and the functions A and B can be expressed by the initial data. The higher derivatives are unbounded. This follows from these fact, that nobody can describe the process for the large time by the linearization. In the case of the problem (10), (11), (13) we can get a suitable representation of the solution, if we use Sobolev's method of the expansion in generalized eigenfunctions. This question was studied in [13]-[15] and the following formulae was obtained

$$u(x,t) = \int\limits_{p(0)}^{p(1)} \left[\varphi_1(\lambda)\cos\sqrt{\lambda}t + \varphi_2(\lambda)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\right] u(x,\lambda)\frac{d\lambda}{K(\lambda)} \tag{15}$$

$$\varphi_i(\lambda) = \int\limits_0^1 \frac{u_i'(x) - u_i'(x_\lambda)}{p(x)-\lambda}, \qquad i = 0,1 \tag{16}$$

$$p(x_\lambda) =\equiv \lambda,$$

$$u(x,\lambda) = \begin{cases} \int\limits_0^x \frac{d\varepsilon}{p(\xi)-\lambda}, & 0 \le x < x_\lambda \\ \\ \int\limits_1^x \frac{d\varepsilon}{p(\xi)-\lambda}, & x_\lambda < x \le 1 \end{cases} \tag{17}$$

V. Fal'chenko got the following formulae

$$K(\lambda) = \frac{1}{p'(x_\lambda)}\left\{\pi^2 + \left[\int\limits_0^1 \frac{d\xi}{p(\xi)-\lambda}\right]^2\right\} \tag{18}$$

and the asymptotical expansion for the function $u(t,x)$. It follows from the obtained results, that the considerable problem is not admissible for discribing the process for the large time.

III. Consider the problem

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial^2 u}{\partial y^2} = 0 \tag{19}$$

with the smooth enough initial data

$$u\Big|_{t=0} = u_0(x,y), \qquad u_t\Big|_{t=0} = u_1(x,y) \tag{20}$$

and one of the two boundary conditions

$$u\Big|_\Gamma = 0 \tag{21}$$

$$\frac{\partial^2}{\partial t^2}\frac{du}{dn} + \frac{\partial u}{\partial y}\cos(n,y)\Big|_\Gamma = 0 \tag{22}$$

We consider the problem in the smooth damain Ω, with the strictly convex boundary Γ. It is easy to establish the solvability and the Hadamard correctness for such problems. There exists a connection between the solutions of the problems (19), (20), (21) and (19), (20), (22) which can be established with the help of analogue of the Cauchy-Riemann system:

$$\frac{\partial v}{\partial x} = -\frac{\partial}{\partial y}\Big(u_{tt} + u\Big), \qquad \frac{\partial v}{\partial y} = -\frac{\partial}{\partial x}u_{tt}. \tag{23}$$

Because of this it is sufficient to discuss only the behaviour of solutions of the problem (19), (20), (21). This problem was been studying in [19]-[29]. On the basis of the facts from [19]-[29] S. L. Sobolev made the conclusion that the solutions of this problem were not suitable for describing any motion of the continuous media for the large time. The analogous situation is relative to the three-dimensional case.

The solutions of the problems considered above are as follows

$$u(x,y,t) = \int\limits_{\mu_1(x,y)}^{\mu_2(x,y)} f(\alpha)\exp\left\{\pm i\sqrt{\frac{\alpha^2}{1+\alpha^2}t}\right\} d\alpha. \tag{24}$$

Here

$$(-1)^k \mu_k \frac{\partial \mu_k}{\partial x} = \frac{\partial \mu_k}{\partial y}, \qquad \mu_1\Big|_\Gamma = \mu_2\Big|_\Gamma. \tag{25}$$

The derivatives of these solutions bahave just in a same way as the derivatives of solutions of the problem (10)-(12).

The solutions of the problem are the normal pressure, entering to the equation of the small vibration of the rotation fluid. If we use the results of the papers, cited above, we can establish that the mathematical description of the motion of the rotating fluid is impossible for the three-dimensional case. In connection with these, I would like to note, that it is incorrect to use the terms like "tornado", "turbulence", "curl" for the physical interpretation of solutions of the discussed linear problems, appearing in some recent publications, devoted to the Sobolev problem.

There arise many problems relating to such mathematical directions such as function theory, group theory, the theory of singular integral equations, numbers theory and so on. It seems to me that the operator, arising here, is the typical representative of operators with the infinite-to-one spectrum.

Precisely this fact can explain Sobolev interest to the problem, discussed above.

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