BIFURCATION AND STABILITY FOR DIFFUSIVE LOGISTIC EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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1. Introduction. In this note we study a semilinear elliptic boundary value problem of one parameter dependence which arises in population genetics, having *nonlinear boundary conditions.* For some cases of sign indefinite weights, we investigate the existence and asymptotic behavior of the *minimal positive solution.* The analysis uses the local bifurcation theory from simple eigenvalues, super-sub-solution method and variational technique.

Let D be a bounded domain of Euclidean space $\mathbf{R}^N$, $N \geq 2$, with smooth boundary ∂D. We here consider the following semilinear elliptic problem with nonlinear boundary conditions:

$$\begin{cases} -\Delta u = \lambda(m(x)u - au^2) & \text{in } D, \\ \dfrac{\partial u}{\partial \mathbf{n}} + g(u)u = 0 & \text{on } \partial D, \end{cases} \tag{$(*)_\lambda$}$$

Here

(1) Δ denotes the usual Laplacian $\sum_{j=1}^N \partial^2/\partial x_j^2$ in $\mathbf{R}^N$, (2) λ is a positive parameter, (3) $m(x)$ is a real-valued Hölder continuous function on the closure $\overline{D}$, which may change its sign but satisfies $m(x_0) > 0$ for some $x_0 \in \overline{D}$, (4) a is a positive constant, (5) $g(t)$ is a real-valued C^2-function on $[0, \infty)$ such that $g(0) = 0$, and (6) $\mathbf{n}$ is the unit outer normal to ∂D.

A function $u \in C^2(\overline{D})$ is called a *positive solution* of $(*)_\lambda$ if u satisfies $(*)_\lambda$ and $u > 0$ in D.

The equation $-\Delta u = \lambda(m(x)u - au^2)$ in D is provided as a model of the population density for some species, where λ represents the reciprocal number of its diffusion rate, $m(x)$ its local growth rate, and a the effect of crowding for the species. For the population density u, the boundary condition $\frac{\partial u}{\partial \mathbf{n}} + g(u)u = 0$ on ∂D means that the rate of inflow migration at the border ∂D is governed *nonlinearly* by $-g(u)u$.

This note is devoted to an investigation of the set of positive solutions of $(*)_\lambda$ in a general class of nonlinear boundary conditions. The discussion of the existence of positive solutions and their stability for semilinear elliptic equations with nonlinear boundary conditions can be found in [12, 6, 1, 11, 15, 17, 18].

To begin with, we consider the following linear eigenvalue problem:

$$\begin{cases} -\Delta \phi = \lambda m \phi & \text{in } D, \\ \dfrac{\partial \phi}{\partial \mathbf{n}} = 0 & \text{on } \partial D. \end{cases} \tag{1.1}$$

It is known that $\lambda = 0$ is a simple eigenvalue of (1.1) with a positive eigenfunction. Brown and Lin [3] has proved under condition $\int_D m dx < 0$ that problem (1.1) possesses a unique eigenvalue $\lambda_1(m) > 0$ having a positive eigenfunction, and that it is simple. Meanwhile, it is also shown in [3] that if $\int_D m dx \geq 0$, then problem (1.1) has no positive eigenvalue with a positive eigenfunction, so that we set $\lambda_1(m) = 0$ in this case.

To solve problem $(*)_\lambda$ means the consideration of the existence of the steady state for the following initial boundary value problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{1}{\lambda} \Delta v + (mv - av^2) & \text{in } (0,\infty) \times D, \\ v(0,x) = u_0(x) & \text{in } D, \\ \dfrac{\partial v}{\partial \mathbf{n}} + g(v)v = 0 & \text{on } (0,\infty) \times \partial D. \end{cases} \tag{1.2}$$

A non-negative solution u of $(*)_\lambda$ is said to be *globally asymptotically stable* if all the global solutions $v(t,x)$ of (1.2), which means that $v(\cdot,x) \in C^1((0,\infty))$, $v(t,\cdot) \in C^2(\overline{D})$ and v satisfies (1.2), tend to u as $t \to \infty$ in the uniform topology of $x \in \overline{D}$ for any non-negative, non-zero initial data $u_0 \in C^2(\overline{D})$.

In the linear boundary condition case, the existence, uniqueness and stability for positive solutions have been discussed by many authors (cf. [4, 2, 5, 9, 16]). The following result for the Neumann case is due to Hess [10].

THEOREM 1. *Suppose $g \equiv 0$. Then the following two assertions hold.*

(1) *Assume $\int_D m\, dx < 0$. Then there exists a unique positive solution $u(\lambda)$ of $(*)_\lambda$ for each $\lambda > \lambda_1(m)$ with the condition that $\|u(\lambda)\|_{C^2(\overline{D})} \to 0$ as $\lambda \downarrow \lambda_1(m)$, and no positive solution for any $0 < \lambda \leq \lambda_1(m)$. In addition, the trivial solution $u \equiv 0$ of $(*)_\lambda$ is globally asymptotically stable for $0 < \lambda < \lambda_1(m)$ and the unique positive solution $u(\lambda)$ is globally asymptotically stable for $\lambda > \lambda_1(m)$.*

(2) *Assume $\int_D m\, dx \geq 0$. Then problem $(*)_\lambda$ admits a unique positive solution for all $\lambda > 0$ and the unique positive solution $u(\lambda)$ satisfies*

$$\left\| u(\lambda) - \frac{\int_D m\, dx}{a|D|} \right\|_{C^2(\overline{D})} \longrightarrow 0 \quad \text{as } \lambda \downarrow 0,$$

where $|D|$ denotes the volume of D. Additionally the unique positive solution $u(\lambda)$ is globally asymptotically stable for $\lambda > 0$.

The purpose of this note is to study the existence of the steady state of (1.2) for non-negative, non-zero, *small* initial data $u_0 \in C^2(\overline{D})$ and to investigate its asymptotic behavior as the diffusion rate $1/\lambda$ increases to an unlimited extent, that is, $\lambda \downarrow 0$. The motivation for the study arises from the fact that, in the case of nonlinear boundary conditions, the uniqueness for positive solutions does not necessarily hold (cf. [1, Theorem 2.6], [13, Theorem 4.6.3] for the uniqueness results).

For each non-negative solution u of $(*)_\lambda$ let $\gamma_1(\lambda, u)$ be the first eigenvalue of the linearized eigenvalue problem (see [1, Theorem 2.2])

$$\begin{cases} -\Delta w = \lambda(m - 2au)w + \gamma(\lambda,u)w & \text{in } D, \\ \frac{\partial w}{\partial \mathbf{n}} + (g'(u)u + g(u))w = \gamma(\lambda,u)w & \text{on } \partial D. \end{cases}$$

A non-negative solution u of $(*)_\lambda$ is called *stable* if $\gamma_1(\lambda, u)$ is positive and *unstable* if $\gamma_1(\lambda, u)$ is negative. Concerning the trivial solution of $(*)_\lambda$, one can show that if $\int_D m dx < 0$, then it is stable for $0 < \lambda < \lambda_1(m)$, and on the other hand, it is unstable for $\lambda > 0$ if $\int_D m dx \geq 0$.

For our purpose we discuss the existence of the minimal positive solution of $(*)_\lambda$ where the trivial solution $u \equiv 0$ is unstable. We say that the minimal positive solution $u(\lambda)$ is *one-side asymptotically stable* if all the global solutions $v(t, x)$ of (1.2) tends to $u(\lambda)$ as $t \to \infty$ in the uniform topology of $x \in \overline{D}$ for any initial data $u_0 \in \{u \in C^2(\overline{D}) : 0 \leq u \leq u(\lambda)\} \setminus \{0\}$.

Now we can formulate our main results.

THEOREM 2. *Suppose that nonlinearity g satisfies the condition*

$$g(0) = 0 \quad \text{and} \quad g'(0) > 0. \tag{G.1}$$

If $\int_D m dx > 0$, then there exists the minimal positive solution $u(\lambda)$ of $(*)_\lambda$ for $\lambda > 0$ small, which is one-side asymptotically stable and satisfies $\|u(\lambda)\|_{C^2(\overline{D})} \longrightarrow 0$ as $\lambda \downarrow 0$.

On the other hand, we have the following:

THEOREM 3. *Suppose that g satisfies the condition $g(0) = 0$. If $\int_D m dx > 0$, then the following assertions hold.*

(1) *Assume that g is strictly negative for $t > 0$, and that there exists a constant $M_0 > 0$ such that*

$$tg(t) \geq -M_0 \quad \text{for } t \geq 0. \tag{1.3}$$

Then the minimal positive solution $u(\lambda)$ of $(*)_\lambda$ exists for all $\lambda > 0$, and it is one-side asymptotically stable and satisfies

$$\|u(\lambda)\|_{C(\overline{D})} \longrightarrow \infty \quad \text{as } \lambda \downarrow 0. \tag{1.4}$$

(2) *Let $m_+(x) = \max(m(x), 0)$. Assume that there exists a constant $t_1 > 0$ such that*

$$\begin{cases} g(t_1) = 0, \\ g(t) < 0 & \text{for all } 0 < t < t_1. \end{cases}$$

Then we can prove the following three assertions:

(2-i) *If t_1 is so* large *that*

$$t_1 \geq \frac{\|m_+\|_{C(\overline{D})}}{a},$$

then the minimal positive solution $u(\lambda)$ of $(*)_\lambda$ exists for all $\lambda > 0$ with the property that $u(\lambda)$ is one-side asymptotically stable and satisfies

$$\|u(\lambda) - t_1\|_{C^2(\overline{D})} \longrightarrow 0 \quad \text{as } \lambda \downarrow 0. \tag{1.5}$$

(2-ii) *On the other hand, if t_1 is so* small *that*

$$t_1 < \frac{\|m_+\|_{C(\overline{D})}}{a},$$

then we have the same conclusion as in (2-i) *whenever* $g(t) > 0$ *for all* $t > t_1$.

(2-iii) *Assume condition* (1.3), *and assume* $g(t) < 0$ *for all* $t > t_1$. *If* t_1 *is so* small *that*

$$t_1 < \frac{\int_D m dx}{a|D|},$$

then problem $(*)_\lambda$ *admits the minimal positive solution* $u(\lambda)$ *for all* $\lambda > 0$, *and it is one-side asymptotically stable and satisfies* (1.4).

Finally we mention case $\int_D m\,dx = 0$. This case is more delicate, where an *a priori* bounds below for positive solutions which we will obtain does not work for the characterization of the behavior of the minimal positive solution, more precisely, the *a priori* bounds is not useful to exclude the existence of the positive solutions u_λ of $(*)_\lambda$ such that $\|u_\lambda\|_{C(\overline{D})} \to 0$ as $\lambda \downarrow 0$. However, using a stability argument, we overcome this difficulty.

Now we have the following:

THEOREM 4. *Suppose that* g *satisfies the condition*

$$g(0) = 0 \quad and \quad g'(0) < 0. \tag{G.2}$$

If $\int_D mdx = 0$, *then for every* $a > 0$ *there exist constants* $\lambda^*(a), t^*(a) > 0$ *such that a positive solution* u *of* $(*)_\lambda$ *is unstable whenever* $0 < \lambda \le \lambda^*(a)$ *and* $u \le t^*(a)$ *on* $\overline{D}$.

As a corollary from Theorem 4, we have the following:

COROLLARY 5. *Suppose condition* (G.2). *Then assertions* (1), (2-i) *and* (2-ii) *of Theorem 3 remain true for case* $\int_D mdx = 0$.

In the next section we give an outline of the proofs of Theorems 2 through 4. For further details, the reader should refer to [19].

2. Outline of proofs. The proof of Theorem 2 relies on the local bifurcation theory due to Crandall and Rabinowitz [7, 8]. Applying the theory to our problem, we have a unique positive solution branch $(\lambda(s), u(s))$ with $s > 0$ small, such that $(\lambda(0), u(0)) = (0, 0)$. Green's formula gives us

$$\lambda'(0) = \frac{g'(0)\sigma(\partial D)}{\int_D mdx},$$

where $\sigma(\partial D)$ is the surface measure of ∂D . This formula and (G.1) characterize the behavior of $u(s)$.

Next we present an outline of the proof of Theorem 3 only for cases (2-i) and (2-iii). Let $\varphi_1(\lambda)$ be the positive eigenfunction, normalized as $\|\varphi_1(\lambda)\|_{C(\overline{D})} = 1$, corresponding to the first eigenvalue $\mu_1(\lambda)$ of the eigenvalue problem

$$\begin{cases} -\Delta\varphi_1(\lambda) = \lambda m \varphi_1(\lambda) + \mu_1(\lambda)\varphi_1(\lambda) & \text{in } D, \\ \frac{\partial \varphi_1(\lambda)}{\partial \mathbf{n}} = 0 & \text{on } \partial D. \end{cases}$$

It follows from [14] that $\mu_1(\lambda) < 0$ for $\lambda > 0$. Therefore we can show that $\varepsilon\varphi_1(\lambda)$ is a sub-solution of $(*)_\lambda$ whenever $\varepsilon > 0$ is sufficiently small. On the other hand, we see that t_1 is a super-solution for case (2-i), and that a large super-solution can be constructed for case (2-iii) by virtue of (1.3). Hence there exists the minimal positive solution by the super-sub-solution method ([1, Theorem 2.1]).

To characterize the behavior of the minimal positive solution, we need to establish an *a priori* bounds below for positive solutions. To construct the sub-solution $\varepsilon\varphi_1(\lambda)$ derives the following *a priori* bounds for positive solutions: For any positive solution u of $(*)_\lambda$ we have

$$u \geq \min\left\{-\frac{\mu_1(\lambda)}{a\lambda},\ t_1\right\}\varphi_1(\lambda) \quad \text{on } \overline{D}. \tag{1.6}$$

Since Green's formula gives us

$$\lim_{\lambda\to 0}\frac{\mu_1(\lambda)}{\lambda} = -\frac{\int_D m\,dx}{|D|}, \tag{1.7}$$

we can exclude the positive solutions of $(*)_\lambda$ that tends to zero as $\lambda \downarrow 0$.

For case (2-i) we know that constant t_1 is a super-solution, which leads to assertion (1.5). For case (2-iii) an analogous one as in (1.6) is given as

$$u \geq -\frac{\mu_1(\lambda)}{a\lambda}\varphi_1(\lambda) \quad \text{on } \overline{D} \tag{1.8}$$

for any positive solution u of $(*)_\lambda$. Assertions (1.7) and (1.8) provide us

$$\|u\|_{C(\overline{D})} > t_1$$

for any positive solution u of $(*)_\lambda$ whenever $\lambda > 0$ is sufficiently small, which leads to assertion (1.4).

Finally we show Theorem 4. As to the first eigenvalue $\gamma_1(\lambda, u)$ we can show

$$\gamma_1(\lambda,u)(|D| + \sigma(\partial D)) \leq 2a\lambda\int_D u\,dx + \int_{\partial D}(g'(u)u + g(u))\,d\sigma.$$

On the other hand, we can verify that

$$\frac{g'(t)t + g(t)}{t} < -\alpha_0 \quad \text{whenever } t > 0 \text{ is small,}$$

with some constant $\alpha_0 > 0$, that

$$\int_{\partial D}\varphi_1(\lambda)\,d\sigma \geq \beta_0 \quad \text{whenever } \lambda > 0 \text{ is small,}$$

with some constant $\beta_0 > 0$, and that

$$\frac{\mu_1(\lambda)}{\lambda^2} < -d_0 \quad \text{whenever } \lambda > 0 \text{ is small,}$$

with some constant $d_0 > 0$. The above three conditions and (1.6) leads to the assertion

$$\gamma_1(\lambda, u) < 0 \quad \text{whenever } \lambda, u > 0 \text{ are sufficiently small.}$$

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