HOMOGENIZATION OF ATTRACTORS OF NON–LINEAR EVOLUTIONARY EQUATIONS

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1. Introduction

The paper is devoted to the homogenization of non–linear evolutionary equations in domains with "traps". Namely, we consider an initial boundary value problem for a semilinear parabolic equation of the form :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathcal{L}_\varepsilon u + f(u) = h, \quad t > 0, \\ u^\varepsilon(x,0) = u_0^\varepsilon(x), \end{cases} \tag{1.1}$$

where $\mathcal{L}_\varepsilon$ is a linear second order differential operator with corresponding boundary condition.

We study three different cases of the definition of this operator :

1) $\mathcal{L}_\varepsilon$ is the Beltrami–Laplace operator Δ_ε defined on the Riemmanian manifold M_ε of a special structure depending on a small parameter ε;

2) $\mathcal{L}_\varepsilon$ is the usual Laplace operator Δ defined in the domain Ω_ε with a large number of perforated spheres, periodically with a period ε distributed in the domain; it is assumed that the diameter of perforation is small with respect to the period of the structure;

3) $\mathcal{L}_\varepsilon$ is an operator with coefficients asymptotically degenerating on a set of periodically with a period ε distributed thin (with respect to the period of the structure) spherical annuluses :

$$\mathcal{L}_\varepsilon = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon(x)\frac{\partial}{\partial x_i}\right).$$

The structure of the operator $\mathcal{L}_\varepsilon$ implies three different approa– ches to the definition of traps in the initial boundary value problem (1.1).

Following the papers [1–5] we assume that the function $f(u)$ in (1.1) belongs to the class $C^2(\mathbf{R})$ and has the following properties :

$$|f'(u)| < C_1; \quad uf(u) \geq C_2u^2 - C_3; \quad \int_0^u f(\xi)d\xi \geq C_4u^2 - C_5 \tag{1.2}$$

with some positive constants $C_j (j = 1, 2, ..., 5)$.

Using the classical technique developed, for instance, in [6, 7] we can prove the existence and uniqueness theorem that allow us to construct an evolution operator S_t^ε in corresponding spaces by the formula $S_t^\varepsilon u_0^\varepsilon = u^\varepsilon(t)$, where $u^\varepsilon(t) = u^\varepsilon(x,t)$ is the solution of problem (1.1) (for the details see [1, 3, 5]). Application of standard methods (see, e.g. [8–11]) makes it possible to prove that the dynamical system in each case for every $\varepsilon > 0$ has a compact global attractor $\mathcal{A}_\varepsilon$ and this attractor has a finite Hausdorff dimension. We study the asymptotical behavior of $\mathcal{A}_\varepsilon$ in the cases 1)–3) as $\varepsilon \to 0$. Our principal goal is to learn how the transition to homogenization description reflects on the long–time dynamics.

The asymptotic behaviour of the solutions $u^\varepsilon(x,t)$ of problem (1.1) in the cases 1)–3) as $\varepsilon \to 0$ was studied for a finite time interval in [1, 2, 4]. It was shown that the homogenization of these problems leads to a system of a semilinear parabolic equation coupled with an ordinary differential equation :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + a_1(u - v) + f(u) = h_1(x), x \in \Omega,\, t > 0; \\ \dfrac{\partial v}{\partial t} + a_2(v - u) + f(v) = h_2(x), x \in \Omega,\, t > 0; \\ \dfrac{\partial u}{\partial n} = 0, x \in \partial\Omega, t > 0; u(x,0) = u_0(x), v(x,0) = v_0(x), \quad x \in \Omega; \end{cases} \tag{1.3}$$

where the coefficients $a_{ij} (i,j = 1, 2, ..., n)$ and $a_k (k = 1, 2)$ are calculated from the solutions of cellular problems and the parameters of the structure.

We consider the long–time dynamics of homogenized system (1.3) and show that it possesses a finite–dimensional global attractor $\mathcal{A}$. We investigate the properties of $\mathcal{A}$ and prove that global attractors $\mathcal{A}_\varepsilon$ tend to $\mathcal{A}$ in a suitable sense as $\varepsilon \to 0$ (see [1, 3, 5]).

We also note that the homogenization problem (1.1) with uniformly non–degenerating elliptic operator was studied by a number of authors (see, e.g. [12–14]).

2. Homogenization results

The aim of the Section 2 is to present the homogeniztion results we use for the proof of the convergence of the global attractors of problem (1.1).

2.1. Homogenization of semilinear parabolic equations on manifolds with complicated microstructure

We consider on n–dimensional $(n \geq 2)$ Riemannian manifold M_ε of complicated microstructure which depends on $\varepsilon > 0$ initial boundary value problem (1.1) with the Neumann boundary condition on the boundary ∂M_ε. We suppose that local structure of the manifold M_ε becames more and more complicated, as ε tends to zero. Now we describe the structure of the manifold M_ε. Let Ω be a smooth bounded domain in $\mathbf{R}^n (n \geq 2)$ and let

$$F_\varepsilon = \bigcup_{j \in N_\varepsilon} F(x^i, a_\varepsilon)$$

be a union of balls $F(x^i, a_\varepsilon)$ of radius $a_\varepsilon << \varepsilon$ $(\lim_{\varepsilon\to 0} a_\varepsilon \varepsilon^{-1} = 0)$ with centers in $x^j = j\varepsilon$ $(j \in \mathbf{Z}^n)$ such that $F(x^i, a_\varepsilon) \in \Omega$. Here N_ε stands for the corresponding set of multiindexes $j \in \mathbf{Z}^n$. In $\mathbf{R}^{n+1}$ we consider the surfaces (below $x = (x_1, ..., x_n) \in \mathbf{R}^n$, $y \in \mathbf{R}$, $(x, y) \in \mathbf{R}^{n+1}$):

$$\Omega_\varepsilon = \{(x; 0) \in \mathbf{R}^{n+1} : x \in \Omega \setminus F_\varepsilon\}$$

and

$$B^j_\varepsilon = (j\varepsilon; 0) + B_\varepsilon, \quad j \in N_\varepsilon \subset \mathbf{Z}^n,$$

where

$$B_\varepsilon = \{(x; y) \in \mathbf{R}^{n+1} : |x|^2 + (y - \sqrt{b^2\varepsilon^2 - a^2_\varepsilon})^2 = b^2\varepsilon^2,\ y \le 0\}.$$

Here b is a parameter such that $a_\varepsilon \varepsilon^{-1} < b < 1$. We assume that

$$M_\varepsilon = \Omega_\varepsilon \bigcup \left(\bigcup_{j \in N_\varepsilon} B^j_\varepsilon \right),$$

i.e. M_ε consists of a piece of flat submanifold in $\mathbf{R}^{n+1}$ with bubbles B^j_ε. We define the Riemannian structure on M_ε by C^∞ metric tensor $g^\varepsilon(x) = \{g^\varepsilon_{\alpha\beta}(x); \alpha, \beta = 1, 2, ..., n\}$, $x \in M_\varepsilon$, and assume the following : (i) the metric coincides with euclidean one on Ω_ε; (ii) the metric is the same for all bubbles B^j_ε, $j \in N_\varepsilon$; (iii) there exist positive constants C_1 and C_2 such that

$$C_1\varepsilon^n|\xi|^2 \le \sum_{\alpha\beta} g^\varepsilon_{\alpha\beta}(x)\xi_\alpha\xi_\beta \le C_2\varepsilon^n|\xi|^2,\ \varepsilon > 0,$$

for all $x \in B^j_\varepsilon$, $j \in N_\varepsilon$ and for all $\xi \in \mathbf{R}^n$.

We consider problem (1.1) on the Riemannian manifold $(M_\varepsilon, g^\varepsilon)$, which can be treated as a model of diffusion in medium with traps. The operator $\mathcal{L}_\varepsilon$ is the Beltrami–Laplace operator Δ_ε of the form

$$\Delta_\varepsilon = \frac{1}{\sqrt{|g^\varepsilon|}} \sum_{\alpha,\beta} \frac{\partial}{\partial x_\alpha} \left(\sqrt{|g^\varepsilon|} g^{\alpha\beta}_\varepsilon \frac{\partial}{\partial x_\beta} \right),$$

where $|g^\varepsilon| = \det g^\varepsilon$ and $g^{\alpha\beta}_\varepsilon$ are the components of the inverse to g^ε tensor.

Now we introduce a parameter for description of asymptotical behaviour of manifolds.

Let $0 \in \Omega$ for the simplicity. We denote

$$G_\varepsilon = \{(x; 0) \in \mathbf{R}^{n+1} : a_\varepsilon \le |x| < \varepsilon/2\}; \quad D_\varepsilon = B_\varepsilon \bigcup G_\varepsilon.$$

From now on $\|\cdot\|_{\mathcal{O}}$ is the norm in $L^2(\mathcal{O})$ and $\|\cdot\|_{1,\mathcal{O}}$ is the norm in the Sobolev space $H^1(\mathcal{O})$. We set

$$\lambda_\varepsilon = \inf \left\{ \frac{\|\nabla_\varepsilon v\|^2_{D_\varepsilon}}{\|v\|^2_{D_\varepsilon}} : v \in H^1_0(D_\varepsilon) \right\}. \tag{2.1}$$

It is clear that λ_ε is the first eigenvalue of the Dirichlet problem

$$\Delta_\varepsilon v + \lambda_\varepsilon v = 0,\ x \in D_\varepsilon;\ v = 0,\ x \in \partial D_\varepsilon. \tag{2.2}$$

Our main assumption concerning to the behaviour of the bubbles B_ε^j (and manifold M_ε) is the existence of the limits

$$\lambda = \lim_{\varepsilon\to 0} \lambda_\varepsilon; \quad \mu = \lim_{\varepsilon\to 0} \varepsilon^{-n} m_\varepsilon > 0, \tag{2.3}$$

where

$$m_\varepsilon = Vol(B_\varepsilon) = \int\limits_{B_\varepsilon} \sqrt{|g^\varepsilon|}dx_1...dx_n.$$

Let P_ε be a bounded operator from $L^2(M_\varepsilon)$ into $L^2(\Omega)$ defined by the formula

$$(P_\varepsilon)(x) = \begin{cases} u(x), & x \in \Omega_\varepsilon; \\ 0, & x \in \Omega \setminus \Omega_\varepsilon. \end{cases}$$

Let $\{x^\alpha = \alpha\varepsilon, \alpha \in \mathbf{Z}^n\}$ be a lattice in $\mathbf{R}^n$ and let Q_ε be a linear interpolation operator that is defined as follows. For each node of the sublattice $\{x^\alpha = \alpha\varepsilon,\ \alpha \in N_\varepsilon\}$ we set

$$(Q_\varepsilon u)(x^\alpha) = \frac{1}{m_\varepsilon}\int\limits_{B_\varepsilon^j} u(x)dx, \quad \alpha \in N_\varepsilon; \tag{2.4a}$$

where m_ε is the volume of the ball B_ε^j. For each node $\{x^\alpha = \alpha\varepsilon,\ \alpha \notin N_\varepsilon\}$ we set $(Q_\varepsilon u)(x^\alpha)$ being equal to a mean between the values of $(Q_\varepsilon u)$ in the nearest nodes of the lattice. In the whole $(Q_\varepsilon u)$ is a poly–linear spline, i.e.

$$(Q_\varepsilon u)(x) = \sum_\alpha (Q_\varepsilon u)(x^\alpha) \prod_{j=1}^n \chi\left(\frac{x_j}{\varepsilon} - \alpha_j\right), \tag{2.4b}$$

where $x = (x_1, ..., x_n) \in \Omega$, $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbf{Z}^n$ and $\chi(\tau) = 0$ for $|\tau| > 1$; $\chi(\tau) = 1-|\tau|$ when $|\tau| \le 1$. It is clear that Q_ε is linear bounded operator from $L^2(M_\varepsilon)$ into $H^1(\Omega)$ for every $\varepsilon > 0$.

Let us assume that the following homogenization conditions hold true :

(A.1) for any $\varepsilon \in (0, \varepsilon_0)$

$$\|u_0^\varepsilon\|_{1,M_\varepsilon} + \|\nabla Q_\varepsilon u_0^\varepsilon\|_\Omega + \|h^\varepsilon\|_{1,M_\varepsilon} + \|\nabla Q_\varepsilon h^\varepsilon\|_\Omega \le C,$$

where the constant C is independent of ε;

(A.2) there exist functions u_0, v_0, h_1, h_2 from $L^2(\Omega)$ such that $P_\varepsilon u_0^\varepsilon \to u_0$, $Q_\varepsilon u_0^\varepsilon \to v_0$, $P_\varepsilon h^\varepsilon \to h_1$, $Q_\varepsilon h^\varepsilon \to h_2$ strongly in $L^2(\Omega)$;

These homogenization conditions along with (2.3) imply that (for the proof see [1])

$$\lim_{\varepsilon\to 0}\{\max_{[0,T]} \|P_\varepsilon u^\varepsilon(t) - u(t)\|_\Omega + \max_{[0,T]} \|Q_\varepsilon u^\varepsilon(t) - v(t)\|_\Omega\} = 0, \tag{2.5}$$

for any $T > 0$, where $U(x,t) = (u(x,t), v(x,t))$ is the solution of the problem (1.3) in the class $\mathcal{W} = \{U(t) : U(t) \in C(\mathbf{R}_+, H^1(\Omega) \times L^2(\Omega)); \quad \frac{d}{dt}U(t) \in L^2(\mathbf{R}_+, L^2(\Omega) \times L^2(\Omega))\}$. The coefficients a_{ij} and a_k in (1.3) are calculated as follows : $a_{ij} = \delta_{ij}$, $a_1 = \lambda\mu$, $a_2 = \lambda$.

2.2. Homogenization of semilinear parabolic equations in domains with spherical traps

Let Ω be a smooth bounded domain in $\mathbf{R}^n$ $(n \geq 3)$ and let $\Gamma_\varepsilon^\alpha$ be $(n-1)$–dimensional sphere centered at the point $x^\alpha \in \Omega$ with radius $r_\varepsilon = r\varepsilon$ $(r < 1/5)$. We denote by D_ε^α the connected subset of $\Gamma_\varepsilon^\alpha$ with the diameter $d_\varepsilon = d\varepsilon^{n/(n-2)}$. Then $S_\varepsilon^\alpha = \Gamma_\varepsilon^\alpha \setminus \bar{D}_\varepsilon^\alpha$ is a $\mathbf{R}^{n-1}$ perforated sphere in the domain Ω and we denote $F_\varepsilon = \bigcup_{\alpha \in N_\varepsilon} S_\varepsilon^\alpha$ the union of such perforated spheres with centers at points $x^\alpha = \alpha\varepsilon$ $(\alpha \in \mathbf{Z}^n)$. Here N_ε stands for the corresponding subset of multiindexes $\alpha \in \mathbf{Z}^n$, such that $F_\varepsilon \subset \Omega$.

We consider initial boundary value problem (1.1) in the domain $\Omega_\varepsilon = \Omega \setminus F_\varepsilon$ with the Neumann boundary condition on $\partial\Omega_\varepsilon$. We set

$$\nu_\varepsilon = \inf \left\{ \frac{\|\nabla v\|^2_{B(R_\varepsilon)}}{\|v\|^2_{B(R_\varepsilon)}}, v \in H^1(B(R_\varepsilon) \setminus S_\varepsilon); v = 0, x \in \partial B(R_\varepsilon) \right\},$$

where $R_\varepsilon = \varepsilon/3$; $\partial B(R_\varepsilon)$ is the surface of the $\mathbf{R}^n$ open ball $B(R_\varepsilon)$; S_ε is the $\mathbf{R}^{n-1}$ perforated sphere of radius $r_\varepsilon = r\varepsilon$ strictly included in the ball $B(R_\varepsilon)$. We assume that both the sphere and the ball are centered at 0 and $0 \in \Omega$ for sake of simplicity. It is clear that ν_ε is the first eigenvalue of some mixed boundary value problem for the Laplace operator in $B(R_\varepsilon) \setminus S_\varepsilon$ (for details see [2]).

We will assume that there exists a limit :

$$\nu = \lim_{\varepsilon \to 0} \nu_\varepsilon. \tag{2.6}$$

The existence of the limit (2.6) is dicussed in [2].

Now we introduce the functions $v_i(x)\,(i = 1, 2, ..., n)$ which are the solutions of the following auxiliary problems :

$$\begin{cases} \Delta v_i = 0, & x \in P = K \setminus \bar{B}; \\ \dfrac{\partial v_i}{\partial n} = (x_i, n), & x \in \partial B; \\ v_i(x), Dv_i(x) \quad are\ K-periodic; \end{cases} \tag{2.7}$$

where n is the unitary normal to ∂B; $K = \{x \in \mathbf{R}^n; \quad |x_i| < 1/2r; \quad i = 1, 2, ..., n\}$; B is a $\mathbf{R}^n$ unit open ball in K. It is known that this problem has a unique solution $v_i(x)$ (up to a constant).

Let us introduce the notation :

$$\mathcal{G}_\varepsilon = \Omega \setminus \bigcup_\alpha \bar{B}_\alpha(r_\varepsilon); \quad \mathcal{B}_\varepsilon = \bigcup_\alpha B_\alpha(r_\varepsilon).$$

Let Q_ε be a polylinear spline defined by (2.4a), (2.4b) with $B_\alpha(r_\varepsilon)$ instead of B_ε^j and let P_ε be a standard linear continuation operator from $\mathcal{G}_\varepsilon$ to Ω. We assume that the following homogenization conditions hold :

(B.1) for any $\varepsilon \in (0, \varepsilon_0)$ and some constant C independent of ε

$$\|\nabla u_0^\varepsilon\|_{\Omega_\varepsilon} + \|\nabla Q_\varepsilon u_0^\varepsilon\|_\Omega + \|\nabla h^\varepsilon\|_{\Omega_\varepsilon} + \|\nabla Q_\varepsilon h^\varepsilon\|_\Omega \le C;$$

(B.2) there exist functions $u_0, v_0, h_1, h_2 \in L^2(\Omega)$ such that $P_\varepsilon u_0^\varepsilon \to u_0$, $Q_\varepsilon u_0^\varepsilon \to v_0$, $P_\varepsilon h^\varepsilon \to h_1$, $Q_\varepsilon h^\varepsilon \to h_2$ strongly in $L^2(\Omega)$;

These homogenization conditions along with (2.6) imply that (for the proof see [2])

$$\lim_{\varepsilon \to 0}\{\max_{[0,T]} \|P_\varepsilon u^\varepsilon(x,t) - u(x,t)\|_\Omega + \max_{[0,T]} \|Q_\varepsilon u^\varepsilon(x,t) - v(x,t)\|_\Omega\} = 0,$$

for any $T > 0$, where $U(x,t) = (u(x,t), v(x,t))$ is the solution of problem (1.3) in the class $\mathcal{W}$.

The coefficients a_{ij} and a_k in (1.3) are calculated from cellular problem (2.7) solutions by

$$a_{ij} = \delta_{ij}\left[1 - \frac{r^n}{1-\theta}\int_P (\nabla v_i, \nabla v_j)dx\right], \quad a_1 = \frac{a_2\theta}{1-\theta}, \qquad a_2 = \nu.$$

Here

$$\theta = \frac{\pi^{n/2} r^n}{\Gamma\left(\frac{n}{2}+1\right)}, \tag{2.8}$$

δ_{ij} is the Kronecker symbol and Γ is the *Gamma* function.

2.3. Homogenization of semilinear parabolic equations with asymptotically degenerating coefficients

Let Ω be a smooth bounded domain from $\mathbf{R}^n$, $(n \ge 2)$. Let us introduce the notation:

$$G_\varepsilon^\alpha = \{x \in \Omega : r_\varepsilon - d_\varepsilon < |x - x^\alpha| < r_\varepsilon\};$$

$$B_\alpha(r_\varepsilon - d_\varepsilon) = \{x \in \Omega : |x - x^\alpha| < r_\varepsilon - d_\varepsilon\};$$

$$\mathcal{G}_\varepsilon = \bigcup_{\alpha \in N_\varepsilon} G_\varepsilon^\alpha; \quad \mathcal{B}_\varepsilon = \bigcup_{\alpha \in N_\varepsilon} B_\alpha(r_\varepsilon - d_\varepsilon); \quad \Omega_\varepsilon = \Omega \setminus (\mathcal{G}_\varepsilon \bigcup \mathcal{B}_\varepsilon);$$

where $x^\alpha = \alpha\varepsilon$ $(\alpha \in \mathbf{Z}^n)$ and N_ε is a set of multi–indices such that $G_\varepsilon^\alpha \subset \Omega$; $r_\varepsilon = r\varepsilon$ $(r < 1/4)$; $d_\varepsilon = d\varepsilon^{2+\gamma}$ $(0 \le \gamma < 1)$.

In the domain Ω we consider the boundary value problem (1.1) with

$$\mathcal{L}_\varepsilon = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon(x)\frac{\partial}{\partial x_i}\right).$$

and the Neumann boundary condition on $\partial\Omega$. The coefficients $a_{ij}^\varepsilon(x)$ in $\mathcal{L}_\varepsilon$ are defined as follows :

$$a_{ij}^\varepsilon(x) = \begin{cases} \delta_{ij}, & x \in \Omega \setminus \mathcal{G}_\varepsilon; \\ a_\varepsilon\delta_{ij} \equiv a\delta_{ij}\varepsilon^{3+\gamma}, (a > 0), & x \in \mathcal{G}_\varepsilon. \end{cases} \tag{2.9}$$

This structure of the diffusion matrix allows us to interpret the problem (1.1) as a rection–diffusion problem in a medium with traps $B_\alpha(r_\varepsilon - d_\varepsilon)$.

Let us introduce the notation :

$$J_\varepsilon(u^\varepsilon) = \frac{1}{2}\left\{\|\nabla u^\varepsilon\|^2_{\Omega_\varepsilon} + a_\varepsilon\|\nabla u^\varepsilon\|^2_{\mathcal{G}_\varepsilon} + \|\nabla u^\varepsilon\|^2_{\mathcal{B}_\varepsilon}\right\};$$

As in the previuos Sections we suppose for sake of simplicity that $0 \in \Omega$. Let Q_ε be a polylinear spline defined by (2.4a), (2.4b) with $B_\alpha(r_\varepsilon - d_\varepsilon)$ instead of B^j_ε and let P_ε be a standard linear continuation operator from Ω_ε to Ω.

Let $u^\varepsilon(x,t)$ be the solution of problem (1.1). We assume that the following conditions hold :

(C.1) for any $\varepsilon \in (0, \varepsilon_0)$

$$\|u^\varepsilon_0\|^2_\Omega + J_\varepsilon(u^\varepsilon_0) + \|\nabla Q_\varepsilon u^\varepsilon_0\|^2_\Omega + \|h^\varepsilon\|^2_{1,\Omega} + \|\nabla Q_\varepsilon h^\varepsilon\|^2_\Omega \le C,$$

where C denotes any constant independent of ε;

(C.2) there exist functions $u_0, v_0, h_1, h_2 \in L^2(\Omega)$ such that $P_\varepsilon u^\varepsilon_0 \to u_0$, $Q_\varepsilon u^\varepsilon_0 \to v_0$, $P_\varepsilon h^\varepsilon \to h_1$, $Q_\varepsilon h^\varepsilon \to h_2$ strongly in $L^2(\Omega)$.

Then we have

$$\lim_{\varepsilon\to 0}\{\max_{[0,T]}\|P_\varepsilon u^\varepsilon(x,t) - u(x,t)\|_\Omega + \max_{[0,T]}\|Q_\varepsilon u^\varepsilon(x,t) - v(x,t)\|_\Omega\} = 0,$$

where the pair of functions $U(x,t) = (u(x,t), v(x,t))$ is the solution of problem (1.3) in the class $\mathcal{W}$. The coefficients a_{ij} and a_k in (1.3) are calculated from cellular problem (2.7) solutions and the structure parameters as follows :

$$a_{ij} = \delta_{ij}\left[1 - \frac{r^n}{1-\theta}\int_P (\nabla v_i, \nabla v_j)dx\right], \quad a_1 = \frac{b_2\theta}{1-\theta}, \quad a_2 = \frac{an}{rd}.$$

Here δ_{ij} is the Kronecker symbol, Γ is the *Gamma* function and the parameter θ is defined by (2.8).

3. Convergence of attractors

It can be proved that if $U_0 \in \mathcal{F} = H^1(\Omega) \times H^1(\Omega)$ and $h_2 \in H^1(\Omega)$ then

$$U(t) \in C(\mathbf{R}_+, \mathcal{F}); \quad \frac{d}{dt}U(t) \in L^2_{loc}(\mathbf{R}_+, L^2(\Omega) \times H^1(\Omega)).$$

So we can define the evolution semigroup S_t in the space $\mathcal{F}$ by the formula $S_tU_0 = U(t)$, where $U(t) = (u(x,t), v(x,t))$ is the solution of the problem (1.3) and $U_0 = (u_0, v_0)$. We prove the following assertion on the existence of finite dimensional weak global attractor for the semigroup S_t in $\mathcal{F}$.

THEOREM 3.1. *Assume that (1.2) is satisfied and*

$$a_2 + \inf\{f'(u)\} > 0, \quad h_2(x) \in H^1(\Omega), \tag{3.1}$$

where inf *in (3.1) is taken over the real functions* u*. Then the dynamical system* $(S_t, \mathcal{F})$ *has weak global attractor* $\mathcal{A}$*. This attractor has finite Hausdorff dimension as a compact set in* $L^2(\Omega) \times L^2(\Omega)$*.*

Recall (see, [8, 9]) that weak global attractor $\mathcal{A}$ is a bounded weakly closed set in $\mathcal{F}$ such that (i) $S_t\mathcal{A} = \mathcal{A}$ for any $t > 0$ and (ii) for any weak neighbourhood $\mathcal{O}$ of $\mathcal{A}$ and for any bounded set $B \subset \mathcal{F}$ we have $S_tB \subset \mathcal{O}$, when $t \geq t_0\,(B, \mathcal{O})$. We also note that assumption (3.1) is of prime importance for the existence of finite dimensional attractor $\mathcal{A}$.

At last using the methods developed in [1], (see also [3, 5]) and some estimates borrowed from [1, 2, 4] we prove our main result.

THEOREM 3.2. *Assume that (1.2), (3.1) and the homogenization conditions (A.1)–(A.2) (conditions (B.1)–(B.2) or (C.1)–(C.2)) dealing with* h^ε *along with (2.3) (along with (2.6) in the case 2)) are satisfied. Then we have*

$$\lim_{\varepsilon\to 0} \sup_{u^\varepsilon\in\mathcal{A}_\varepsilon} \{ \inf_{(u,v)\in\mathcal{A}} (\|P_\varepsilon u^\varepsilon - u\|_\Omega + \|Q_\varepsilon u^\varepsilon - v\|_\Omega)\} = 0,$$

where operators P_ε *and* Q_ε *are defined for the cases 1)–3) in the Sections 2.1–2.3.*

Thus the space structure of the attractor $\mathcal{A}_\varepsilon$ becomes more and more complicated in homogenization process ($\varepsilon \to 0$). Besides, in fact, we observe a limiting spliting of each element from $\mathcal{A}_\varepsilon$ into two–component function. In particular, the same effect is valid for stationary solutions of (1.1) (for the details see [5]).

COROLLARY 3.1. *Let the conditions of Theorem 3.2 are valid. Then for each stationary solution* $u^\varepsilon(x)$ *of (1.1) we have that*

$$\lim_{\varepsilon\to 0}\{ \inf_{(u,v)\in Z} (\|P_\varepsilon u^\varepsilon - u\|_\Omega + \|Q_\varepsilon u^\varepsilon - v\|_\Omega)\} = 0,$$

where Z *is the set of stationary solutions of (1.3).*

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