Self-stochasticity in deterministic boundary value problems

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1 Introduction

This paper presents the experience of applying dynamical systems theory to an investigation into nonlinear boundary value problems for partial differential equations (PDE for short) in the case that their solutions become chaotic with time. To describe the long time behavior of such solutions, the concept of self-stochasticity had been suggested (10–12).

The results reported in this work are concerned linear systems of PDE with nonlinear boundary conditions; general ideas on the manner in which chaotic solutions may be described are set forth by the example of several simplest boundary value problems. A fuller treatment can be found in (5–14). It takes considerable efforts to extend these results for nonlinear systems of PDE.

Definition 1. *By **self-stochasticity in deterministic dynamical systems** we mean the fact that: for a deterministic system there exists a completion of its phase space with random functions such that the system has a "massive" set of trajectories whose ω-limit sets contain **random functions**. Each such a trajectory is referred to as **self-stochastic**.*

It is obvious that self-stochasticity phenomenon may occur only if the starting phase space of a dynamical systems is noncompact. In this case, the cardinal problem is to find a suitable metric which allows to compare deterministic functions with random ones.

By the term ***"massive set"*** is implied either a set of positive measure or a set of full measure ("almost all (in measure)") and either an everywhere dense set or a set of second category ("almost all") or whatever, as required by the problem. Thereby we necessitate a system to have a great deal of self-stochastic trajectories (for instance, if there were a few self-stochastic trajectories among all the trajectories, then almost surely (with probability 1) these self-stochastic trajectories should escape detection by a computer).

To reveal that self-stochasticity phenomenon is actually feasible and to do the concept of self-stochasticity more clear, we discuss some dynamical systems induced by nonlinear boundary value problems for PDE.

Among nonlinear boundary value problems which serve as mathematical models for chaotic processes, one can separate two fundamental classes:

1. problems for PDE of parabolic type (such as Navier–Stokes equation much used in hydrodynamics);

2. problems for PDE of hyperbolic type (which are characteristic of electrodynamics).

In many cases boundary value problems from both these classes induce infinite dimensional dynamical systems of the form

$$\left\{ C^k(D,E), T, S^t \right\} \tag{1}$$

where $C^k = C^k(D,E)$ is the space of C^k-functions $\varphi : D \to E$, $D \subset \mathbf{R}^l$, $E \subset \mathbf{R}^m$ $(k,l,m \geq 1)$, $T = \mathbf{R}^+$ or $\mathbf{Z}^+$, and S^t is the shift operator along solutions.

The phase space C^k equipped a priori with the C^k-metric is noncompact. It may therefore occur that for some (and even for almost all) $\varphi \in C^k$ the corresponding trajectory $S^t[\varphi]$ is noncompact. As a consequence, its ω-limit set $\omega[\varphi]$ is found to be either empty or, if not, noncompact and hence this ω-limit set characterizes the asymptotic behavior of $S^t[\varphi]$ incompletely. For boundary value problems from the first class, trajectories are generally compact and, in contrast, for those of the second class, trajectories are in many cases noncompact. Every so often it is the noncompactness of trajectories that is responsible for initiation of chaos in problems from the second class.

Let us consider the simplest nonlinear problem

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}, \quad x \in [0,1], \quad t \in \mathbf{R}^+, \tag{2}$$

$$u\,|_{x=1} = f(u)\,|_{x=0}, \tag{3}$$

$$u\,|_{t=0} = \varphi(x), \tag{4}$$

with $f \in C^1(I,I)$ being an irreversible map of a closed interval I into itself and $\varphi \in C^1\Big([0,1], I\Big)$ complying with the consistency relations $\varphi(1) = f\Big(\varphi(0)\Big)$ and $\dot\varphi(1) = \dot f\Big(\varphi(0)\Big)\,\dot\varphi(0)$ (which insure the bounded solutions of the problem to be C^1-smooth).

Pr. (2)-(4) is the simplest nonlinear problem that one can envision. We have here taken it as a model problem for representation of self-stochasticity phenomenon because in this case, explanations are quite simple.

Pr. (2)–(4) is reduced to a difference equation. Indeed, on substituting the general solution of (2), which has the form $u(x,t) = w(x+t)$, into (3), we arrive at

$$w(\tau+1) = f(w(\tau)), \quad \tau \in \mathbf{R}^+. \tag{5}$$

Then from (4) and the consistency relations it follows that the solution u_φ of Pr. (2)–(4) can be presented in the form

$$u_\varphi(x,t) = w_\varphi(x+t), \tag{6}$$

where w_φ is a solution of Eq. (5) such that

$$w_\varphi(\tau) = \varphi(\tau) \quad \text{for } \tau \in [0,1]. \tag{7}$$

Consequently,

$$\begin{array}{lcll}
u_\varphi(x,t) & = & \varphi(x+t), & \text{if} \quad 0 \leq x+t < 1, \\
u_\varphi(x,t) & = & (f \circ \varphi)(x+t-1), & \text{if} \quad 1 \leq x+t < 2, \\
\multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
u_\varphi(x,t) & = & (f^n \circ \varphi)(x+t-n), & \text{if} \quad n \leq x+t < n+1,
\end{array}$$

and hence $u_\varphi(x,t)$ can be written in the form

$$u_\varphi(x,t) = f^{[t+x]}\Big(\varphi(\{t+x\})\Big), \quad t \in \mathbf{R}^+,\ x \in [0,1], \tag{8}$$

with f^n standing for the n-th iteration of f (i.e., $f^n = f \circ f^{n-1}$ and $f^0 = id$), $[\cdot]$ and $\{\cdot\}$ standing for the integral and fractional parts of a number.

Thus Pr. (2) and (3) generates the dynamical system

$$\Big\{C^k\left([0,1], I\right), T, S^t\Big\} \tag{9}$$

with

$$S^t[\varphi](x) = \left(f^{[t+x]} \circ \varphi\right)(\{t+x\})\,. \tag{10}$$

The formula (10) indicates that the long time behavior of a trajectory $S^t[\varphi]$ is dictated by the asymptotic (as $n \to \infty$) properties of the iteration f^n of the map f, which specifies the nonlinearity in the boundary condition.

What consequences may the noncompactness of $S^t[\varphi]$ ***bring about ?*** The situations prove to be possible that although in theory every trajectory $S^t[\varphi]$ is determined uniquely by φ, ***it is not practical to predict the values of the function*** $S^t[\varphi](x)$ ***for*** t ***large enough***.

To make clear why the aforesaid is valid, we consider Pr. (2)–(4) with the nonlinearity in the boundary condition being given by the map

$$f : z \longmapsto 4z(1-z), \quad z \in [0,1], \tag{11}$$

which is often referred to as chaotic parabola. The map (11) has the following property: for any open interval $J \subset [0,1]$ there exists $n^* > 0$ such that $f^n(J) = [0,1]$ if $n > n^*$. As for Pr. (2)–(4) and (11), this implies that for any $x \in [0,1]$ and any $\varepsilon > 0$ there exists $t^* > 0$ such that

$$S^t[\varphi]\left(V_\varepsilon(x)\right) = [0,1] \quad \text{for } t > t^*, \tag{12}$$

where $V_\varepsilon(\cdot)$ is for the ε-neighborhood of a point. If ε is not over a computer precision, then the computer recognizes the neighborhood $V_\varepsilon(x)$ as a single point and by virtue of (12) the value of $S^t[\varphi]$ calculated at a point x for $t > t^*$ might appear to be equal to anyone of numbers from $[0,1]$ according to which implementation of f in code is used. In this case, $S^t[\varphi]$ ***is said to be out of predictability horizon*** (at least at given x for $t > t^*$).

Let us summarize. ***There exist boundary value problems such that the predictability of their solutions sooner or later breaks down and, in this sense, nonrandom solutions behave much like random processes*** (this fact is a powerful argument in support of that self-stochasticity phenomenon is really a possibility). In such situations, it is reasonable to turn a probabilistic description: rather than ask "what is the value of a solution at a point (x,t) ?", we should ask ***"what is the probability of the value of a solution at a point*** (x,t) ***falling into a certain set ?"*** If the answer to the latter question can be provided (of course, where a solution becomes unpredictable with time), we have called such a situation ***self-stochasticity***, in contrast to ***chaoticity*** as a behavior with no regularity at all.

2 Construction of a Metric

Let us consider the general dynamical system (1) and assume that it possesses a wealth of noncompact trajectories with unpredictable long time behavior. To describe Syst. (1), we need complete its phase space C^k via an appropriate metric so as the trajectories to be compact in a new phase space.

As we have seen just now, the ordinary sup-metric, which effects pointwise comparison, is unfit for Syst. (1). The sought-for metric must allow for the values of functions φ_1, φ_2 not only at a point x but also in some neighborhood of x (this approach is conformable to ideas developed in statistical physics [1, 3, 4]).

In other words, ***the sought-for metric must involve the distributions of the values of*** φ_1 ***and*** φ_2 ***near a point*** x or more specifically, the ensemble-averaged distributions

$$F^\varepsilon_{\varphi_i}(x,z) = \frac{1}{mes\, V_\varepsilon(x)} \int_{V_\varepsilon(x)} F_{\varphi_i}(y,z)dy, \quad i = 1,2, \tag{13}$$

where $F_\varphi(x,z)$ is the distribution function $\varphi \in C^k(X,Z)$ with Z being bounded. Inasmuch as φ is deterministic,

$$F_\varphi(x,z) = \chi_{(-\infty,z)}\Big(\varphi(x)\Big) \tag{14}$$

with $\chi_A(\cdot)$ being the indicator function of a set A.

Thus we arrive at the following metric

$$\varrho(\varphi_1,\varphi_2) = \sup_{\varepsilon>0} \min\left\{\varepsilon,\ \frac{1}{\text{mes}\, Z} \sup_{x\in X} \int_Z \Big|F^\varepsilon_{\varphi_1}(x,z) - F^\varepsilon_{\varphi_2}(x,z)\Big|\, dz\right\}. \tag{15}$$

One can extend the formula for ϱ to a set of functions that is essentially wider then $C^k(X,Z)$, namely, to the set of random processes $\psi : X \to Z$ that are specified by all their finite-dimensional distributions. Write $\Re(X,Z)$ for this set.

It is clear that ϱ is a semimetric on $\Re(X,Z)$, whereas it is a metric on $C^k(X,Z)$. To turn ϱ into a metric on $\Re(X,Z)$, it is necessary to involve all finite-dimensional distributions into (15). Thus we are led to the metric

$$\varrho^\#(\psi_1,\psi_2) = \sup_{\varepsilon>0}\min\Big\{\varepsilon, \tag{16}$$

$$\sum_{r=1}^{\infty} \frac{1}{2^r \textit{mes}\, Z^r} \sup_{(x_1,\dots,x_r)\in X^r} \int\limits_{Z^r} \Big|F^\varepsilon_{\psi_1}(x_1,\dots,z_r) - F^\varepsilon_{\psi_2}(x_1,\dots,z_r)\Big|\, dz_1 \dots dz_r\Big\},$$

where

- $F^\varepsilon_\psi(x_1\dots z_r) = \dfrac{1}{m_\varepsilon(x_1\dots x_r)} \displaystyle\int\limits_{V_\varepsilon(x_1\dots x_r)} F_\psi\,(y_1\dots z_r) dy_1 \dots dy_r;$
- given $(x_1,\dots,x_r)\in X^r$ and $(z_1,\dots,z_r)\in Z^r$, the collections $(x_1,\dots,x_r; z_1,\dots,z_r)$ is written $(x_1\dots z_r)$;
- $F_\psi(x_1\dots z_r)$ is the r-dimensional distribution of a function $\psi\colon X\to Z$;
- $V_\varepsilon(x_1\dots x_r)$ is the ε-neighborhood of $(x_1,\dots,x_r)\in X^r$;
- $m_\varepsilon(x_1\dots x_r) = \textit{mes}\, V_\varepsilon(x_1\dots x_r)$ for $(x_1,\dots,x_r)\in X^r$.

The metric $\varrho^\#$ is applicable both to deterministic functions and to random functions. A little though reveals that $\varrho^\#$ is well-defined on the subset of those functions $\psi\in\Re(X,Z)$ such that

$$F^\varepsilon_\psi(x_1\dots z_r) \longrightarrow F_\psi(x_1\dots z_r) \quad \text{as } \varepsilon\to 0 \tag{17}$$

for all points $(x_1,\dots,z_r)\in X^r\times Z^r$ outside of a set of Lebesgue measure zero.

3 Some Results

Let $C^\# = C^\#(D,E)$ be the completion of the phase space $C^k(D,E)$ via the metric $\varrho^\#$ with random and deterministic functions having the property (17). The space $C^\#$ is not compact. But there exist classes of boundary value problems (among which is Pr. (2)-(4)) such that, under certain conditions, the corresponding dynamical system possesses a massive set of trajectories that are compact in $C^\#$.

As for Pr. (2)-(4), this conditions, which we will refer to as (IM), are as follows:

- the map f has an invariant measure μ with the support being a cycle of intervals $J_1, J_2, \ldots, J_p$ $\Longleftrightarrow$ $supp\,\mu = \bigcup_{i=1}^{p} J_i$, where $f(J_i) = J_{i+1(mod\,p)}$, and $\mu(f^{-1}(A)) = \mu(A)$;
- the measure μ is equivalent to Lebesgue measure on $supp\,\mu$ $\Longleftrightarrow$ $mes\,A = 0 \iff \mu(A) = 0$;
- the map f^p has the property of intermixing $\Longleftrightarrow$ $\mu(A \cap f^{-pj}(B)) \to \mu(A) \cdot \mu(B)$ as $j \to \infty$, $A, B \subset J_i$, $1 \le i \le p$;
- the map f is nonsingular (with respect to Lebesgue measure) $\Longleftrightarrow$ $mes\,A = 0 \implies mes\,f^{-1}(A) = 0$, $mes\,f(A) = 0$.

These conditions are, in particular, fulfilled if f is a unimodal map with negative Schwarzian derivative $S_f := f'''/f' - 3/2\,(f''/f')^2$ and satisfies Collet-Eckmann's conditions $\lim_{n\to\infty}\inf \frac{1}{n}\log\left|\frac{d}{dt}f^n(c)\right| > 0$, where c is the (unique) extreme point of f. For ease of subsequent formulations, we assume f to be such a map.

Syst. (9) and (10), generated by Pr. (2) and (3), being uniformly continuous with respect to $\varrho^\#$, induces a dynamical system on the extended space $C^\#$, namely,

$$\left\{C^\#\left(D, E\right), T, S^t\right\} \tag{18}$$

with

$$S^t[\psi](x) = \left(f^{[t+x]} \circ \psi\right)\left(\{t+x\}\right), \tag{19}$$

where by the superposition $g \circ \psi$ of deterministic and random functions we mean the random function specified by the infinite-dimensional distributions

$$F_{g\circ\psi}(x_1, \ldots, x_r; z_1, \ldots, z_r) \;:=$$

$$\int\limits_{g^{-1}\left((-\infty,z_1)\right)} \cdots \int\limits_{g^{-1}\left((-\infty,z_r)\right)} \frac{\partial^r F_\psi\left(x_1, \ldots, x_r; y_1, \ldots, y_r\right)}{\partial y_1 \ldots \partial y_r} dy_1 \ldots dy_r,\ r = 1, 2, \ldots\ .$$

With Syst. (18) and (19) in hand, we are in a position to characterize the ω-limit set of the trajectory $S^t[\varphi]$ of Syst. (9) and (10), which corresponds to the solution u_φ of Pr. (2)-(4).

Theorem 1. *For every nonsingular $\varphi \in C^1$, the ω-limit set of the trajectory $S^t[\varphi]$ consists of random functions that combine into a cycle of Syst. (18) and (19) with period p, more precisely,*

$$\omega[\varphi] = \bigcup_{t\in[0,p)} S^t\left[f^\# \circ \varphi\right],$$

where $f^\#$ is the purely random process specified by the distribution function

$$F_{f^\#}(u, z) := p \cdot \mu\left(J_i \cap (-\infty, z)\right) \quad \textit{for } u \in \hat{J}_i$$
$$\textit{with } \hat{J}_i = \textstyle\bigcup_{j\ge 0} \int f^{-jp}(J_i), \quad 1 \le i \le p,$$

and by the superposition $f^{\#}\circ\varphi$ is meant the random process specified by the distributions

$$F_{f^{\#}\circ\varphi}(x_1,\dots,x_r;z_1,\dots,z_r) \;:=\; F_{f^{\#}}(\varphi(x_1),\dots,\varphi(x_r);z_1,\dots,z_r),\quad r=1,2,\dots\ .$$

(Recall that a stochastic process $Y(x)$ is said to be purely random, if for any x_1 and x_2 random variables $Y(x_1)$ and $Y(x_2)$ are mutually independent. In this case, $Y(x)$ is completely determinated by its distribution function F_Y.) From now on we assume φ to be nonsingular (without specifying this fact when no confusion can arise).

As Th. 1 implies, if $p>1$ trajectories $S^t[\varphi_1]$ and $S^t[\varphi_2]$ with $\varphi_1\not\equiv\varphi_2$ have, in general, distinct ω-limit sets, and if $p=1$ all the trajectories $S^t[\varphi]$ with φ being nonsingular are attracted by the single fixed point $f^{\#}\circ id$ of Syst. (18) and (19).

As an illustration of Th. 1 we refer to the dynamical system generated by Pr. (2), (3) and (11). The map $f: z\mapsto 4z(1-z)$, $z\in[0,1]$, satisfies the above conditions, in particular, f has the invariant measure

$$\mu_f(dz)=\frac{1}{\pi}\frac{dz}{\sqrt{z(1-z)}}$$

and $supp\,\mu_f=[0,1]$ is of period 1. Every trajectory $S^t[\varphi]$ tends to the single fixed point $f^{\#}\circ id$, which is the random function given by the distribution

$$F_{f^{\#}}(x,z)=\frac{1}{\pi}\int_0^z\frac{dy}{\sqrt{y(1-y)}}=\frac{2}{\pi}\arcsin\sqrt{z}.$$

What information about the solutions of Pr. (2)-(4) does Th. 1 provide? If we want to go from a trajectory $S^t[\varphi]$ to the solutions $u_\varphi(x,t)$, we should associate with u_φ the random process

$$P_\varphi(x,t)=\Big(f^{[t+x]}\circ f^{\#}\circ\varphi\Big)\Big(\{t+x\}\Big),\tag{20}$$

which gives a statistical description for u_φ in the following sense.

Theorem 2. *For any $\sigma>0$, there are $\varepsilon_1=\varepsilon_1(\sigma)>0$, $\varepsilon_2=\varepsilon_2(\sigma)>0$ and $T=T(\sigma)>0$ such that*
for any $r\geq 1$ and all points $(x_1,t_1,\dots,x_r,t_r;z_1,\dots,z_r)$ outside of a set of Lebesgue measure zero,

$$\Big|F^{\varepsilon}_{u_\varphi}(x_1,t_1,\dots,x_r,t_r;z_1,\dots,z_r)-F_{P_\varphi}(x_1,t_1,\dots,x_r,t_r;z_1,\dots,z_r)\Big|<\sigma$$

if $\varepsilon\in(\varepsilon_1,\varepsilon_2)$ and $t_i>T$, $i=1,2,\dots r$.

Th. 2 shows that statistical properties of a deterministic solution u_φ are asymptotical exactly reproduced by the random process P_φ. We can thus say that P_φ is ***the approximate statistical image*** of u_φ.

Properties of the approximate statistical image P_φ:

- P_φ is a continuous random process;
- $P_\varphi(x,t)$ is p-periodic if $p>1$ and stationary if $p=1$;

- $P_\varphi(x,t)$ is autocorrelated only at those points (x_1,t_1) and (x_2,t_2) such that $\{t_1+x_1\}=\{t_2+x_2\}$.

The expressions for the distribution densities and autocorrelation function of P_φ are rather unwieldy and we here omit them.

Unlike the case considered, it may occur that self-stochastic trajectories are generate only by part of initial functions φ. An example of such is given by Eq. (2) with the boundary condition

$$\frac{\partial u}{\partial t}\mid_{x=1}= h(u)\frac{\partial u}{\partial t}\mid_{x=0} \tag{21}$$

and the initial condition (4). The condition (21) integrates to the expression

$$u\mid_{x=1}= f(u)\mid_{x=0}+\lambda,$$

where f is an antiderivative of h and λ is an arbitrary constant.

Thus we arrive at a problem with parameter, which is given by Eq. (2) with the initial condition (4) and the boundary conditions

$$u\mid_{x=1}= f_\lambda(u)\mid_{x=0}, \quad \text{where } f_\lambda\colon z\longmapsto f(z)+\lambda, \quad \lambda\in\mathbf{R}. \tag{22}$$

For reasons of continuity, with each $\varphi\in C^1([0,1],\mathbf{R})$ there is, in general, associated only one of the conditions (22), namely, the condition with

$$\lambda=\gamma[\varphi]:=\varphi(1)-f(\varphi(0)). \tag{23}$$

Thus Pr. (2) and (22) presets a measure ν on C^1, namely, for any open set $\Phi\subset C^1$

$$\nu(\Phi)=mes\,\{\lambda:\ \lambda=\gamma[\varphi],\,\varphi\in\Phi\}\,.$$

Let us assume h to satisfy the conditions

- h is a C^2-smooth function with $|h'(z)|>L>0$,
- $(z-z_0)h''(z)\geq 0$ for $z\in\mathbf{R}$ and some $z_0\in\mathbf{R}$.

Under these conditions, for any h, there exists a (semiopen) interval $\Lambda(h)$ such that with $\lambda\in\Lambda(h)$, the map f_λ has bounded nontrivial invariant intervals, the largest of which, denoted by I_λ, contains all these intervals. Moreover there exist a bounded (generally speaking, noninvariant) interval $I=I(h)$ such that $I_\lambda\subset I$ for any $\lambda\in\Lambda(h)$. Therefore the bounded nonconstant solutions of Pr. (2) and (22) arise from the set

$$B_1(h)=\left\{\varphi\in C^1([0,1],I):\ \gamma[\varphi]\in\Lambda(h)\text{ and }\varphi(x)\in I_{\gamma[\varphi]}\text{ for }x\in[0,1]\right\}.$$

Pr. (2) and (22) induces the dynamical system

$$\left\{B_1(h),T,S^t\right\} \tag{24}$$

with

$$S^t[\varphi](x)=\left(f_{\gamma[\varphi]}^{[t+x]}\circ\varphi\right)\left(\{t+x\}\right), \tag{25}$$

which in its turn induces the extended dynamical system

$$\left\{C_1^\#(h),T,S^t\right\}, \tag{26}$$

where $C_1^\#(h)$ is the completion of the phase space $B_1(h)$ with functions from $\Re([0.1],I)$ via the metric $\varrho^\#$ and S^t is given by (25).

A peculiarity of this type of problems is that the action S^t depends on φ. This fact results in that the trajectories can differ radically in their long time behavior. In particular, if for some $\varphi \in B_1(h)$, the map $f_{\gamma[\varphi]}$ satiefies the conditions (IM), then the corresponding trajectory $S^t[\varphi]$ is self-stochastic. A precise formulation is given by the following theorem.

Theorem 3. *There exist two sets $\Phi_d, \Phi_r \subset B_1(h)$, of positive ν-measure each, such that the ω-limit set $\omega[\varphi]$ of a trajectory $S^t[\varphi]$ of Syst. (24) and (25) consists of*

- *deterministic functions if $\varphi \in \Phi_d$,*
- *random functions if $\varphi \in \Phi_r$.*

Moreover, for any integer $n \geq 1$, there exist $\varphi'_{(n)} \in \Phi_d$ and $\varphi''_{(n)} \in \Phi_r$ such that $\omega[\varphi'_{(n)}]$ and $\omega[\varphi''_{(n)}]$ are cycles of period n (of the extended system).

For $\varphi \in \Phi_d$, functions $\psi \in \omega[\varphi]$ are, in general, step functions; the set of discontinuities $\mathcal{D}_\psi$ of ψ may be infinite but $mes\, \mathcal{D}_\psi = 0$.

For $\varphi \in \Phi_r$, functions $\psi \in \omega[\varphi]$ are described similar to that in Th. 1, namely, in terms of the invariant measure of the map $f_{\gamma[\varphi]} : z \mapsto f(z) + \gamma[\varphi]$.

It is not improbable for a self–stochastic trajectory that "points" of its ω-limit set $\omega[\varphi] \subset C^{\#}(D, E)$, as functions $D \to E$, are random over one subdomain of D and deterministic over another. An example of such is provided by the problem

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_2}, \qquad (27)\\ \frac{\partial u_2}{\partial t} &= -\frac{\partial u_2}{\partial x_1} - \frac{\partial u_2}{\partial x_2}, \quad x = (x_1, x_2) \in \mathbf{R} \times [0,1], \quad t \in \mathbf{R}^+; \end{aligned}$$

$$u_1 = u_2 \mid_{x_2=0}, \quad \frac{\partial u_1}{\partial t} = h(u_2) \frac{\partial u_2}{\partial t} \mid_{x_2=1} . \qquad (28)$$

$$u_1 \mid_{t=0} = \varphi_1(x), \quad u_2 \mid_{t=0} = \varphi_2(x). \qquad (29)$$

As well as in the above case, one can introduce

$$\gamma[\varphi](x_1) = \varphi_1(x_1, 1) - f(\varphi_2(x_1, 1)), \quad x_1 \in \mathbf{R},$$

and find $B_2(h) \subset C^1(\mathbf{R} \times [0,1], \mathbf{R}^2)$ such that the bounded nonconstant solutions are generated by those, and only those, φ such that $\varphi \in B_2(h)$ and construct the completion $C_2^{\#}(h)$ of $B_2(h)$ with functions from $\Re$. Corresponding to Pr. (27) and (28) is the dynamical system $\left\{B_2(h), T, S^t\right\}$ and the extended dynamical system $\left\{C_2^{\#}(h), T, S^t\right\}$, where for both these systems, S^t is given by (10) with f being replaced with $f_{\gamma[\varphi](x_1)}$.

Theorem 4. *There exist two open (in C^1) sets $\Phi_d, \Phi_r \subset B_2(h)$ such that the ω-limit set $\omega[\varphi]$ of a trajectory $S^t[\varphi]$ of the dynamical system $\left\{B_2(h), T, S^t\right\}$ consists of*

- *deterministic functions (which combine into a cycle of the extended system) if $\varphi \in \Phi_d$;*
- *random functions (which combine into a family of almost periodic trajectories of the extended system) if $\varphi \in \Phi_r$.*

Moreover, for $\varphi \in \Phi_r$ the domain $\mathbf{R} \times [0,1]$ falls into subdomains $D_1(\varphi)$ and $D_2(\varphi)$, of positive Lebesgue measure each, such that any $\psi \in \omega[\varphi]$ is a random function on D_1 and a deterministic function on D_2.

It should be noted that ***self-stochasticity phenomenon is not exotic*** because $mes\, \Lambda_{acim} > 0$, where $\Lambda_{acim}(h) = \{\lambda : f_\lambda$ has a smooth invariant measure$\}$ [2].

4 Universal Properties

The theorems 3-4 can essentially be widened through the use of a number of properties of 1-D dynamical systems. Here we present several statements for Pr. (2) and (21), which involve, in particular, Feigenbaum's constants $\delta = 4.6992\ldots$ and $\alpha = 2.502\ldots$.

Let $\lambda(n)$ be for the lower bound of those λ such that f_λ has a cycle of period n; $\beta(n)$ be for the lower bound of those λ such that f_λ has a smooth invariant measure with the support consisting of n intervals.

As known, with ordering for natural numbers

$$1 \prec 2 \prec 2^2 \prec \ldots \prec 5\cdot 2^2 \prec 3\cdot 2^2 \prec \ldots \prec 5\cdot 2 \prec 3\cdot 2 \prec \ldots \prec 9 \prec 7 \prec 5 \prec 3,$$

the following relations hold:

$$\begin{aligned}
&\lambda(n) < \lambda(n') \quad \text{for } n \prec n',\\
&\lambda(n) < \beta(n) < \lambda(n') \quad \text{for } n \prec n', \quad \text{if } n \neq 2^i, \quad \text{and}\\
&\lambda(6n) < \beta(n) < \lambda((2s+1)n) \quad \text{for any } s \geq 1, \quad \text{if } n = 2^i,\ i = 0,1,2,\ldots\ .
\end{aligned}$$

Moreover, $\dfrac{\lambda\left(2^i s\right) - \lambda\left(2^{i-1} s\right)}{\lambda\left(2^{i+1} s\right) - \lambda\left(2^i s\right)} \to \delta$ as $i \to \infty$, whatever $s \geq 1$.

Let φ_ξ, $\xi \in (\xi_1, \xi_2)$, be a family of functions from $B_1(h)$ that depend on the parameter ξ continuously, and let $\lambda_j = \gamma[\varphi_{\xi_j}]$, $j = 1, 2$.

Theorem 5. *Let $n_1 \prec n \prec n_2$. If the map f_{λ_1} has no cycles of period n_1 and the map f_{λ_2} has a cycle of period n_2, then*

- *there exists an interval $\Xi_n \subset (\xi_1, \xi_2)$ such that for $\xi \in \Xi_n$ the ω-limit set $\omega[\varphi_\xi]$ is a cycle of period n;*
- *if $n_1 \neq 2^i$ there exists $\xi' \in \Xi_n$ such that the ω-limit set $\omega[\varphi_{\xi'}]$ is a cycle of period n, whose points are random functions.*

Set

$$\begin{aligned}
\mathcal{P}_i &= \left\{\varphi \in B_1(h) :\ \lambda(2^i) < \gamma[\varphi] < \lambda(2^{i+1})\right\},\\
\mathcal{Q}_i &= \left\{\varphi \in B_1(h) :\ \lambda(2^i) < \gamma[\varphi] < \beta(2^{i-1})\right\},\\
\mathcal{R}_i &= \left\{\varphi \in \mathcal{Q}_i :\ \gamma[\varphi] \in \Lambda_{acim}(h)\right\}, \quad i = 0,1,2,\ldots\ .
\end{aligned}$$

Theorem 6. *The ω-limit set of a trajectory $S^t[\varphi]$ of Syst. (24), (25) is*

- *a cycle of the period equal to 2^i, which consists of deterministic functions, for $\varphi \in \mathcal{P}_i$;*
- *a cycle of the period divisible by 2^i, which consists of deterministic functions, for almost all $\varphi \in \mathcal{Q}_i \setminus \mathcal{R}_i$;*
- *a cycle of the period divisible by 2^i, which consists of random functions, for almost all $\varphi \in \mathcal{R}_i$.*

Furthermore, the following universal relations hold:

$$\lim_{i\to\infty} \frac{\nu(\mathcal{P}_i)}{\nu(\mathcal{P}_{i+1})} = \lim_{i\to\infty} \frac{\nu(\mathcal{Q}_i)}{\nu(\mathcal{Q}_{i+1})} = \lim_{i\to\infty} \frac{\nu(\mathcal{R}_i)}{\nu(\mathcal{R}_{i+1})} = \delta.$$

Theorem 7. *There exists a constant $C > 0$ such that for any $\varepsilon > 0$ and any $\varphi \in Q_n$ one can find a 2^n-periodic step function q_φ with the property that for the solution u_φ of Pr. (2) and (21) there is a $\theta(\varphi, \varepsilon) > 0$ such that*

$$mes\left\{(x,t) : |u_\varphi(x,t) - q_\varphi(x,t)| > C\alpha^{-n} \text{ for } \theta < t < \theta + 1\right\} < \varepsilon$$

$$\text{for } \theta > \theta(\varphi, \varepsilon).$$

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