HOMOGENIZATION OF DEGENERATE NONLINEAR DIRICHLET PROBLEMS IN PERFORATED DOMAINS OF GENERAL STRUCTURE

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Abstract. It is considered a family of boundary value problems for degenerate nonlinear elliptic second order equations in divergence form in a sequence of domains $\Omega^{(s)} \subset \Omega \subset \mathbb{R}^n, s = 1, 2, \ldots$. The class of equations under consideration is characterized by energetic space $W_p^1\left(\Omega^{(s)}, w\right), 2 \leq p < n$, where function $w(x)$ belongs to the certain Muckenhoupt class. Any geometric hypothesis on the structure of domains $\Omega^{(s)}$ are not presupposed. It is assumed that there exists a sequence r_s tending to zero, as $s \to \infty$, such that the inequality $\operatorname{cap}_{p,w}\left(K(x_0, r)\backslash\Omega^{(s)}\right) \leq Aw(K(x_0, r))$ is valid for every $x_0 \in \Omega$ and every $r \geq r_s$. Here $K(x_0, r)$ is the cube with centre x_0 and edge $2r$, $w(K(x_0, r))$ denotes the weighted measure of $K(x_0, r)$ and $\operatorname{cap}_{p,w}$ denotes the weighted $(p, w)-$ capacity. The strong convergence in $W_m^1(\Omega, w), m < p$, of solutions $u_s(x)$ of problems under consideration is established and the limit boundary value problem is constructed.

In this paper we study the homogenization of a family of Dirichlet problems for degenerate nonlinear elliptic second order equations in perforated domains of general structure provided that the weight function belongs to the certain Muckenchoupt class.

Let Ω be any bounded open set in the n-dimensional Euclidean space $\mathbb{R}^n$ and let $\Omega^{(s)}, s = 1, 2, \ldots$, be an arbitrary sequence of open subsets of Ω. In the domain $\Omega^{(s)}$ we consider a nonlinear elliptic boundary value problem

$$\sum_{j=1}^{n} \frac{d}{dx_j} a_j\left(x, u, \frac{\partial u}{\partial x}\right) = a_0\left(x, u, \frac{\partial u}{\partial x}\right), \qquad x \in \Omega^{(s)}, \tag{1}$$

$$u(x) = f(x), \qquad x \in \partial\Omega^{(s)}. \tag{2}$$

Our conditions on coefficients $a_j(x, u, q), j = 0, 1, \ldots, n,$ and function $f(x)$ provide the existence of solution $u_s(x) \in f(x) + \overset{\circ}{W}{}_p^1\left(\Omega^{(s)}, w\right)$ of problem (1), (2) for every s and the boundedness of the sequence $u_s(x)$ in $W_p^1\left(\Omega^{(s)}, w\right)$, where the function $w(x)$ belongs to the certain Muckenhoupt class. In the present paper we establish the strong

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convergence of the sequence $u_s(x)$ in $W^1_m(\Omega, w)$ for every $m < p$, as $s \to \infty$, and construct a boundary value problem satisfied by the limit function.

Similar problems under suitable geometric assumptions on the sets $\Omega^{(s)}$ have been considered by the author in [1-3]. The asymptotic behaviour of the solutions of degenerate nonlinear Dirichlet problems has been studied for a family of domains with a finely granulated boundary and for perforated domains with more complicated microstructure (in particular, a smallness of diameters of cavities relative to distances between them has not been assumed). In this paper we consider a homogenization of degenerate nonlinear elliptic boundary value problems (1), (2) without any geometric assumptions on the structure of the sets $\Omega \backslash \Omega^{(s)}$.

In the nondegenerate case (i.e. $w(x) \equiv 1$) the linear problems of homogenization under similar assumptions on the structure of perforated domains have been investigated in [4,5], and the nonlinear problems of homogenization have been investigated in [6,7].

1. Let w be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < w < \infty$ almost everywhere. We say that w belongs to the Muckenhoupt class A_t ($w \in A_t(\mathbb{R}^n)$), $1 < t < \infty$, if there is a constant $c_{t,w}$ such that

$$\frac{1}{|B|}\int\limits_B w dx \le c_{t,w}\left(\frac{1}{|B|}\int\limits_B w^{1/(1-t)}dx\right)^{1-t}$$

for all balls B in $\mathbb{R}^n$. By $|E|$ we denote the Lebesgue n-measure of measurable set $E \subset \mathbb{R}^n$.

Definition and basic properties of Muckenhoupt class A_t were explicitly studied in [8].

2. We assume that the functions $a_j(x,u,q), j = 0,1,\dots,n,$ are defined for $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^1, q \in \mathbb{R}^n$, and satisfy the following conditions:

A_1) functions $a_j(x,u,q)$ are continuous in u, q for almost every $x \in \mathbb{R}^n$, measurable in x for all $u \in \mathbb{R}^1, q \in \mathbb{R}^n; a_j(x,u,0) = 0$ for $x \in \mathbb{R}^n, u \in \mathbb{R}^1, j = 1,\dots,n$;

A_2) there exist positive constants $\nu_1, \nu_2, \varepsilon$, such that for $2 \le p < n$ and all $x \in \mathbb{R}^n$, $u, v \in \mathbb{R}^1, g, q \in \mathbb{R}^n$ the inequalities

$$\sum_{j=1}^n [a_j(x,u,g) - a_j(x,u,q)](g_j - q_j) \ge \nu_1 |g-q|^p w(x),$$

$$a_0(x,u,g)u \ge -(\nu_1 - \varepsilon)|g|^p w(x) - \varphi(x)(1+|u|)w(x),$$

$$\sum_{j=1}^n |a_j(x,u,g) - a_j(x,v,q)| \le \nu_2 \left(|u|^{p_1} + |v|^{p_1} + |g|^p + |q|^p\right)^{(p-2)/p}(|u-v| + |g-q|)w(x),$$

$$|a_0(x,u,g)| \le \nu_2\left(|u|^{p_1} + |g|^p\right)^{(p_1-1)/p_1} w(x) + \varphi(x)w(x)$$

hold, where $w(x) \in A_{p-1+\frac{p}{n}}(\mathbb{R}^n), [w(x)]^{-\frac{1}{p-1}} \in A_{\frac{p}{p-1}(1-\frac{1}{n})}(\mathbb{R}^n)$.

From [8] it follows that if $w(x) \in A_{p-1+\frac{p}{n}}(\mathbb{R}^n)$, then $w(x) \in A_{p_0}(\mathbb{R}^n)$, with $1 < p_0 < p - 1 + \frac{p}{n}$. We assume that the constant p_1 and the function $\varphi(x)$ in assumption A_2) satisfies the conditions

$$p \le p_1 < \frac{npp_0}{np_0 - p},$$

$$\varphi(x) \in L_r(\Omega, w), \qquad r > \frac{np_0}{p}.$$

Remark. Let us choose $w(x) = |x|^\alpha, -n+p < \alpha < n(p-1)-(n-p), 2 \leq p < n$. In this case $w(x)$ satisfies assumption $A_2)$. This is easily verified by a direct computation.

For the purpose of formulation of the condition imposed on the sets $\Omega^{(s)}$ let us introduce the notion of weighted (p,w)-capacity $\text{cap}_{p,w}$ ([8]).

Let us fix a bounded open set $\Omega_0 \subset \mathbb{R}^n$ such that $\overline{\Omega} \subset \Omega_0$. The number

$$\text{cap}_{p,w}(F) = \inf \int\limits_{\Omega_0} \left| \frac{\partial v(x)}{\partial x} \right|^p w(x)dx \tag{3}$$

is called the (p,w)-capacity of the compact set $F \subset \Omega_0$. The infimum in (3) is taken over all functions $v(x) \in C_0^\infty(\Omega_0)$ which satisfy the condition $v(x) = 1$ for $x \in F$.

We assume that the following condition is fulfilled:

$B)$ there exist a positive number A and a sequence $r_s > 0$, tending to zero as $s \to \infty$, such that the inequality

$$\text{cap}_{p,w}\left(K(x_0,r)\backslash\Omega^{(s)}\right) \leq Aw\left(K(x_0,r)\right)$$

holds for every $r \geq r_s$ and for every $x_0 \in \Omega$ such that $\text{dist}\,(x_0, \partial\Omega) > 2\sqrt{n}r_s$. Here

$$w\left(K\left(x_0,r\right)\right) = \int\limits_{K(x_0,r)} w(x)dx,$$

and $K\left(x_0,r\right) = \{x \in \mathbb{R}^n : |x_j - x_j^{(0)}| \leq r, j = 1,\ldots,n\}$ is the closed cube with centre $x_0 = (x_1^{(0)},\ldots,x_n^{(0)})$ and side $2r$.

The study of homogenization of nonlinear problems essentially distinguishes from the linear case because by the construction of the limit boundary value problem we need in some strong convergence of gradients of solutions of the problems (1), (2). The proof of such strong convergence is based on special asymptotic expansion by which solutions of nonlinear problems (1), (2) are approximated near sets $\Omega\backslash\Omega^{(s)}$ by some special auxiliary functions $v_r^{(s)}(x,x_0,k)$ which are defined as the solutions of some model degenerate nonlinear boundary value problems.

Let $\psi(x)$ be a function of class $C^\infty(\mathbb{R}^n)$, equal to zero outside of $K(0,1)$ and to one in $K(0,1/2)$. For an arbitrary real k and for arbitrary $x_0 \in \Omega$ under $r < 1/2$ we define $v_r^{(s)}(x,x_0,k)$ as the function belonging to $k\psi(x-x_0) + \overset{\circ}{W}{}_p^1(D_s(x_0,r),w)$ which satisfies the integral identity

$$\sum_{j=1}^{n} \int\limits_{D_s(x_0,r)} a_j\left(x, 0, \frac{\partial}{\partial x} v_r^{(s)}(x,x_0,k)\right) \frac{\partial\varphi(x)}{\partial x_j} dx = 0 \tag{4}$$

for every $\varphi(x) \in \overset{\circ}{W}{}_p^1(D_s(x_0,r),w)$. Here $D_s(x_0,r) = K(x_0,1)\backslash\left\{K(x_0,r)\backslash\Omega^{(s)}\right\}$.

Definitions and properties of weighted Sobolev spaces $W_p^1(\Omega, w), \overset{\circ}{W}{}_p^1(\Omega, w)$ were explicitly studied in [8,9].

The existence and uniqueness of the function $v_r^{(s)}(x, x_0, k)$ follows from the global theory of monotone operators (see, e.g. [10]). We extend $v_r^{(s)}(x, x_0, k)$ to $\mathbb{R}^n$ by setting $v_r^{(s)}(x, x_0, k) = k\psi(x - x_0)$ outside $D_s(x_0, r)$.

Main role by the study of the asymptotic behaviour of solutions of nonlinear elliptic problems (1), (2) and by the construction of the limit boundary value problem play several integral and pointwise estimates of solutions of the model problems (4). Corresponding results have been obtained by I.V.Skrypnik and author in [11].

In order to formulate one more condition on the sets $\Omega \backslash \Omega^{(s)}$ we introduce a capacity connected with differential equation (1), defined for every compact set $K(x_0, r) \backslash \Omega^{(s)}$ and for every real number k by

$$C_A\left(K(x_0, r) \backslash \Omega^{(s)}, k\right) = \frac{1}{k} \sum_{j=1}^{n} \int\limits_{D_s(x_0, r)} a_j\left(x, 0, \frac{\partial}{\partial x} v_r^{(s)}(x, x_0, k)\right) \frac{\partial v_r^{(s)}(x, x_0, k)}{\partial x_j} dx.$$

We assume that the following condition which ensure the possibility of constructing of the limit boundary value problem is satisfied:

C) there exists a function $c(x, k)$, continuous in $(x, k) \in \Omega \times \mathbb{R}^1$, such that for an arbitrary $x_0 \in \Omega$ and an arbitrary number $k \in \mathbb{R}^1$ the equality

$$\lim_{r \to 0} \left\{ \lim_{s \to \infty} \frac{C_A\left(K(x_0, r) \backslash \Omega^{(s)}, k\right)}{w(K(x_0, r))} \right\} = c(x_0, k) \tag{5}$$

holds and the convergence to the limit in (5) being uniform with respect to $x_0 \in \Omega$ and k on any bounded interval of values of k.

3. Given $f(x) \in W_p^1(\Omega, w)$, a solution of the boundary value problem (1), (2) is a function $u(x) \in W_p^1\left(\Omega^{(s)}, w\right)$, satisfying $u(x) - f(x) \in \overset{\circ}{W}{}_p^1\left(\Omega^{(s)}, w\right)$ such that the integral identity

$$\sum_{j=1}^{n} \int\limits_{\Omega^{(s)}} a_j\left(x, u(x), \frac{\partial u(x)}{\partial x}\right) \frac{\partial \varphi(x)}{\partial x_j} dx + \int\limits_{\Omega^{(s)}} a_0\left(x, u(x), \frac{\partial u(x)}{\partial x}\right) \varphi(x) dx = 0$$

holds for an arbitrary function $\varphi(x) \in \overset{\circ}{W}{}_p^1\left(\Omega^{(s)}, w\right)$.

Using some methods of the global theory of monotone operators it is easy to prove the existence of a solution of problem (1), (2) (see, e.g. [10]). We can state

THEOREM 1. *Assume that the conditions $A_1), A_2), f(x) \in W_p^1(\Omega, w)$ are satisfied. Then for every s the problem (1), (2) has at least one solution $u_s(x)$. Moreover, there is a constant R independent of s, such that the estimate*

$$||u_s(x)||_{W_p^1(\Omega^{(s)}, w)} \leq R$$

holds for all s.

In what follows, $u_s(x)$ stands for one of the possible solutions of the problem (1), (2), satisfying the above estimate. Therefore, the sequence $\{u_s(x)\}$ will be considered fixed. The functions $u_s(x)$, defined for $x \in \Omega^{(s)}$, are extended to Ω by setting $u_s(x) = f(x)$ for $x \in \Omega \backslash \Omega^{(s)}$. The functions $u_s(x)$, obtained in this way, are defined for $x \in \Omega$, belong to $W_p^1(\Omega, w)$ and satisfy the estimate

$$||u_s(x)||_{W_p^1(\Omega,w)} \leq R_1 \tag{6}$$

with a constant R_1 independent of s. By (6), the sequence $\{u_s(x)\}$ contains a weakly convergent subsequence and, passing to a subsequence if necessary, we may assume that $u_s(x)$ converges weakly in $W_p^1(\Omega, w)$ to a function $u_0(x)$.

It is easy to prove, by Mozer's method, that the sequence $\{u_s(x)\}$ is uniformly bounded. More precisely, the following result holds.

THEOREM 2. *Assume that the conditions* $A_1), A_2), f(x) \in W_q^1(\Omega, w) \cap L_\infty(\Omega, w)$, $q > np_0$, *are satisfied. Let* $u_s(x)$ *be a sequence of solutions of the problem* (1), (2) *satisfying* (6). *Then there exists a constant* M *independent of* s, *such that the estimate*

$$\operatorname*{ess\,sup}_{x \in \Omega} |u_s(x)| \leq M$$

holds for all s.

The main result of this paper is the following

THEOREM 3. *Assume that the conditions* $A_1), A_2), B), C), f(x) \in W_q^1(\Omega_0, w) \cap$ $\cap L_\infty(\Omega_0, w)$, $q > np_0$, *are satisfied. Let* $u_s(x)$ *be a sequence of solutions of the problem* (1), (2) *which converges weakly in* $W_p^1(\Omega, w)$ *to* $u_0(x)$. *Then the sequence* $u_s(x)$ *converges to* $u_0(x)$ *strongly in* $W_m^1(\Omega, w)$ *for every* $m \in (1, p)$, *and* $u_0(x)$ *belongs to* $f(x) + \overset{\circ}{W}{}_p^1(\Omega, w)$ *and satisfies the integral identity*

$$\sum_{j=1}^{n} \int_\Omega a_j \left(x, u_0(x), \frac{\partial u_0(x)}{\partial x}\right) \frac{\partial \varphi(x)}{\partial x_j} dx + \int_\Omega a_0 \left(x, u_0(x), \frac{\partial u_0(x)}{\partial x}\right) \varphi(x) dx =$$

$$= \int_\Omega c(x, f(x) - u_0(x)) \varphi(x) w(x) dx$$

for every $\varphi(x) \in \overset{\circ}{W}{}_p^1(\Omega, w)$.

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