ON GLOBAL SOLUTIONS TO A NONLINEAR ELLIPTIC BOUNDARY PROBLEM IN AN UNBOUNDED DOMAIN

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Abstract. We consider solutions to the elliptic linear equation

$$Lu := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) = 0$$

of second order in an unbounded domain

$$\{x = (x', x_n) \ : |x'| < Ax_n^\sigma + B, \ 0 < x_n < \infty\}, \ \ 0 \leq \sigma \leq 1,$$

in $\mathbf{R}^n$. We study the asymptotic behavior as $x_n \to \infty$ of the solutions of (1) satisfying the nonlinear boundary condition

$$\frac{\partial u}{\partial N} - b(x)|u(x)|^{p-1}u(x) \geq 0$$

on the lateral surface

$$S = \{x \in \partial Q, \ 0 < x_n < \infty\},$$

where $p > 0, \ b(x) \geq b_0 > 0$.

We show that a global solution of the problem can exist not for all values of parameters p, σ and indicate these values.

The boundary problem in the cylinder was studied by us in [4], [5]. The obtained results generalize some results of B. Hu in [3].

1. Introduction. We consider the solutions to the elliptic second order linear equation

$$Lu := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) = 0 \tag{1}$$

of second order in an unbounded domain

$$Q = \{x = (x', x_n) \ : |x'| < Ax_n^\sigma + B, \ 0 < x_n < \infty\}, \ \ 0 \leq \sigma \leq 1,$$

in $\mathbf{R}^n$. We study the asymptotic behavior as $x_n \to \infty$ of the solutions of (1) satisfying nonlinear boundary condition

$$\frac{\partial u}{\partial N} - b(x)|u(x)|^{p-1}u(x) \geq 0 \tag{2}$$

on the lateral surface

$$S = \{|x'| = Ax_n^\sigma + B, \ 0 < x_n < \infty\},$$

where $p > 0, \ b(x) \geq b_0 > 0$ and

$$\frac{\partial u}{\partial N} = \sum_{i=1}^{n} a_{ij}(x) \, \frac{\partial u}{\partial x_j} \, \cos\theta_i,$$

θ_i is the angle between the axis x_i and the outer normal vector.

We suppose that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq c_0|\xi|^2, c_0 > 0, \quad x \in Q,$$

and that $|a_{ij}(x)| \leq C$ for $i, j = 1, ..., n$ and for all $x \in Q$. We don't assume that a_{ij} are continuous.

Let us denote Ω_T and Σ_T the sections of the domain Q and the boundary S by the plane $x_n = T$, and Q_T and S_T the parts of Q and S between the planes $x_n = 1$ and $x_n = T$.

We consider weak solutions u (1) of equation satisfying (2). It means that $u \in H^1_{loc}(Q) \cap L_{p+1,loc}(S)$,

$$\int_Q \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial \psi}{\partial x_i}dx = 0,$$

$$\int_Q \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i}dx + \int_S b(x)|u(x)|^{p-1}u(x)\varphi(x)dS \geq 0 \tag{3}$$

for all functions $\psi \in H^1_0(Q)$, and positive functions $\varphi(x) \in H^1(Q)$vanishing at $x_n = 0$ and in a neighborhood of $x_n = \infty$.

We are looking for the necessary conditions of global existence of solutions to the problem (1)-(2).

2. Auxiliary results.

LEMMA 1. (Harnack's inequality). *Suppose that $u > 0$ is a weak solution to the equation $Lu = 0$ in a domain $\Omega \subset \mathbf{R}^n$ and $\Omega' \subset\subset \Omega$. There exists a positive constant C such that*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u.$$

If $\Omega = R \cdot \Omega_1, \Omega' = R \cdot \Omega'_1$, then the constant C is independent of R for $R \geq 1$.

Proof. See Theorem 8.1 in [1], p. 237. □

LEMMA 2. (Harnack's inequality in a closed domain). *Let u be a weak solution to the equation $Lu = 0$ and $u > 0$ in a domain $\Omega \subset \mathbf{R}^n$ with smooth (of class C^1) boundary, $\partial u/\partial N = 0$ on $\partial\Omega \cap \overline{\Omega}'$, where Ω' is a bounded subdomain in $\mathbf{R}^n$ with boundary of class C^1, $\overline{\Omega}' \subset\subset \overline{\Omega}$. Then there exists a positive constant C such that*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u. \tag{4}$$

If $\Omega = R \cdot \Omega_1, \Omega' = R \cdot \Omega'_1$, then the constant C is independent of R for $R \geq 1$.

Proof. Let us fix a point $y_0 \in \Gamma$. There exists a neighborhood ω of this point and a system of local coordinates $(y_1, \ldots, y_n)$ such that

1) $\omega \cap \Omega' \subset \{y_1 > 0\}$; $\omega \cap \Gamma \subset \{y_1 = 0\}$;

2) operator L has the form

$$Lv = \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}(b_{ij}(y)\frac{\partial v}{\partial y_j}).$$

We can suppose that this neighborhood is of the form $\omega = \{y : |y - y_0| < r\}$. The operator $L_1(y, D)$ coincides with L in $\omega^+ = \omega \cap \Omega'$ and with $L(-y_1, y', -D_1, D')$ in $\omega \setminus \Omega'$. Let $u_1(y) = u(y)$ in $\omega \cap \Omega'$ and $u_1(y) = u(-y_1, y')$ in $\omega \setminus \Omega'$. Then

$$L_1(y, D)u_1(y) = 0$$

in $\omega \setminus \{y_1 = 0\}$ and $\partial u(y)/\partial y_1 = 0$ for $y_1 = 0^+$. Let $v(y)$ be the solution of the equation $L_1 v(y) = 0$ in ω, coinciding with u_1 on the boundary of ω. Obviously, $v(y) = u_1(y)$ in ω.

Let $\omega_1 = \{y; |y - y_0| < r/2\}$. By Lemma 1, we have

$$\sup_{\omega_1} v(y) \le C \inf_{\omega_1} v(y),$$

and therefore,

$$\sup_{\omega_1^+} u(y) \le C \inf_{\omega_1^+} u(y).$$

Choosing a finite covering of the boundary Γ by the sets of the form as ω_1 and applying once again Lemma 1, we obtain the inequality (4). The proof is complete. □

LEMMA 3. (Hardy's inequality) *Let $n \ge 3$. Let $u \in C(\overline{Q}) \cap C^1(Q)$ and $u = 1$ if $x_n = 1$, $\sigma(n-1) \ge 1$, and $u = 0$ if $x_n = 1$, $\sigma(n-1) < 1$. There exists a constant C such that*

$$\int_Q \frac{u^2}{|x|^2} dx \le C \int_Q |\nabla u|^2 dx.$$

Proof. Put $u = 1$ if $x_n \le 1$, $\sigma(n-1) \ge 1$ and $u = 0$ if $x_n \le 1$, $\sigma(n-1) < 1$. Let $y' = x', y_n = x_n^\sigma$. The domain Q in y-coordinates has the form

$$|y'| \le Ay_n + B.$$

Using the Hardy inequality

$$\int_Q u^2 |y|^{-1-1/\sigma} dy \le c_1 \int_Q |\nabla_y u|^2 |y|^{1-1/\sigma} dy,$$

(see, for example [6]), we obtain in x-coordinates the inequality

$$\int_Q u^2 x_n^{-2} dy \le c_2 \int_Q (x_n^{2(\sigma-1)} |\nabla_{x'} u|^2 + u_{x_n}^2) dx,$$

what implies the inequality

$$\int_Q \frac{u^2}{|x|^2} dx \le c_3 \int_Q |\nabla u|^2 dx.$$

□

LEMMA 4. *Let $\sigma(n-1) \ge 1, \sigma \le 1$,*

$$Q = \{x = (x', x_n) \ : |x'| < Ax_n^\sigma + B,\ 1 < x_n < \infty\},\ A > 0, B \ge 0.$$

There exists a weak solution to the following problem :

$$LE = 0 \text{ in } Q,\ \frac{\partial E}{\partial N} = 0 \text{ on } S,\ E = 1 \quad \text{for } x_n = 1,$$

from the class $H^1_{loc}(Q)$, such that $E(x) \ge c x_n^{\sigma(2-n)}$, $c > 0$. Moreover,

$$\lim_{x_n \to \infty} E(x) = 0,\ \left| \int_{\Omega_1} \sum_{j=1}^n a_{nj}(x) \frac{\partial E}{\partial x_j} dx' \right| = c_0 \ne 0.$$

Proof. Consider the problem of minimization of the functional

$$\int_Q \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} dx$$

in the class of functions u from $H^1(Q)$, equal 1 for $x_n = 1$. By the Hardy inequality of Lemma 3 we have

$$\int_Q \frac{u^2}{|x|^2} dx \le C \int_Q |\nabla u|^2 dx, \tag{5}$$

so it is easy to prove the existence of the minimizing function E and its uniqueness.

This function is positive and satisfies the equality

$$\int_Q \sum_{i,j=1}^n a_{ij}(x) \frac{\partial E}{\partial x_j} \frac{\partial \varphi}{\partial x_i} dx = 0 \tag{6}$$

for all functions φ from $H^1(Q)$, equal 0 in a neighborhood of infinity and for $x_n = 1$.

The function E cannot be equal to a constant in Q, since for $\sigma \geq 1/(n-1)$

$$\int_Q \frac{1}{|x|^2} dx = \int_0^1 x_n^{\sigma(n-1)-2} dx_n = \infty$$

and it contradicts to (5).

The same arguments show that $E(x)$ cannot be estimated from below with a positive constant. The maximum principle implies then that

$$\lim_{x_n \to +\infty} E(x) = 0.$$

Note that the function E is the limit in H^1 of the sequence of the functions u_k from $C^1(\bar{Q})$, equal 0 for $x_n > k$ and 1 for $x_n = 1$. Therefore,

$$\int_{\Omega_1} \sum_{j=1}^n a_{nj}(x) \frac{\partial E}{\partial x_j} dx' = \int_{\Omega_1} u_k(x) \sum_{j=1}^n a_{nj}(x) \frac{\partial E}{\partial x_n} dx'$$

$$= -\int_Q \sum_{i,j=1}^n a_{ij}(x) \frac{\partial E}{\partial x_j} \frac{\partial u_k}{\partial x_i} dx.$$

The limit of the last integral as $k \to \infty$ is equal to

$$-\int_Q \sum_{i,j=1}^n a_{ij}(x) \frac{\partial E}{\partial x_j} \frac{\partial E}{\partial x_i} dx$$

and therefore, is not equal to 0. Moreover, using Theorem 7.1 from [1], we can show that $u_k \to E$ uniformly on each compact subset of Q. Thus,

$$\left| \int_{\Omega_1} \sum_{j=1}^n a_{nj}(x) \frac{\partial E}{\partial x_j} dx' \right| = c_0 \neq 0.$$

Let us fix a point $x_0 \in Q$ and set $x_{0n} = T$. Let $E(x_0) = m$. By Lemma 2,

$$c_1 m \leq E(x) \leq c_2 m,$$

if $x \in Q$, $T - R \leq x_n \leq T + R$ and $c_1 > 0$, if $R = \min(T^\sigma, T/2)$. Using the homothety $x = Ry$, we obtain the equation $L_R E = 0$ in a bounded domain of diameter ≤ 2, and the coefficients of the operators L_R are uniformly bounded and these operators are uniformly elliptic.

By Lemma 2, the constants c_1, c_2 do not depend on T. Therefore,

$$\int_{T-R \leq x_n \leq T+R} E(x)^2 dx \leq C_1 m^2 T^{\sigma n}.$$

Let $h \in C^1(\mathbf{R})$, $h(x_n) = 1$ if $T - R/2 \leq x_n \leq T + R/2$, $h(x_n) = 0$ if $x_n \leq T - R$ or $x_n \geq T + R$, $|h'(x_n)| \leq C_2/R$. Put in (6) $\varphi(x) = h(x_n)E(x)$. We obtain

$$\int_{T-R \leq x_n \leq T+R} h \sum_{i,j=1}^n a_{ij}(x) \frac{\partial E}{\partial x_j} \frac{\partial E}{\partial x_i} dx$$

$$+\int_{T-R/2\geq x_n\geq T-R}\sum_{j=1}^{n}a_{nj}(x)\frac{\partial E}{\partial x_j}h'(x_n)E(x)dx$$

$$+\int_{T+R/2\leq x_n\leq T+R}\sum_{j=1}^{n}a_{nj}(x)\frac{\partial E}{\partial x_j}h'(x_n)E(x)dx=0,$$

and therefore,

$$\int_{T-R/2\leq x_n\leq T+R/2}|\nabla E|^2dx\leq C_3m^2T^{\sigma(n-2)}.$$

There exists a constant $\theta\in]T-R/2,T+R/2[$ such that

$$\int_{x_n=\theta}|\nabla E|^2dx'\leq C_3m^2T^{\sigma(n-3)}.$$

Put in (4) the function $\varphi(x)=h_\varepsilon(x)$, where $h_\varepsilon\in C^1(\mathbf{R})$, $h_\varepsilon(x_n)=1$ if $1+\varepsilon\leq x_n\leq\theta-\varepsilon$, $h_\varepsilon(x_n)=0$ if $x_n\leq 1$ or $x_n\geq\theta$, and pass to the limit for $\varepsilon\to 0$. We obtain

$$|\int_{x_n=\theta}\sum_{j=1}^{n}a_{nj}(x)\frac{\partial E}{\partial x_j}dx'|=c_0$$

for all $\theta>1$. We have

$$c_0=|\int_{x_n=T}\sum_{j=1}^{n}a_{nj}(x)\frac{\partial E}{\partial x_j}dx'|$$

$$\leq C_4(\int_{x_n=\theta}|\nabla E|^2dx')^{1/2}T^{\sigma(n-1)/2}\leq C_5mT^{\sigma(n-2)},$$

so that

$$m\geq C_6T^{\sigma(2-n)},$$

i.e.

$$E(x)\geq C_6x_n^{\sigma(2-n)},$$

q.e.d. □

LEMMA 5. *Let* $0\leq\sigma(n-1)<1$, $n>2$,

$$Q=\{x=(x',x_n)\ :|x'|<Ax_n^\sigma+B,\}\ 1<x_n<\infty,\ A>0,\ B\geq 0.$$

There exists a weak solution E of the problem :

$$LE=0\text{ in }Q,\ \ \frac{\partial E}{\partial N}=0\text{ on }S,\ \ E=0\ \text{ at }x_n=1,$$

from the class $H^1_{loc}(Q)$, satisfying the estimate $E(x)\geq c>0$ for $x_n\geq 2$.

Proof. Let $u_{T,\lambda}$ be the solution to the equation $Lu=0$ in Q_T, satisfying the boundary condition $\frac{\partial E}{\partial N}=0$ on S and such that

$$u_{T,\lambda}(x',1)=0,\ \ u_{T,\lambda}(x',T)=\lambda. \tag{7}$$

We can find this solution, minimizing the functional

$$F_T(u)=\int_{Q_T}\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\frac{\partial u(x)}{\partial x_i}dx$$

in the class of positive functions u from $C^\infty(Q_T)$, satisfying (7). The minimizing function $u_{T,\lambda}$ is positive, $0<u_{T,\lambda}(x)<\lambda$ in Q_T. Choosing the value of λ we can obtain $u_{T,\lambda}(y_0)=1$, where y_0 is a fixed point from Q, whose n-th coordinate is equal to 2.

Moreover,

$$\int_Q \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u_{T,\lambda}}{\partial x_j} \frac{\partial \varphi}{\partial x_i} dx = 0 \tag{8}$$

for all functions $\varphi(x) \in H^1(Q)$, equal 0 for $x_n \le 1$ and for $x_n = T$. The function $u_{T,\lambda}(x)$ is continuous in $\overline{Q_T}$, see [1].

Let K be a compact subset in $\bar{Q}$, and let T_0 be such that $K \subset \bar{Q}_{T_0}$. By Lemma 2, we have $|u_{T,\lambda}(x)| \le C(T_0)$ in Q_{T_0} for $T > T_0 + 1$.

Let $h(x_n)$ be such a piece-wise linear function that $h(x_n) = 1$ for $1 < x_n < T_0 - 1$, $h(x_n) = x_n$ for $0 < x_n < 1$, $h(x_n) = 0$ for $x_n > T_0$.

Put in (8) the function $\varphi(x) = h(x_n)u_{T,\lambda}(x)$, where $T > T_0$. We obtain

$$J(T,T_0) \equiv \int_Q \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u_{T,\lambda}}{\partial x_j} \frac{\partial u_{T,\lambda}}{\partial x_i} h(x_n) dx$$

$$= -\int_Q \sum_{j=1}^n a_{nj}(x) \frac{\partial u_{T,\lambda}}{\partial x_j} h'(x_n) u_{T,\lambda}(x) dx \le C_1 J(T,T_0)^{1/2} (\int_{Q_{T_0}} u_{T,\lambda}(x)^2 dx)^{1/2}$$

$$\le C_1 J(T,T_0)^{1/2} (\int_{Q_{T_0}} dx)^{1/2} \le C(T_0) J(T,T_0)^{1/2}.$$

Thus $J(T,T_0) \le C_1(T_0)$ and $C_1(T_0)$ does not depend on λ.

Therefore, the set of bounded functions $u_{T,\lambda}$ on K is weakly compact, and there exists a subsequence $\{u_{T_k,\lambda_k}\}$, which converges weakly in $H^1(K)$ to a function E. Considering a sequence of compact sets K_m, converging to Q and using diagonalization, we can find a sequence $\{u_k\}$, converging weakly in the space $H^1_{loc}(Q)$ to the function E from this space. Moreover, using Theorem 7.1 from [1], we can show that $u_k \to E$ uniformly on each compact subset of Q.

We have also

$$\int_Q \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u_k}{\partial x_j} \frac{\partial \varphi}{\partial x_i} dx = 0$$

for all functions $\varphi(x) \in H^1(Q)$, equal to 0 for $x_n = 0$ and for $x_n = T_k$.Passing to the limit we obtain that

$$\int_Q \sum_{i,j=1}^n a_{ij}(x) \frac{\partial E}{\partial x_j} \frac{\partial \varphi}{\partial x_i} dx = 0$$

for all functions $\varphi(x) \in H^1(Q)$, equal to 0 for $x_n = 0$ and in a neighborhood of infinity, i.e. E is a weak solution of the Neumann problem. Moreover, $E \ge 0$ in Q and $E(y_0) = 1$, so that $E \not\equiv 0$.

Let us show that the function E cannot be bounded. Indeed, let $E(x) \le C$ in Q. Let $h(x_n)$ be a smooth function such that $h(x_n) = 1$ if $1 < x_n < T$, $h(x_n) = 0$ for $x_n > 2T$. One can suppose that $|h'(x_n)| \le C_1 T^{-1}$, $|h''(x_n)| \le C_2 T^{-2}$. We have

$$\int_Q \sum_{i,j=1}^n h(x_n) a_{ij}(x) \frac{\partial E}{\partial x_j} \frac{\partial E}{\partial x_i} dx = \int_Q \sum_{j=1}^n h'(x_n) E(x) a_{nj}(x) \frac{\partial E}{\partial x_j} dx$$

$$\le C_3 (\int_Q \sum_{i,j=1}^n h(x_n) a_{ij}(x) \frac{\partial E}{\partial x_j} \frac{\partial E}{\partial x_i} dx)^{1/2} T^{-1/2+\sigma(n-1)/2}.$$

Therefore,

$$\int_Q \sum_{i,j=1}^n h(x_n)a_{ij}(x)\frac{\partial E}{\partial x_j}\frac{\partial E}{\partial x_i}dx \le C_4 T^{-1+\sigma(n-1)},$$

and we obtain a contradiction when $T \to \infty$.

Thus there exists a subsequence y_k, tending to ∞ and such that $E(y_k) \ge c_1$, where $c_1 = min_{x_n=2}E(x) > 0$. By the maximum principle and Lemma 2, it follows that $E(x) \ge c_0$ in Q for $x_n \ge 2$, q.e.d. □

3. Existence of positive solutions. Consider a weak solution $v(x)$ of equation (1), with boundary condition (2).

THEOREM 1. *Let*

$$Q = \{x = (x', x_n) \ : |x'| < Ax_n^\sigma + B, \ \ 1 < x_n < \infty\}, \ \frac{1}{n-1} < \sigma \le 1.$$

Suppose that $v(x)$ *satisfies* (1), (2) *and* $v(x) \ge 0$ *in* Q. *If*

$$1 < p \le 1 + \frac{2-\sigma}{\sigma(n-2)},$$

then $v(x) \equiv 0$. *If* $\sigma(n-1) = 1, 1 < p < 1 + \frac{2-\sigma}{\sigma(n-2)}$, *then also* $v(x) \equiv 0$.

Proof. The maximum principle implies that $v(x) \ge cE(x)$ for $x_n \ge 1$, where E is the function, constructed in Lemma 4, $c > 0$, or $v(x) \equiv 0$. Indeed, put $w(x) = v(x) - cE(x)$, where c is so small that $w(x) \ge 0$, when $x_n = 1$. Since $Lw = 0$ in Q, $\frac{\partial w}{\partial N} \ge 0$ on S, the function w cannot have a minimum point x_1 in Q_T if $x_{1n} > 1$. On the other hand, by the maximum principle,

$$\lim_{N\to\infty} \inf_{x_n > N} w(x) \ge 0.$$

Therefore,

$$w(x) \ge 0 \text{ for } x \in Q_T,$$

i.e.

$$v(x) \ge cE(x) \ge c_1 x_n^{\sigma(2-n)}.$$

Put in (3)

$$\varphi(x) = h(x_n)/v(x),$$

where $h(x_n) = 1$ for $1 < x_n < T$, $h(x_n) = 0$ for $2T < x_n$ and for $x_n < 1/2$, h is a smooth function for $x_n > 1$,

$$|h'(x_n)| \le c_1 T^{-1}, \quad |h''(x_n)| \le c_1 T^{-2}.$$

We see that

$$0 \ge J(T) + \int_{Q_T} h'(x_n)\frac{1}{v}\sum_{j=1}^n a_{nj}(x)\frac{\partial v(x)}{\partial x_j}dx + \int_{\Omega_1} \frac{1}{v}\sum_{j=1}^n a_{nj}(x)\frac{\partial v(x',1)}{\partial x_j}dx',$$

where

$$J(T) = \int_{Q_T} h(x_n)\frac{1}{v^2}\sum_{i,j=1}^n a_{ij}(x)\frac{\partial v(x)}{\partial x_j}\frac{\partial v(x)}{\partial x_i}dx + \int_{S_T} b(x)h(x_n)|v(x)|^{p-1}dS.$$

Using the inequality $|h'(x_n)|^2 \le c_1 h(x_n)T^{-2}$ for $x_n > 1$, we see that

$$|\int_{Q_T} h'(x_n)\frac{1}{v}\sum_{j=1}^n a_{nj}(x)\frac{\partial v(x)}{\partial x_j}dx|^2 \le C_1 T^{\sigma(n-1)-1}J(T).$$

Therefore,

$$J(T) \le -\int_\Omega \sum_{j=1}^n a_{nj}(x)\frac{\partial v(x',1)}{\partial x_j}dx' + C_2T^{\sigma(n-1)/2-1/2}\sqrt{J(T)}$$

$$= C_3 + C_4T^{\sigma(n-1)/2-1/2}\sqrt{J(T)} \tag{9}$$

and

$$J(T) \le C_5T^{\sigma(n-1)-1} \quad \text{for} \quad T \ge 1. \tag{10}$$

However,

$$J(T) \ge \int_{S_T} b(x)h(x_n)|v(x)|^{p-1}dS \ge C_6T^{-(p-1)\sigma(n-2)+1+\sigma(n-2)}, \tag{11}$$

$C_6 > 0$, and this inequality leads to a contradiction for large T if

$$1 < p < 1 + \frac{2-\sigma}{\sigma(n-2)}.$$

Therefore, in this case $v(x) \equiv 0$.

If

$$p = 1 + \frac{2-\sigma}{\sigma(n-2)},$$

then (10) implies that the integral

$$\int x_n^{1-\sigma(n-1)}|\nabla v|^2 v^{-2}dx$$

is converging and therefore,

$$T^{1-\sigma(n-1)}\int_{Q_T\cap \mathrm{supp}\ h'}|\nabla v|^2v^{-2}dx \to 0$$

for $T \to \infty$.

On the other hand, from (11) and (9) we have

$$0 < C_6 \le T^{(p-1)\sigma(n-2)-1-\sigma(n-2)}J(T) = T^{1-\sigma(n-1)}J(T)$$

$$\le C_7T^{1-\sigma(n-1)} + C_8\left(T^{1-\sigma(n-1)}\int_{Q_T\cap\ \mathrm{supp}\ h'}\frac{1}{v^2}(\sum_{j=1}^n a_{nj}(x)\frac{\partial v(x)}{\partial x_j})^2dx\right)^{1/2}.$$

Since the right hand side of the last inequality tends to 0 as $T \to \infty$ if $\sigma(n-1) > 1$, we obtain a contradiction, i.e. in this case also $v(x) \equiv 0$.

The proof is complete. □

THEOREM 2. *Let*

$$Q = \{x = (x', x_n) : |x'| < Ax_n^\sigma + B, 1 < x_n < \infty\}, 0 \le \sigma(n-1) < 1.$$

Suppose that $v(x)$ satisfies (1), (2) and $v(x) \ge 0$ in Q. If $p > 1$, then $v(x) \equiv 0$.

Proof. Let us show that $v(x) > c_1 > 0$ for $x_n > 1$ or $v(x) \equiv 0$. Indeed, let $c > 0$ be so small that $v(x) > cE(x)$ at $x_n = 1$, where E is the function found in Lemma 5. Put

$$z(x) = \max(cE(x) - v(x), 0).$$

Then $z(x) = 0, \nabla z(x) = 0$ at $x_n = 1$, $z(x)\partial z(x)/\partial N \le 0$ on S. Let h be a smooth function for $x_n > 1$, $h(x_n) = 1$ for $x_n < T$, $h(x_n) = 0$ for $x_n > 2T$. Therefore,

$$-\int_Q h(x_n)\sum_{j=1}^n a_{ij}(x)\frac{\partial z(x)}{\partial x_j}\frac{\partial z(x)}{\partial x_i}dx + \int_S h(x_n)z(x)\frac{\partial z(x)}{\partial N}dS$$

$$-\int_{Q_T} h'(x_n)z(x)\sum_{j=1}^{n} a_{nj}(x)\frac{\partial z(x)}{\partial x_j}dx \geq 0.$$

Therefore,

$$\int_Q h(x_n)\sum_{j=1}^{n} a_{ij}(x)\frac{\partial z(x)}{\partial x_j}\frac{\partial z(x)}{\partial x_i}dx \leq C\int_{Q\cap supph'} h'(x_n)^2 z(x)^2 dx$$

$$\leq C_1T^{-2}\int_{Q\cap supph'} E(x)^2dx \leq C_2\int_{Q\cap supph'} x_n^{-2}E(x)^2dx.$$

Since by Lemma 3, the integral $\int_Q x_n^{-2}E(x)^2dx$ is converging, the integral

$$\int_{Q\cap supph'} x_n^{-2}E(x)^2dx$$

tends to 0 as $x_n \to \infty$. In particular, it means that $v(x) > cE(x) > c_1 > 0$ for $x_n \geq 1$.

Let $h(x_n) = 1$ for $x_n < T-1$, $h(x_n) = 0$ for $T < x_n$, h be a smooth function for $x_n > 1$,

$$|h'(x_n)| \leq c_1, \quad |h''(x_n)| \leq c_1.$$

Put in the definition of weak solutions the function $\varphi(x) = h(x_n)/v(x)$. We have

$$0 \geq J(T) - \int_{Q_T} h'(x_n)\frac{1}{v}\sum_{j=1}^{n} a_{nj}(x)\frac{\partial v(x)}{\partial x_j}dx + \int_{\Omega_1}\frac{1}{v}\sum_{j=1}^{n} a_{nj}(x)\frac{\partial v(x',1)}{\partial x_j}dx',$$

where

$$J(T) = \int_{Q_T} h(x_n)\frac{1}{v^2}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial v(x)}{\partial x_j}\frac{\partial v(x)}{\partial x_i}dx + \int_{S_T} b(x)h(x_n)|v(x)|^{p-1}dS.$$

Using the inequality $|h'(x_n)|^2 \leq 2c_1h(x_n)$ for $x_n > 1$,we see that

$$|\int_{Q_T} h'(x_n)\frac{1}{v}\sum_{j=1}^{n} a_{nj}(x)\frac{\partial v(x)}{\partial x_j}dx|^2 \leq C_1J(T)T^{\sigma(n-1)/2}.$$

Therefore,

$$J(T) \leq \left|\int_\Omega \sum_{j=1}^{n} a_{nj}(x)\frac{\partial v(x',1)}{\partial x_j}dx'\right| + C_2\sqrt{J(T)}T^{\sigma(n-1)/2}$$

$$= C_3 + C_2\sqrt{J(T)}T^{\sigma(n-1)/2}.$$

and $J(T) \leq C_4T^{\sigma(n-1)}$. However,

$$J(T) \geq \int_{S_T} b(x)h(x_n)|v(x)|^{p-1}dS \geq C_5T^{1+\sigma(n-2)},\ C_5 > 0,$$

and the inequality $C_5T^{1+\sigma(n-2)} \leq C_4T^{\sigma(n-1)}$, following from the above inequality, is impossible for large T. Therefore, $v(x) \equiv 0$. □

THEOREM 3. *Let*

$$Q = \{x = (x', x_n)\ : |x'| < Ax_n^\sigma + B,\ 1 < x_n < \infty\},\ 0 \leq \sigma(n-1) \leq 1.$$

Suppose that $v(x)$ *satisfies* (1) *and* $v(x) \geq 0$ *in* Q. *Suppose that*

$$\sum_{j=1}^{n} a_{nj}(x)\frac{\partial v(x)}{\partial x_j} \leq 0 \quad \text{as} \quad x_n = 1$$

and $\partial v(x)\partial N \geq 0$ on S, $p > 0$. Then $v(x) \equiv 0$.

Proof. Let ε be a small positive number. Put in (3) $\varphi(x) = h(x_n)/(v(x)+\varepsilon)$, where $h(x_n) = 1$ for $1 < x_n < T$, $h(x_n) = 0$ for $2T < x_n$, h is a smooth function for $x_n > 1$. We have

$$J(T) \equiv \int_{Q_T} h(x_n)\frac{1}{(v+\varepsilon)^2}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial v(x)}{\partial x_j}\frac{\partial v(x)}{\partial x_i}dx + \int_{S_T} b(x)h(x_n)\frac{v(x)^p}{(v+\varepsilon)^2}dS$$

$$\leq \int_{Q_T} h'(x_n)\frac{1}{v+\varepsilon}\sum_{j=1}^{n} a_{nj}(x)\frac{\partial v(x)}{\partial x_j}dx,$$

i.e.

$$J(T) \leq C\int_{Q_T} T^{-2}dx \leq C_1 T^{-2+1+\sigma(n-1)}.$$

We see tending $T \to \infty$ that $v(x) \equiv 0$.

The proof is complete. □

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