ON STABILIZATION OF SOLUTIONS OF THE CAUCHY PROBLEM FOR PARABOLIC EQUATIONS ON THE NETS

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ABSTRACT. This note is devoted to study sufficient conditions for stabilization of the difference $\lim\limits_{t\to\infty} |u(x,t)-v(x,t)| = 0$, $x \in S$ — nets, where $u(x,t)$ is solution of the Cauchy problem for parabolic equation which is definite on $S \times [0,+\infty)$, S – nets in $\mathbf{E}^N$ and $v(x,t)$ is solution of the Cauchy problem with averaged constant matrix which is definite in all point $x \in \mathbf{E}^N$, $t \geq 0$.

1. Definitions and the statement of the problems. In the Euclidean space $\mathbf{E}^N$ ($N \geq 2$) we consider the net S, which is a union at all of the lines, parallel to coordinate axes, and knots of network $(n_1, n_2, \dots, n_N)$, $n_i \in \mathbf{Z}$, $(i = 1, \dots, N)$. From this it follows that S is a union of edges of a unit cubes $\square_i = \{i \leq x_k \leq i+1,\ i \in \mathbf{Z},\ k = 1, \dots, N\}$. Under μ we define the linear Lebesque measure on S, (that is μ is linear measure on the edges of cubes), with normalizing coefficient $1/N$.

In the half space $\{t \geq 0\} \equiv \{x \in S, t \geq 0\}$ we consider the Cauchy problem for parabolic equations of divergence form

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}(a(x)\nabla u), (x,t) \in \{t > 0\}, \\ u\Big|_{t=0} = \varphi(x), x \in S \end{cases} \tag{1}$$

where we assume that the real function $a(x)$ is defined on S for all $t > 0$ and is periodic on each variables with period 1 and satisfies the condition

$$\frac{1}{\lambda} \leq a(x) \leq \lambda, \quad \lambda > 0, \quad x \in \mathbf{E}^N. \tag{2}$$

The symbol ∇ desinate here the differential operator on the net S, which is coincident with $\partial/\partial x_i$ on lines x_i $(i = 1, \dots, N)$ parallel to axis x_i.

Also we assume that the initial function $\varphi(x)$ is definite on the net S and is bounded function on S.

The Cauchy problem (1) we understand in usual weak sense, that is in sense of integral identity:

$$\int\limits_0^{+\infty}\int\limits_S u\frac{\partial \eta}{\partial t}\,d\mu\,dt + \int\limits_S \varphi(x)\eta(x,0)\,d\mu = \int\limits_0^{+\infty}\int\limits_S (a\nabla u, \nabla \eta)\,d\mu\,dt, \tag{3}$$

for all functions $\eta(x,t) \in C_0^\infty(\{t > 0\})$ where function $u(x,t)$ is definite on $\{t > 0\}$ and belonging to $L^2\{S \times [0,T], d\mu \cdot dt\}$ $\forall T > 0$, and $\nabla u(x,t) \in L^2\{S \times [0,T], d\mu \cdot dt\}$ $\forall T > 0$.

Key words and phrases. Heat equation, Parabolic equation, Cauchy problem, Stabilization, Nets, Knots, Average theory.

This work was supported by Russian Grant no. 99-01-00072.

The solutions of the problem (1) we takes from class of uniqueness, that is solutions is bounded in each strip $\{0 < t \le T\} \equiv \{S \times (0, T]\}$.

2. Example. If $a(x) = 1$, $N = 2$, then the problem (1) we can interpret in the following equivalence sense

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u, \quad (x,t) \in \{t > 0\}, \\ u\Big|_{t=0} = \varphi(x), \quad x \in S, \end{cases} \tag{1'}$$

where we assume that

1) S is the usual square net on the plane $\mathbf{E}^2$ with natural linear measure $\mu[i \le x_1 \le i+1, j \le x_2 \le j+1]$ on edges of square;

2) $u(x,t)$ is continuous function on net S together with knots (n_1, n_2), $n_i \in \mathbf{Z}$;

3) on horizontal and vertical units function $u(x,t)$ have first and second derivative, with is square integrable.

4) derivative du/dx_1 (on horizontal units), du/dx_2 (on vertical units) they can have discontinuity on nodes of network, but a jump of derivative du/dx_1 + jump of derivative $du/dx_2 = 0$ in each nodes of network.

Under this conditions we have, that

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u \equiv \begin{cases} \dfrac{d^2u}{dx_1^2} \text{ on horizontal unit}, \ t > 0, \\ \dfrac{d^2u}{dx_2^2} \text{ on vertical unit}, \ t > 0, \end{cases} \\ u\Big|_t = \varphi(x), \quad x \in S, \end{cases} \tag{1'}$$

$\varphi(x)$ is bounded initial function on net S.

For this definition of Laplace operator Δ on net S see [1].

Together with problem (1) we consider usual Cauchy problem

$$\begin{cases} \dfrac{\partial u^0}{\partial t} = L^0 u^0, \quad (x,t) : x \in \mathbf{E}^N, \ t > 0 \\ u^0\Big|_{t=0} = \widetilde{\varphi}(x), \quad x \in \mathbf{E}^N, \end{cases} \tag{4}$$

where $L^0 = \sum\limits_{i,j=1}^{N} a_{ij}^0 \partial^2/(\partial x_i \partial x_j)$, $\|a_{ij}^0\|_{N \times N}$ — so called averaged matrix with constant coefficient [2], $\widetilde{\varphi}(x)$ — is bounded initial function on $\mathbf{E}^N$.

The averaged matrix $a^0 = \|a_{ij}^0\|_{N\times N}$ is also simmetric and satisfies the elliptic conditions

$$\frac{1}{\lambda}|\xi|^2 \le \sum_{i,j=1}^{N} a_{ij}^0 \xi_i \xi_j \le \lambda |\xi|^2, \quad \lambda > 0.$$

The initial function $\widetilde{\varphi}(x)$ in (4) is fulfillment of initial function $\varphi(x)$ in (1) on S.

We assume that fulfilment function $\widetilde{\varphi}(x)$ $x \in \mathbf{E}^N$, is bounded and satisfies conditions

$$\int\limits_{\Box^i} \varphi(x)\, d\mu(x) = \int\limits_{\Box^i} \widetilde{\varphi}(x)\, dx \tag{5}$$

on each cell $\Box^i = \{i \le x_k \le i+1;\ k = 1, \ldots, N,\ i \in \mathbf{Z}\}$.

We have the following assertions

Theorem 1. *The solutions of the Cauchy problems* (1), (4) *satisfies the following property: exist the limit of difference*

$$\lim_{t\to\infty}(u(x,t)-u^0(x,t))=0, \tag{6}$$

on each $x\in S$.

From this closeness theorem we can to obtain the criterium for stabilizations of the solutions of the Cauchy problem (1).

$$\lim_{t\to\infty}u(x,t)=0,\quad x\in S$$

from well known pointwize criterium of stabilization of the solutions of the Cauchy problem (4).

$$\lim_{t\to\infty}u^0(x,t)=0,\quad x\in S\subset \mathbf{E}^N$$

(see [3]–[6]).

Theorem 2. *If the fulfilment function* $\tilde{\varphi}(x)$ *in (4) is connected with initial function* $\varphi(x)$ *in* (1) *by conditions* (5), *then the solutions* $u(x,t)$ *of the Cauchy problem* (1) *stabilizes on* S

$$\lim_{t\to\infty}u(x,t)=0,\quad x\in S$$

if and only if the following limit of ellipcoidal averaged value of initial function $\varphi(x)$ *exist*

$$\lim_{R\to\infty}\frac{1}{\gamma_N\mathbf{R}^N}\int\limits_{(By,y)\le\mathbf{R}^2}\varphi(y)\,d(\mu)=0,$$

where B — *in inverse matrix for averaged matrix* a^0, γ_N — *is volume of the inut ellipsoid in* $\mathbf{E}^N$.

3. Outline of proofs. For fixed $\varepsilon>0$ we consider the compressed net S_ε with variables x/ε, $x\in S$ and definite the Cauchy problem for parabolic equation (1)

$$\frac{\partial u^\varepsilon}{\partial t}=\operatorname{div}(a^\varepsilon(x)\nabla u^\varepsilon), \tag{1_ε}$$

with initial function

$$u^\varepsilon(x,0)=f^\varepsilon(x),\quad f(x)\in C_0^\infty(\mathbf{E}^N)$$

Applying the real Laplace transform to solution of the problem (1_ε) on variable $t>0$, we obtain following problem in $W^{1,2}(\mathbf{E}^N,d\mu^\varepsilon)$

$$-\operatorname{div}(a^\varepsilon(x)w^\varepsilon)+pw^\varepsilon=f^\varepsilon,\quad p>0, \tag{7}$$

where $W^{1,2}(\mathbf{E}^N,d\mu^\varepsilon)$ is a closure of functions $w\in C_0^\infty(\mathbf{E}^N)$ in the norm

$$||w||_{W^{1,2}(\mathbf{R}^N,d\mu^\varepsilon)}=\left[\int\limits_{\mathbf{E}^N}(|w^\varepsilon|^2+|\nabla w^\varepsilon|^2)\,d\mu^\varepsilon\right]^{1/2}$$

and $w^\varepsilon(x,\rho)=\int\limits_0^\infty e^{-pt}u^\varepsilon(x,t)\,dt$, $p>0$ the Laplace transform of function $u^\varepsilon(x,t)$. Applying the well-known average theorem 6.3 from [1], we obtain, that the solution w^ε of the problem (7) satisfies the following limit relating: for any $\eta\in C_0^\infty(\mathbf{E}^N)$ the limit exists

$$\lim_{\varepsilon\to0}\int\limits_{\mathbf{E}^N}\eta(x)w^\varepsilon(x,p)\,d\mu^\varepsilon=\int\limits_{\mathbf{E}^N}\eta(x)w^0(x,p)\,dx \tag{8}$$

$$\lim_{\varepsilon\to 0+}\int\limits_{\mathbf{E}^N}[w^\varepsilon(x,p)]^2\,d\mu^\varepsilon=\int\limits_{\mathbf{E}^N}[w^0(x,p)]^2\,dx \tag{9}$$

where w^0 is the solution of the average problem in $W^{1,2}(\mathbf{E}^N,dx)$

$$-\mathrm{div}\,(a^0\nabla w^0)+pw^0=f^0, \tag{10}$$

where a^0 is average constant matrix, $f^0\in C_0^\infty(\mathbf{E}^N)$.

After that we can to apply the well-known Trotter-Kato theorem [7], which imply that the following limit exist

$$\lim_{\varepsilon\to 0}\int\limits_{\mathbf{E}^N}\eta(x)u^\varepsilon(x,t)\,d\mu^\varepsilon=\int\limits_{\mathbf{E}^N}\eta(x)v^0(x,t)\,dx,\quad \forall\eta\in C_0^\infty(\mathbf{E}^N) \tag{11}$$

$$\lim_{\varepsilon\to 0+}\int\limits_{\mathbf{E}^N}[u^\varepsilon(x,t)]^2\,d\mu^\varepsilon=\int\limits_{\mathbf{E}^N}[v^0(x,t)]^2\,dx,\quad \forall\eta\in C_0^\infty(\mathbf{E}^N) \tag{12}$$

for any fixed $t>0$, where v^0 is the solution of the Cauchy problem (4) with average matrix a^0 and initial function $f^0\in C_0^\infty(\mathbf{E}^N)$.

This is main property whish we can obtain from standard average theory. But in order to prove theorem 1 we must to bring some refinements in average theory. We have following result

THEOREM 3. *If initial function f in the Cauchy problem (1_ε) satisfies limit condition $f^\varepsilon\in L^\infty(\mathbf{E}^N,d\mu^\varepsilon)$ and the following limit exist*

$$\lim_{\varepsilon\to 0}\int\limits_{\mathbf{E}^N}f^\varepsilon(x)\eta(x)\,d\mu^\varepsilon=\int\limits_{\mathbf{E}^N}f^0(x)\eta(x)\,dx \tag{13}$$

for any $\eta(x)\in C_0^\infty(\mathbf{E}^N)$, then the following limit exist

$$\lim_{\varepsilon\to 0}\int\limits_{\mathbf{E}^N}u^\varepsilon(x,t)\eta(x)\,d\mu^\varepsilon=\int\limits_{\mathbf{E}^N}u^0(x,t)\eta(x)\,dx,\quad t>0 \tag{14}$$

for any $\eta(x)\in C_0^\infty(\mathbf{E}^N)$, where $u^0(x,t)$ is the solutions of the Cauchy problem (4), $t>0$, with initial function $u^0(x,0)=f^0(x)$:

$$\frac{\partial u^0}{\partial t}=L^0u^0,\quad u^0\Big|_{t=0}=f^0(x). \tag{15}$$

Proof of the theorem 3. Let us assume that limit conditions (13) holds for any function $\eta(x)\in C_0^\infty(\mathbf{E}^N)$. From hypothesis $|f^\varepsilon(x)|<M$ it follows that exists subsequence $\{f^\varepsilon\}$, which is weakly convergence to f^0 ($f^\varepsilon\xrightarrow{\varepsilon\to 0}f^0$ weakly in $L^2(\mathbf{E}^N,d\mu^\varepsilon)$).

Now by applaying Green formula for solutions of the Cauchy problem

$$\frac{\partial u^\varepsilon}{\partial t}=Lu^\varepsilon,\quad u^\varepsilon\Big|_{t=0}=f^\varepsilon(x) \tag{16}$$

and

$$\frac{\partial v^\varepsilon}{\partial t}=L^0v^\varepsilon,\quad v^\varepsilon\Big|_{t=0}=\eta(x), \tag{17}$$

where $\eta(x)\in C_0^\infty(\mathbf{E}^N)$, we have, that following equality

$$\int\limits_{\mathbf{E}^N}u^\varepsilon(x,t_0)\eta(x)\,d\mu^\varepsilon=\int\limits_{\mathbf{E}^N}v^\varepsilon(x,t_0)f^\varepsilon(x)\,d\mu^\varepsilon,\quad t_0>0 \tag{18}$$

holds. Passing $\varepsilon \to 0$ in the left of (18) we have

$$\lim_{\varepsilon\to 0}\int\limits_{\mathbf{E}^N} u^\varepsilon(x,t_0)\eta(x)\,d\mu^\varepsilon = \int\limits_{\mathbf{E}^N} u^*(x,t_0)\eta(x)\,dx, \tag{19}$$

where $u^*(x,t_0)$ — some limit point (in weak seance) of sequence $u^\varepsilon(x,t_0)$. Applying the Trotter–Kato theorem [7] in right of (16) we have that following limit exist

$$\lim_{\varepsilon\to 0+}\int\limits_{\mathbf{E}^N} v^\varepsilon(x,t_0)f^\varepsilon(x)\,d\mu^\varepsilon = \int\limits_{\mathbf{E}^N} v^0(x,t_0)f^0(x)\,dx, \tag{20}$$

where $v^0(x,t)$ is the solution of the average Cauchy problem (17). From (19), (20) it follows that for any $\eta(x)\in C_0^\infty(\mathbf{E}^N)$

$$\int\limits_{\mathbf{E}^N} u^*(x,t_0)\eta(x)\,dx = \int\limits_{\mathbf{E}^N} v^0(x,t_0)f^0(x)\,dx. \tag{21}$$

Applying Green formula in the right side of (21), we have

$$\int\limits_{\mathbf{E}^N} u^*(x,t_0)\eta(x)\,dx = \int\limits_{\mathbf{E}^N} u^0(x,t_0)\eta(x)\,dx, \tag{22}$$

for any $\eta(x)\in C_0^\infty(\mathbf{E}^N)$, $t_0>0$. From last equality it is easy to see that

$$u^*(x,t_0) = u^0(x,t_0), \quad t_0>0$$

where $u^0(x,t_0)$ is the solutions of the Cauchy problem (15).

Theorem 3 is proved.

The following statement play very important role in the proof of theorem 1.

LEMMA 1. *If initial fulfillment function $\tilde\varphi(x)$ in the Cauchy problem* (4) *and initial function $\varphi(x)$ in the Cauchy problem* (1) *satisfies property* (5), *than limit exist*

$$\lim_{\varepsilon\to 0}\int\limits_{\mathbf{E}^N} \eta(x)\varphi^\varepsilon(x)\,d\mu^\varepsilon = \int\limits_{\mathbf{E}^N} \eta(x)\varphi^0(x)\,dx \tag{23}$$

if and only if the following limit exist

$$\lim_{\varepsilon\to 0}\int\limits_{\mathbf{E}^N} \eta(x)\tilde\varphi^\varepsilon(x)\,dx = \int\limits_{\mathbf{E}^N} \eta(x)\varphi^0(x)\,dx \tag{24}$$

for any $\eta(x)\in C_0^\infty(\mathbf{R}^N)$.

The proof of lemma 1 is straighforward and is left to reader. For proof theorem 1 we consider two Cauchy problem

$$\frac{\partial u^\varepsilon}{\partial t} = Lu^\varepsilon, \quad u^\varepsilon\Big|_{t=0} = \varphi^\varepsilon(x), \tag{25}$$

$$\frac{\partial v^\varepsilon}{\partial t} = L^0 v^\varepsilon, \quad v^\varepsilon\Big|_{t=0} = \tilde\varphi^\varepsilon(x), \tag{26}$$

where $\tilde\varphi^\varepsilon(x)$ is some fulfilment of initial function $\varphi(x)$, and conditions (5) are holds. Now we put $\varepsilon = \frac{1}{\sqrt{t}}$, $t>0$. From condition (5) and lemma 1 it follows that the sequances $\{\varphi^\varepsilon(x)\}$ and $\{\tilde\varphi^\varepsilon(x)\}$ have the same weak limit:

$\varphi^\varepsilon \xrightarrow{\varepsilon\to 0} \varphi^0$ weakly in $L^2(\mathbf{E}^N, d\mu^\varepsilon)$,

$\tilde\varphi^\varepsilon \xrightarrow{\varepsilon\to 0} \varphi^0$ weakly in $L^2(\mathbf{E}^N, d\mu^\varepsilon)$.

It is known [8] that solution $\{u^\varepsilon(x,t)\}$ of the Cauchy problem satisfies uniform Holder conditions, with constant which does not depend on ε. From this condition and theorem 3 it follows that following limit

$$\lim_{t\to\infty} u^{\frac{1}{\sqrt{t}}}(0,1) = u^0(0,1) \tag{27}$$

exist. Now we must to apply Poisson formula for solution of the Cauchy problem (26) with constant coefficient, i. e.

$$v^\varepsilon(0,1) = \int_{\mathbf{E}^N} K_0(x,0,1)\tilde{\varphi}(\varepsilon^{-1}x)\,dx$$

where $K_0(x,y,t)$ is fundamental solutions of (28). Passing $\varepsilon \to 0$ we have

$$\lim_{t\to\infty} v^{\frac{1}{\sqrt{t}}}(0,1) = u^0(0,1). \tag{28}$$

From (27), (28) it follows that theorem 1 is proved.

Proof of the theorem 1 is omitted, and it follows straightforward from theorem 1 and well known criterium of stabilization of the solution of the Cauchy problem for heat equation [3].

After this proof the theorem 2 may be made very easy as in the book [2].

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