# WEYL'S SPECTRAL ASYMPTOTIC FORMULA FOR DIRICHLET KOHN–LAPLACIAN

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## 1. Introduction

The counting function $N(\lambda) = N(\lambda, \Omega)$ of the Dirichlet Laplacian on a bounded open set $\Omega \subset R^d$ is a defined as the number of eigenvalues less than a given $\lambda$. The problem of the asymptotic behaviour of the counting function as $\lambda \to +\infty$ has been extensively studied by mathematicians and physicists almost 100 years. The first mathematical results in this direction belong to H.Weyl who showed in 1911 that for domains with smooth boundaries

$$N(\lambda, \Omega) \sim (2\pi)^{-d} \omega_d |\Omega|^{d/2} \ as \ \lambda \to +\infty, \tag{1}$$

where $\omega_d$ is the volume of the unit ball in $R^d$. Formula (1) was then extended to arbitrary open sets in $R^d$ with finite volume and generalized to higher-order elliptic operators with constant coeficients, see [1], [2].

Our aim is the corresponding spectral asymptotics for the Kohn-Laplacian $\Delta_H$ associated to the Heisenberg group. This operator $\Delta_H$ is of Hörmander type, not strongly elliptic, and invariant with respect to translations on the Heisenberg group. More exactly, we consider the eigenvalue problem

$$-\Delta_H u = \lambda u, \qquad u|_{\partial\Omega} = 0, \tag{2}$$

where $\Omega$ is a bounded domain in the odd-dimensional space $R^{2n+1}$, and get the asymptotic formula

$$N(\lambda, \Omega) \sim C_n |\Omega| \lambda^{d_s/2}, \qquad d_s = 2(n+1). \tag{3}$$

The exponent $d_s$ $(> d = 2n+1)$ depends on $n$ and we say $d_s$ is the *spectral dimension* relative to our problem.

## 2. Preliminaries

Let us recall the definition of the operator $\Delta_H$ in dimension $d = 3$. The definitions and statements in any odd dimension $d = 2n + 1$ will be given later.

Consider the two linear operators $X^1$ and $Y^1$:

$$X^1 = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}, \quad Y^1 = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial z}$$

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and introduce the gradient $\nabla_H$ by

$$\nabla_H = (X^1, Y^1) = \sigma \nabla,$$

where $\nabla$ is the standard gradient: $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, and $\sigma$ is the following matrix

$$\sigma = \begin{pmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x \end{pmatrix}.$$

Then the operator $\Delta_H$ is given by

$$\Delta_H = (X^1)^2 + (Y^1)^2 =$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (4y^2 + 4x^2)\frac{\partial^2}{\partial z^2} + 4y\frac{\partial^2}{\partial z \partial x} - 4x\frac{\partial^2}{\partial z \partial y} = div(\sigma^T \sigma \nabla),$$

where

$$\sigma^T \sigma = \begin{pmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x \\ 2y & -2x & 4y^2 + 4x^2 \end{pmatrix}.$$

The operator $\Delta_H$ is elliptic (i.e. $\sigma^T \sigma \xi \cdot \xi \geq 0$ for any $\xi \in R^3$) but clearly not strongly elliptic, because the first eigenvalue of $\sigma^T \sigma$ is zero and the rank of $\sigma^T \sigma$ is two in every point. However we have the following condition on commutator

$$[X^1, Y^1] = X^1 Y^1 - Y^1 X^1 = -4\frac{\partial}{\partial z}. \tag{4}$$

As a consequence of (4), $\Delta_H$ is an Hörmander type operator, and enjoys nice properties like hipoellipticity, subelliptic estimates, the maximum principle, Poincaré's inequality.

The space $R^3$ becomes a group if the group law $+$ define as following:

$$for\ vectors\ \xi = (x, y, z),\ \xi' = (x', y', z')\ we\ set$$

$$\xi' + \xi = (x + x', y + y', z + z' - 2(x'y - xy')).$$

Notice that $\xi + \xi' \neq \xi' + \xi$ and the Lebesgue measure is invariant with respect to these right or left translations. The operator $\Delta_H$ is invariant with respect to the left translations, i.e. for fixed $\xi'$,

$$\Delta_H(u(\xi' + \cdot)) = (\Delta_H(u))(\xi' + \cdot).$$

The similar definitions can be given for any odd-dimensional space $R^{2n+1}$. Let $\xi = (x^1, x^2, \dots x^n, y^1, y^2, \dots y^n, z) = (x, y, z)$, where $x, y \in R^n$. Consider the operators

$$X^j = \frac{\partial}{\partial x^j} + 2y^j\frac{\partial}{\partial z},\ Y^j = \frac{\partial}{\partial y^j} - 2x^j\frac{\partial}{\partial z},\ j = 1, 2, \dots n,$$

and set

$$\nabla_H = (X^1, X^2, \dots X^n, Y^1, Y^2, \dots Y^n),$$

$$\Delta_H = \sum_{j=1}^{n} (X^j)^2 + (Y^j)^2.$$

Then all properties of $\Delta_H$ remain the same.

## 3. Counting function of the Dirichlet Kohn-Laplacian

Let $\Omega$ be a bounded domain in $R^{2n+1}$.

We denote by $D_H^0(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\left(\int\limits_\Omega |\nabla_H v|^2 d\xi + \int\limits_\Omega v^2 d\xi\right)^{1/2}.$$

The Poincaré inequality

$$\int\limits_\Omega v^2 d\xi \leq C(\Omega)\int\limits_\Omega |\nabla_H v|^2 d\xi \qquad for\ any\ v \in C_0^\infty(\Omega),$$

and Lax-Milgram lemma give the unique solvability of the problem:

$$\begin{cases} -\Delta_H u = f\ in\ \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{5}$$

where $f \in L^2(\Omega)$, i.e. the existence and the uniqueness of a function $u \in D_H^0(\Omega)$ such that

$$\int\limits_\Omega \nabla_H u \cdot \nabla_H \varphi d\xi = \int\limits_\Omega f\varphi d\xi \qquad for\ any\ \varphi \in C_0^\infty(\Omega).$$

Consider the collection of all the solutions of problem (5) for $f$ varying in $L^2(\Omega)$. This set is a domain of $-\Delta_H$ as a positive self-adjoint operator in $L^2(\Omega)$. By definition we have

$$\int\limits_\Omega (-\Delta_H)u\varphi d\xi = \int\limits_\Omega \nabla_H u \cdot \nabla_H \varphi d\xi \qquad for\ any\ \varphi \in D_H^0(\Omega).$$

Remark that the inverse operator $(-\Delta_H)^{-1}$ is compact. It is clearly from the following subelliptic estimate:

$$\|\varphi\|_{H^{1/2}(\Omega)} \leq C\left(\int\limits_\Omega |\nabla_H v|^2 d\xi + \int\limits_\Omega v^2 d\xi\right)^{1/2} \quad for\ any\ \varphi \in C_0^\infty(\Omega),$$

where $\|\cdot\|_{H^{1/2}(\Omega)}$ is the classical $H^{1/2}$ norm.

So for any bounded open set, the spectrum of $-\Delta_H$ consists of a countable sequence of positive eigenvalues $\lambda_j(\Omega)(j = 1, 2, \dots)$:

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \dots \leq \lambda_j(\Omega) \leq \dots,\ \lambda_j(\Omega) \to \infty\ as\ j \to \infty.$$

**Definition.** Let $\lambda$ be a given positive number. We denote by $L(\lambda) = N(\lambda, \Omega)$ the number of eigenvalues less than $\lambda$.

The function $N(\lambda, \Omega)$ is called the *counting function* of the Dirichlet Kohn-Laplacian on $\Omega$.

Let us formulate the main result.

**Theorem.** *Assume $\Omega$ is measurable in the sense of Jordan. Then asymptotic relation (3) holds with*

$$C_n = \frac{1}{(n+1)\Gamma(n+1)(4\pi)^{n+1}} \int\limits_0^\infty \left(\frac{\Theta}{sh\Theta}\right)^n d\Theta,$$

*where $\Gamma(\alpha)$ is the Euler gamma-function.*

Clearly that it is sufficient to proof formula (3) for smooth domains $\Omega$ only.

## 4. Sketch of the proof

We apply Carleman's analytic aproach or "parabolic equation method".

Let $K(\xi, \xi', t)$ be a fundamental solution associated to the parabolic operator

$$\frac{\partial}{\partial t} - \Delta_H.$$

One can prove the following properties:

$$K(t, \xi, \xi) = K(t, 0, 0) = \frac{1}{(4\pi t)^{n+1}} \int\limits_0^\infty \left(\frac{\Theta}{sh\Theta}\right)^n d\Theta \;\div\; \frac{A_n}{t^{n+1}} \tag{6}$$

$$0 \le K(t, \xi, \xi') \le \frac{c_1}{t^{n+1}} \exp(-\frac{c_2}{t}\rho^2(\xi, \xi')), \tag{7}$$

where $c_1, c_2 > 0$ and

$$\rho(\xi, \xi') =$$

$$= \left[((x - x')^2 + (y - y')^2)^2 + (z - z' - 2(x' \cdot y - x \cdot y'))^2\right]^{1/4}.$$

By $G(\xi, \xi', t)$ $(\xi, \xi' \in \Omega)$ denote a Green function of the parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_H u = 0 \; in \; \Omega \times (0, \infty), \\ u|_{\partial\Omega} = 0. \end{cases}$$

Then $G$ is continuous on $\Omega \times \Omega \times (0, \infty)$; moreover, from estimate (7) and the maximum principle we have

$$G(\xi, \xi, t) \le \Gamma(\xi, \xi, t) \; for \; any \; \xi \in \Omega, t > 0,$$

$$\Gamma(\xi, \xi) \le G(\xi, \xi) + c(\delta)t \qquad if \; \xi \in \Omega \; and \; \rho(\xi, \partial\Omega) \ge \delta > 0.$$

It follows that

$$\frac{A_n\,|\Omega|}{t^{n+1}} = \int\limits_\Omega \Gamma(\xi, \xi, t) d\xi \sim \int\limits_\Omega G(\xi, \xi, t) d\xi \; as \; t \to +0.$$

Let $\varphi_j(\xi)$ be a eigenfunction corresponding to the eigenvalue $\lambda_j$ and normalized by $\int\limits_{\Omega} \varphi_j^2 d\xi = 1$. Then we have

$$\varphi_j \in C^\infty(\Omega), \ \sum_{\lambda_j < \lambda} \varphi_j^2(\xi) \leq C\lambda^{n+1},$$

$$G(\xi, \xi', t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} \varphi_j(\xi)\varphi_j(\xi').$$

As a result, we obtain the important relation

$$\int\limits_0^\infty e^{-\lambda t} dN(\lambda) = \int\limits_0^\infty G(\xi, \xi, t) d\xi \sim \frac{A_n\, |\Omega|}{t^{n+1}} \ as \ t \to +0. \tag{8}$$

Now it is sufficient to apply the classical Tauberian theorem of Hardy-Littelwood.

**Tauberian Theorem (see [3]).** *Assume that* $N(\lambda)$ *is a nondecreasing function on* $[0, \infty)$ *and*

$$\int\limits_0^\infty e^{-\lambda t} dN(\lambda) < \infty \ for \ any \ t > 0.$$

*Then the relations*

$$N(\lambda) \sim c\lambda^\alpha \ as \ \lambda \to \infty \ (\alpha > 0),$$

$$\int\limits_0^\infty e^{-\lambda t} dN(\lambda) \sim \frac{\alpha\Gamma(\alpha)c}{t^\alpha} \ as \ t \to +0$$

*are equivalent.*

Now from (6), (8) we get asymptotic formula (3).

## References


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