# ON TWO NONLOCAL ELLIPTIC PROBLEMS

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ABSTRACT. We study here the stationary solutions of the following system

$$u_t = \nabla \cdot \left( \frac{m-1}{m} \nabla u^m + u \nabla \varphi \right), \qquad m > 1,$$
$$\Delta \varphi = \pm u,$$

defined in a bounded domain $\Omega$ of $\mathbb{R}^n$. The physical interpretation of the above system comes from the porous medium theory and semiconductor physics.

The temporal evolution of the spatial density $u(x,t)$ ($x \in \mathbb{R}^n, t \geq 0$) of free carriers in semiconductors or in electrolytes is described by the parabolic-elliptic system of equations [7], [11], [13], which simplified form reads

$$u_t = \nabla \cdot (\nabla u + u \nabla \varphi) = \Delta u + \nabla u \cdot \nabla \varphi + u \Delta \varphi, \tag{1}$$

$$\Delta \varphi = -u. \tag{2}$$

Here $\varphi$ is an electric potential generated by the density $u$.

In this model we assume that the flow of particles caused by thermal chaotic movement is proportional to the gradient of density $\nabla u$ (Fick's law), and velocity of each carrier is proportional to the gradient of the electric potential. The last assumption is consistent with the character of frictional forces acting on each particle.

We propose, following [12], to change the continuity equation (1) so, that the term $\frac{m-1}{m}\Delta u^m$, $m > 1$, replaces $\Delta u$. This kind of term appears in the equations describing the flow in porous media ([1]). In this way, we consider free carriers in semiconductors (ions in electrolyte, respectively) as a gas of self-interacting particles moving in a porous medium. Our problem assumes now the form

$$u_t = \nabla \cdot \left( \frac{m-1}{m} \nabla u^m + u \nabla \varphi \right), \quad m > 1, \tag{3}$$

$$\Delta \varphi = -u. \tag{4}$$

Instead of (4) we can assume another relation between the density and potential. For example, considering self-gravitating system, (4) should be replaced by

$$\Delta \varphi = u, \tag{5}$$

where $\varphi$ is the gravitational potential generated by $u$.

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The natural boundary condition which guarantees the conservation of total mass $M = \int_\Omega u(x,t)\,dx$ is "no-flux" condition, i.e.

$$\frac{m-1}{m}\frac{\partial u^m}{\partial \nu} + u\frac{\partial \varphi}{\partial \nu} = 0. \tag{6}$$

We put for $\varphi$ zero boundary condition

$$\varphi|_{\partial\Omega} = 0, \tag{7}$$

which, in the Coulomb case, says that the boundary is grounded.

The system is supplemented with the initial condition

$$u(x,0) = u_0(x). \tag{8}$$

The problem (1)-(2) was considered in the series of papers: [2] – [10], [14] – [17]. It was shown that the global existence of solution, the existence of stationary solutions and blow up phenomena depend on the character of interaction between the particles, total mass of the system and geometry of the domain $\Omega$.

Here we are interested in the problem of existence of a stationary solution $\langle U, \Phi\rangle$ of (3), (4) (or (5)), (6)-(7) with a given total charge (mass) of particles $\int_\Omega U = M$.

Stationary solutions $\langle U, \Phi\rangle$ of (3), (4) (or (5)) fulfill the system

$$\nabla \cdot \left(\frac{m-1}{m}\nabla U^m + U\nabla\Phi\right) = 0, \tag{9}$$

$$\Delta\Phi = -U, \quad (\Delta\Phi = U). \tag{10}$$

From the first equation we get the following relationship between $U$ and $\Phi$

$$\Phi + U^{m-1} = C, \tag{11}$$

with some constant $C > 0$ (note that $\Phi = 0$ and $U > 0$ on the boundary $\partial\Omega$).

Putting $U = (C - \Phi)^{1/(m-1)}$ into (10) we reduce the question of the existence of stationary solutions of (3), (4) (or (5)), (6)-(7) with a given total charge (mass) $M$ to the nonlocal elliptic problem

$$\Delta\Phi = -(C-\Phi)^\alpha, \qquad \alpha = \frac{1}{m-1} > 0, \tag{12}$$

in the Coulomb case or

$$\Delta\Phi = (C-\Phi)^\alpha, \tag{13}$$

in the gravitational one.

On the boundary $\partial\Omega$ we have

$$\Phi = 0, \tag{14}$$

and the unknown constant $C$ is connected with $M = \int_\Omega U$ by the relation

$$\int_\Omega (C-\Phi)^\alpha = M. \tag{15}$$

For a given $C > 0$ the problem (12) ((13)), (14) will be called *problem* $(C)$ and $\Phi_C$ will denote its solution.

The proof of the existence of a solution to the *problem* $(C)$ will be based on the theory of sub- and supersolutions of elliptic problems (cf. [18], [19]).

In the Coulomb case we prove the following theorem.

THEOREM 1. *For every $M > 0$ and each domain $\Omega$ there exists a unique solution of the problem* (12), (14), (15).

*Proof:* We proceed as follows. First we prove the existence of a unique solution $\Phi_C$ of the *problem (C)* for all $C > 0$. Next we check the monotonicity and continuity of the function $M(C) = \int_\Omega (C - \Phi_C)^\alpha$. The theorem will be proved if we show that $M((0, +\infty)) = \mathbb{R}^+$.

The proof is divided into two parts: $\alpha \geq 1$ (here we obtain classical solutions) and $0 < \alpha < 1$ (solutions will be weak).

The existence of the solution $\Phi_C$ for $C > 0$ is obvious since we have $\underline{\Phi_C} = 0$ as a subsolution and $\overline{\Phi_C} = C$ as a supersolution for the *problem* $(C)$. To show the continuity $M(C) = \int_\Omega (C - \Phi_C)^\alpha$ note that for $\alpha > 0$ and $C > C_0$ ($C < C_0$ resp.) the function $\Phi_{C_0} + C - C_0$, ($\Phi_{C_0}$ resp.) is a supersolution to the *problem* $(C)$. To prove monotonicity of the function $M(C)$ note that for $m \in (1, 2]$ ($\alpha \geq 1$) and all $C > C_0$ the function $\frac{C}{C_0}\Phi_{C_0}$ is a subsolution for the *problem* $(C)$. Hence we have

$$M(C) = -\int_\Omega \Delta\Phi_C = -\int_\Omega \frac{\partial}{\partial\nu}\Phi_C > -\int_\Omega \frac{\partial}{\partial\nu}\frac{C}{C_0}\Phi_{C_0} = \frac{C}{C_0}M(C_0). \tag{16}$$

The last inequality implies that $M(C) \to \infty$ as $C \to \infty$. Note that $M(C) = \int_\Omega (C - \Phi_C)^\alpha \leq |\Omega| C^\alpha$. Hence $M(C) \to 0$ if $C \to 0$, and the theorem has been proved for $\alpha \geq 1$.

The proof for $\alpha \in (0, 1)$ is similar but we use weak equivalents.

In the gravitational problem (13), (14) we consider three cases: $\alpha \in (0, 1)$, $\alpha = 1$, $\alpha > 1$.

THEOREM 2. *For $m > 2$ ($\alpha \in (0, 1)$), $n \geq 1$ and for any domain there exists a unique solution of* (13), (14), (15) *if only $M > M_0$ where $M_0$ is some positive constant depending on $\Omega$ and $m$, $M_0 = M_0(\Omega, m) > 0$.*

*Proof:* For $\alpha \in (0, 1)$ the idea of the proof is exactly the same as in Coulomb case. The only difference lies in the fact that $M((0, \infty)) = (M_0, +\infty)$, $M_0 = M_0(\Omega, m) > 0$. To prove it we use the function $\underline{\Phi_C} = -w(2n)^{-1}(R^2 - |x|^2)$ as a subsolution of the *problem* $(C)$ for $w \geq w_C$, where $w_C = (C + \frac{w_C R^2}{2n})^\alpha$ and the function $\overline{\Phi_C} = 0$ as a supersolution.

THEOREM 3. *For $m = 2$ ($\alpha = 1$), $n \geq 1$, $M > 0$ and for any domain for which the first eigenvalue $\lambda_1$ of $-\Delta$ is greater than 1 there exists a unique solution of* (13), (14), (15). *There is no solution for domains with $\lambda_1 \leq 1$.*

*Proof:* Since for $\alpha = 1$ $\Phi_C = C\Phi_1$ we have to consider only the problem

$$\Delta\Phi_1 = 1 - \Phi_1, \tag{17}$$

$$\Phi_1|_{\partial\Omega} = 0. \tag{18}$$

We distinguish two cases:

*1)* $\lambda_1 > 1$

The Fredholm alternative implies that there exists exactly one solution $\Phi_1$ of the equation (17), (18) in $\Omega$. What we must show is that $\Phi_1 \leq 0$ ($1 - \Phi_1$ is a density). Since for $\psi_1$ (the first eigenfunction) $(\Delta + \mathrm{Id})\psi_1 = (1 - \lambda_1)\psi_1$ we can apply in $\Omega$ the maximum principle. The solution $\Phi_1$ on the boundary $\partial\Omega$ equals 0, $(\Delta + \mathrm{Id})\Phi_1 = 1 > 0$ so we get that $\Phi_1 \leq 0$ in $\Omega$.

*2)* $\lambda_1 \leq 1$

Let $\psi_1$ be the first eigenfunction for $-\Delta$ ($\Delta\psi_1 = -\lambda_1\psi_1$, $0 < \lambda_1 \le 1$). Multiplying the equation (17) by $\psi_1$ and integrating over $\Omega$ by parts we get $(1-\lambda_1)\int_\Omega \Phi_1\psi_1 = \int_\Omega \psi_1$. Since $1-\lambda_1 \ge 0$ and $\psi_1$ is constant sign function, then the function $\Phi_1$ cannot be negative.

**Theorem 4.** *For $m \le 2n/(n+2)$ ($\alpha > (n+2)/(n-2)$), $n > 2$ and the star-shaped domain there exists no solution of* (13), (14), (15) *with mass greater than* $M_1(\Omega, m)$.

*Proof:* To prove the nonexistence result for large $M$ we first prove the nonexistence of the solution of (13), (14) for $C$ large enough.

Assume that for $C \ge (\lambda_1)^{1/(\alpha-1)}$ there exists a solution $\Phi_C$ in $\Omega$. Multiplying our equation by $\psi_1$ ( $\psi_1 \ge 0$ ) we get $\int_\Omega(-\lambda_1\Phi_C-(C-\Phi_C)^\alpha)\psi_1 = 0$. But $-\lambda_1\Phi_C-(C-\Phi_C)^\alpha < (C-\Phi_C)(\lambda_1 - (\lambda_1^{\frac{1}{alpha-1}})^{\alpha-1}) \le 0$. Hence there is no solution of the *problem* $(C)$ for $C > (\lambda_1)^{1/(\alpha-1)}$. Assuming that $\Omega$ is a star-shaped domain in $\mathbb{R}^n$, we use the Pokhozhaev identity to show that there is no solution of (13), (14), (15) for $M$ large enough.

Indeed from the relation

$$\int_{\partial\Omega}\left|\frac{\partial\Phi_C}{\partial\nu}\right|^2 (x\cdot\nu)dx = 2n\frac{1}{\alpha+1}\int_\Omega\left((C-\Phi_C)^{\alpha+1}-C^{\alpha+1}\right)-(n-2)\times \left(\int_\Omega(C-\Phi_C)^{\alpha+1}-CM\right)$$

we infer

$$M^2 \le \left(\int_{\partial\Omega}|\frac{\partial\Phi_C}{\partial\nu}|^2(x\cdot\nu)dx\right)\left(\int_{\partial\Omega}(x\cdot\nu)^{-1}dx\right).$$

Since $\int_{\partial\Omega}(x\cdot\nu)^{-1}dx \le C(\Omega)d^{n-2}$, where $d =$ diam$(\Omega)$ and $C(\Omega)$ depends on the shape of $\Omega$ only (not on the size of $\Omega$), we obtain

$$M^2 \le C(\Omega)d^{n-2}\left((2n\frac{1}{\alpha+1}-n+2)\int_\Omega(C-\Phi_C)^{\alpha+1}+(n-2)CM\right).$$

If $m \le \frac{2n}{n+2}$ ($n > 2$), then $2n\frac{1}{\alpha+1}-n+2 < 0$ hence

$$M \le C(\Omega)d^{n-2}(n-2)C,$$

which together with the upper bound for $C$ gives us the nonexistence of solutions of (13), (14), (15) for $M > M_2(\Omega,\alpha) = C(\Omega)d^{n-2}(n-2)(\lambda_1)^{\frac{1}{\alpha-1}}$.

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