# ON ONE CLASS OF NONLINEAR OPERATOR EQUATIONS IN HILBERT SPACES

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Abstract. One special class of nonlinear operator equations depended from two real numerical parameters in Hilbert spaces are considered. Non-local conditions of existence and uniqueness of solutions are established. The conditions of admissibility of of solutions of linearized equation for description of solutions of considered equation are established.

Let $\mathbf{H}$ be some Hilbert space with the scalar product $\langle , \rangle$ and zero element $\theta$. Let linear operator $D$ maps $\mathbf{H}$ to some Banach space $B$. Let operator's $D$ domain of definition is everywhere dense in $\mathbf{H}$. Let $\mathcal{H} \subset \mathbf{H}$ be a closure of $Ker(D)$ in $\mathbf{H}$ at the norm of $\mathbf{H}$. Let operator $D$ can be expanded to the closed operator $\widetilde{D}$, which kernel is $\mathcal{H}$. Operator $\widetilde{D}^* : B^* \to \mathbf{H}$ is designated as $\mathcal{G}$. Then exists some linear bounded closed operator $\mathcal{R} : \mathbf{H} \overset{on}{\to} \mathcal{H}$ and $Ker\,(\mathcal{R}^*) = Im\,(\mathcal{G})$. Operator $\mathcal{R}$ can be constructed not by the unique method. In the simplest case operator $\mathcal{R}$ be a projector of $\mathbf{H}$ on $\mathcal{H}$.

Theorem 1. $\mathbf{H} = \mathcal{H} \oplus Im\,(\mathcal{G})$.

*Proof.* Sets $\mathcal{H}$ and $Im\,(\mathcal{G})$ are closed by the it construction. So it are subspaces of $\mathbf{H}$. At first will be proven, that $\mathcal{H}$ and $Im\,(\mathcal{G})$ are orthogonal. Let $\mathbf{h}$ be an arbitrary element of $\mathcal{H}$. Then $\widetilde{D}\mathbf{h} = \theta \in B$. From here follows, that for any $g \in \mathcal{B}^*$:

$$0 = g\left(\widetilde{D}\mathbf{h}\right) = \left\langle \mathbf{h},\ \widetilde{D}^* g\right\rangle = \langle \mathbf{h},\ \mathcal{G}g\rangle .$$

This means that $\mathcal{H}$ and $Im\,(\mathcal{G})$ are orthogonal.

Let $g \in \mathbf{H}$ and $g \notin \mathcal{H}$. Then for any $\mathbf{h} \in \mathcal{H}$ exists element $\mathbf{h}_R \in \mathbf{H}$ such that $\mathbf{h} = \mathcal{R}\mathbf{h}_R$. Than

$$0 = \langle g,\ \mathbf{h}\rangle = \langle g,\ \mathcal{R}\mathbf{h}_R\rangle = \langle \mathcal{R}^* g,\ \mathbf{h}_R\rangle .$$

Because $\mathbf{h}$ be an arbitrary element of $\mathcal{H}$ from here follow that $\mathcal{R}^* g = \theta$. So $g \in Ker\,(\mathcal{R}^*) = Im\,(\mathcal{G})$. Since $\mathbf{H} = \mathcal{H} \oplus Im\,(\mathcal{G})$. Theorem is proven.

From this theorem and properties of $\mathcal{R}$ follows, that on $\mathcal{H}$ operator $\mathcal{R}^*$ has bounded inverse $(\mathcal{R}^*)^{-1}$. By means of $\mathcal{R}^*$ a Hilbert structure in $\mathcal{H}$ can be constructed. A scalar product in $\mathcal{H}$ can be defined by the next way: for every $\mathbf{h}_1,\ \mathbf{h}_2 \in \mathcal{H}$

$$[\mathbf{h}_1,\ \mathbf{h}_2] = \langle \mathcal{R}^*\mathbf{h}_1,\ \mathcal{R}^*\mathbf{h}_2\rangle .$$

In this case all axioms of the scalar product are executed. Operator $\mathcal{R}$ may be not unique. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ are two different constructions of operator $\mathcal{R}$. Then it generate in $\mathcal{H}$ equivalent norms. Really, on subspace $\mathcal{H}$ operators $\mathcal{R}_1^*$, $\mathcal{R}_2^*$ have bounded inverse

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$(\mathcal{R}_1^*)^{-1}$, $(\mathcal{R}_2^*)^{-1}$. Then

$$\|\mathcal{R}_1^*\mathbf{h}\|_{\mathbf{H}}^2 = \langle \mathbf{h},\ \mathcal{R}_1\mathcal{R}_1^*\mathbf{h}\rangle = \left\langle (\mathcal{R}_2^*)^{-1}\mathcal{R}_2^*\mathbf{h},\ \mathcal{R}_1\mathcal{R}_1^*(\mathcal{R}_2^*)^{-1}\mathcal{R}_2^*\mathbf{h}\right\rangle \le$$
$$\le \left\|(\mathcal{R}_2^*)^{-1}\mathcal{R}_2^*\mathbf{h}\right\|_{\mathbf{H}} \left\|\mathcal{R}_1\mathcal{R}_1^*(\mathcal{R}_2^*)^{-1}\mathcal{R}_2^*\mathbf{h}\right\|_{\mathbf{H}} \le$$
$$\le \left\|(\mathcal{R}_2^*)^{-1}\right\| \left\|\mathcal{R}_1\mathcal{R}_1^*(\mathcal{R}_2^*)^{-1}\right\| \|\mathcal{R}_2^*\mathbf{h}\|_{\mathbf{H}}^2 = C_1^2 \|\mathcal{R}_2^*\mathbf{h}\|_{\mathbf{H}}^2 .$$

By the analogues with this estimate can be proven estimate

$$\|\mathcal{R}_2^*\mathbf{h}\|_{\mathbf{H}}^2 \le C_2^2 \|\mathcal{R}_1^*\mathbf{h}\|_{\mathbf{H}}^2 .$$

From this estimates follows equivalence of the norms, generated by means of operators $\mathcal{R}_1^*$, $\mathcal{R}_2^*$.

**1. Multiplicative inequalities.** Let $\mathbf{H}$ be a direct sum of some it subspaces $\mathbf{H}^{(1)}$, $\mathbf{H}^{(2)}$ which don't coincide with $\mathcal{H}$. Later on $D_i$ are restrictions of $\widetilde{D}$ on $\mathbf{H}^{(i)}$ $(i = 1,2)$. Then for each $\mathcal{H} \ni \mathbf{h} = \mathbf{h}_1 + \mathbf{h}_2$ $\left(\mathbf{h}_i \in \mathbf{H}^{(i)}\right)$ elements $\mathbf{h}_1,\ \mathbf{h}_2$ are connected by following expression: $D_1\mathbf{h}_1 = -D_2\mathbf{h}_2$. From this expression follows that elements $\mathbf{h}_i \notin \mathcal{H}$ $(i = 1,2)$ can be expressed through the $\mathbf{h}_j$ $(i \ne j = 1,2)$ : $\mathbf{h}_i = D_{ij}\mathbf{h}_j$, where $D_{ij} : \mathbf{H}^{(i)} \to \mathbf{H}^{(j)}$ are some linear operators, which coinside with the unit operators on $\mathbf{H}^{(j)}$. Operators $D_{ij}$ have non trivial kernels (if $\mathbf{h}_i = \theta$, then $\mathbf{h}_j \in Ker(D_{ij})$). Let $P_i$ are projectors of $D_{ij}\mathbf{H}$ on $Ker(D_i)$, $Q_i = I - P_i$. Then $\widetilde{D}_{ij} = Q_iD_{ij}$ are bounded operators and

$$\|\mathbf{h}_i\|_{\mathbf{H}}^2 = \left|\left\langle \mathbf{h}_i,\ \widetilde{D}_{ij}\mathbf{h}_j\right\rangle\right| = \left|\left\langle \widetilde{D}_{ij}^*\mathbf{h}_i,\ \mathbf{h}_j\right\rangle\right| \le \|\mathbf{h}_j\|_{\mathbf{H}} \left\|\widetilde{D}_{ij}^*\mathbf{h}_i\right\|_{\mathbf{H}} \le C^2 \|\mathbf{h}_j\|_{\mathbf{H}} \|\mathbf{h}\|_{\mathcal{H}} ,$$
$$C^2 = C_H \max_{ij=1,2} \left\|\widetilde{D}_{ij}^*\right\| ,$$

where $C_H$ be an embedding constant of $\mathcal{H}$ into $\mathbf{H}$. From here follows inequality

$$\|\mathbf{h}_i\|_{\mathbf{H}} \le C \|\mathbf{h}_j\|_{\mathbf{H}}^{1/2} \|\mathbf{h}\|_{\mathcal{H}}^{1/2} . \qquad (1)$$

Let $\mathbf{H} \supseteq E_\alpha$, $0 \le \alpha \le 1$ be a scale of Banach spaces, and $E_0 = \mathbf{H}$. Let, at last, a Hilbert space $\mathcal{H}$ is contained in all $E_\alpha$. From the properties of the Banach spaces scale follows multiplicative inequality

$$\|\mathbf{h}_i\|_{E_\alpha} \le \beta_\alpha \|\mathbf{h}_i\|_{\mathbf{H}}^{1-\alpha} \|\mathbf{h}\|_{\mathcal{H}}^{\alpha}, \quad \alpha \ge 0; \quad i,j = 1,2. \qquad (2a)$$

From (1) and (2a) for elements of $\mathbf{H} \ni \mathbf{h} = \mathbf{h}_1 + \mathbf{h}_2$, $\mathbf{h}_i \in \mathbf{H}^{(i)}$ follows next analogues of multiplicative inequality:

$$\|\mathbf{h}_i\|_{E_\alpha} \le \eta_\alpha \|\mathbf{h}_j\|_{\mathbf{H}}^{(1-\alpha)/2} \|\mathbf{h}\|_{\mathcal{H}}^{(1+\alpha)/2}, \quad i,j = 1,2, \quad i \ne j. \qquad (2b)$$

If $\alpha = 0$ index $_\alpha$ in all formulas will be missed.

**2. Operator equation. Existence of solutions.** Let linear operators $A, C, T$ and bilinear operator $B(\cdot,\cdot)$ map spaces $E_\alpha, \mathbf{H}$ into itself and $B, C, T$ are completely continuous in the topologies of these spaces. Later on will be supposed that operators $A, B, C, T$ satisfy to next conditions: for any $\mathbf{v}, \mathbf{w} \in \mathbf{H}$

$$\begin{gathered} \langle \mathbf{w},\ A\mathbf{w}\rangle \ge m_A \|\mathbf{w}\|_{\mathcal{H}}^2; \\ \langle \mathbf{w},\ B(\mathbf{v},\mathbf{w})\rangle \le 0; \\ \langle \mathbf{w},\ C\mathbf{w}\rangle_{\mathbf{H}} \le M_C \|\mathbf{w}\|_{\mathcal{H}} \|\mathbf{w}\|_{E_\alpha},\ 0 < \alpha \le 1/2; \\ \langle \mathbf{w},\ T\mathbf{w}\rangle \ge m_T \|w_1\|_{\mathbf{H}}^2. \end{gathered} \qquad (*)$$

Let consider operator equation

$$A\mathbf{w} = \lambda_1 \left[B(\mathbf{w},\mathbf{w}) + C\mathbf{w}\right] - \lambda_2^2 T\mathbf{w} + \mathbf{f}(\lambda_1,\lambda_2), \qquad (3)$$

where $\lambda_1 \ge 0$, $\lambda_2 > 1$ are real numerical parameters, $\mathbf{f}(\lambda_1,\lambda_2) \in \mathbf{H}$ and

$$\|\mathbf{f}(\lambda_1,\lambda_2)\|_{\mathbf{H}} \le C_1(\lambda_1) + C_2(\lambda_1)\sqrt{\lambda_2} + C_3(\lambda_1)\lambda_2,$$

where $C_i(\lambda_1)$, $i = 1,2,3$ are some bounded at $0 \le \lambda_1 < \infty$ functions. Later on dependence from parameter $\lambda_1$ in different functions will be omitted. Non local in the sense of values of parameters $\lambda_1, \lambda_2$ conditions of existence of the solution to equation (3) can be established for any $0 < \alpha < 1$.

THEOREM 2. *Let operators $A, B, C, T$ map the spaces $E_\alpha$, $\alpha > 0$ and $\mathbf{H}$ into itself and $B, C, T$ are completely continuous in the topologies of these spaces. Let for any elements $\mathbf{H} \ni \mathbf{v}, \mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, $\mathbf{u}_i \in \mathbf{H}_i$, conditions (*) are realized. Let values of parameter $\lambda_2 > \lambda_{20}(\lambda_1)$ where*

$$\lambda_{20} = \left[\frac{M_C}{3m_T}(1-\alpha)\lambda_1\right]^{1/2} \left\{2\beta_\alpha \left[m_A^{-1}\lambda_1 M_C \beta_\alpha (1+\alpha)\right]^{(1+\alpha)/(1-\alpha)} + \right. \\ \left. + \eta_\alpha \left[m_A^{-1}\lambda_1 M_C \eta_\alpha (3+\alpha)\right]^{(3+\alpha)/(1-\alpha)}\right\}^{1/2}. \tag{4}$$

*Then equation (3) has at least one solution in $\mathbf{H}$, for which a priori estimates*

$$\|\mathbf{w}\|_{\mathcal{H}} \le 4m_A^{-1}\|\mathbf{f}(\lambda_1,\lambda_2)\|_{\mathbf{H}} \le C_{1w} + C_{2w}\sqrt{\lambda_2} + C_{3w}\lambda_2, \\ \|\mathbf{w}_1\|_{\mathbf{H}} \le 4m_A m_T^{-1/2}\|\mathbf{f}(\lambda_1,\lambda_2)\|\lambda_2^{-1} \le C_{11}/\lambda_2 + C_{11}/\sqrt{\lambda_2} + C_{12}. \tag{5}$$

*are valid.*

*Proof.* Both parts of equation (3) are scalar multiplied in the space $\mathbf{H}$ on the element $\mathbf{w}$. Then, by virtue of conditions of theorem, next "energy" inequality is valid

$$m_A\|\mathbf{w}\|_{\mathcal{H}}^2 + m_T\lambda_2^2\|\mathbf{w}_1\|_{\mathbf{H}}^2 \le \lambda_1 M_C \|\mathbf{w}\|_{\mathcal{H}}\|\mathbf{w}\|_{E_\alpha} + \|\mathbf{f}(\lambda_1,\lambda_2)\|\|\mathbf{w}\|_{\mathbf{H}}. \tag{6}$$

In this inequality the product $\|\mathbf{w}\|_{\mathcal{H}}\|\mathbf{w}\|_{E_\alpha}$ is estimated by means of multiplicative and Jung's inequalities:

$$\|\mathbf{w}\|_{\mathcal{H}}\|\mathbf{w}\|_{E_\alpha} = \left(\|\mathbf{w}_1\|_{E_\alpha} + \|\mathbf{w}_2\|_{E_\alpha}\right)\|\mathbf{w}\|_{\mathcal{H}} \le \\ \le \|\mathbf{w}_1\|_{E_\alpha}\|\mathbf{w}\|_{\mathbf{H}} + \eta_\alpha\|\mathbf{w}_1\|_{\mathcal{H}}^{(1-\alpha)/2}\|\mathbf{w}\|_{\mathcal{H}}^{(3+\alpha)/2} \le \\ \le \beta_\alpha\|\mathbf{w}_1\|_{\mathbf{H}}^{1-\alpha}\|\mathbf{w}\|_{\mathcal{H}}^{1+\alpha} + \eta_\alpha\|\mathbf{w}_1\|_{\mathbf{H}}^{(1-\alpha)/2}\|\mathbf{w}\|_{\mathcal{H}}^{(3+\alpha)/2} \le \\ \le \tfrac{1}{2}\beta_\alpha\left[(1+\alpha)\varepsilon_1^{2/(1+\alpha)}\|\mathbf{w}\|_{\mathcal{H}}^2 + (1-\alpha)\varepsilon_1^{-2/(1-\alpha)}\|\mathbf{w}_1\|_{\mathbf{H}}^2\right] + \\ + \tfrac{1}{4}\eta_\alpha\left[(3+\alpha)\varepsilon_2^{4/(3+\alpha)}\|\mathbf{w}\|_{\mathcal{H}}^2 + (1-\alpha)\varepsilon_2^{-4/(1-\alpha)}\|\mathbf{w}_1\|_{\mathbf{H}}^2\right].$$

This estimate is substituted into the inequality (6):

$$\left\{m_A - \lambda_1\frac{M_C}{4}\left[2(1+\alpha)\beta_\alpha\varepsilon_1^{2/(1+\alpha)} + \eta_\alpha(3+\alpha)\varepsilon_2^{4/(3+\alpha)}\right]\right\}\|\mathbf{w}\|_{\mathcal{H}}^2 + \\ + \lambda_2^2\left\{m_T - \lambda_1\lambda_2^{-2}\frac{M_C}{4}(1-\alpha)\left[2\beta_\alpha\varepsilon_1^{-2/(1-\alpha)} + \eta_\alpha\varepsilon_2^{-4/(1-\alpha)}\right]\right\}\|\mathbf{w}_1\|_{\mathbf{H}}^2 \le \|\mathbf{f}(\lambda_1,\lambda_2)\|\|\mathbf{w}\|_{\mathbf{H}}.$$

Values of $\varepsilon_1, \varepsilon_2$ in obtained inequality are chosen from the condition of the positiveness of the multiplier at $\|\mathbf{w}\|_{\mathcal{H}}^2$. For example:

$$\varepsilon_1 = \left[\lambda_1(1+\alpha)\beta_\alpha\frac{M_C}{m_A}\right]^{-(1+\alpha)/2}, \qquad \varepsilon_2 = \left[\lambda_1(3+\alpha)\eta_\alpha\frac{M_C}{m_A}\right]^{-(3+\alpha)/4}.$$

Value of parameter $\lambda_2$ is chosen from the condition of positiveness of multiplier at $\|\mathbf{w}_1\|_{\mathbf{H}}^2$ and the chosen above values of $\varepsilon_i$. Then from the last inequality and embedding of $\mathcal{H}$ into $\mathbf{H}$ follows estimates (5):

$$m_A\|\mathbf{w}\|_{\mathcal{H}}^2 + \lambda_2^2 m_T\|w_1\|_{\mathbf{H}}^2 \le 4\|\mathbf{f}(\lambda_1,\lambda_2)\|\|\mathbf{w}\|_{\mathbf{H}}; \\ \|\mathbf{w}\|_{\mathcal{H}} \le 4m_A^{-1}\|\mathbf{f}(\lambda_1,\lambda_2)\|, \quad \lambda_2^2 m_T\|w_1\|_{\mathbf{H}}^2 \le 16 m_A^{-2}\|\mathbf{f}(\lambda_1,\lambda_2)\|^2.$$

For the completion of the proof of the theorem remains to notice that from the generalized Leray-Schauder principle [1] and estimate (5) follows, that the equation has solutions in $\mathbf{H}$.

From the estimates (5) and multiplicative inequalities follow estimates to components of solutions of the equation (3)

$$\begin{gathered}
\|\mathbf{w}_1\|_{E_\alpha} \le \beta_\alpha \|\mathbf{w}_1\|_{\mathbf{H}}^{1-\alpha} \|\mathbf{w}\|_{\mathcal{H}}^{\alpha} \le \\
\le \beta_\alpha \left[C_{1w} + C_{2w}\sqrt{\lambda_2} + C_{3w}\lambda_2\right]^{\alpha} \left[C_{11}/\lambda_2 + C_{12}/\sqrt{\lambda_2} + C_{13}\right]^{1-\alpha} \quad \le \\
= M_1 \lambda_2^{\alpha} + \varphi_1(\lambda_2), \quad |\varphi_1(\lambda_2)| \le M_{\varphi 1} \lambda_2^{1/2-\alpha}. \\
\|\mathbf{w}_2\|_{E_\alpha} \le \eta_\alpha \|\mathbf{w}_1\|_{\mathbf{H}}^{(1-\alpha)/2} \|\mathbf{w}\|_{\mathcal{H}}^{(1+\alpha)/2} \le \\
\le \eta_\alpha \left[C_{1w} + C_{2w}\sqrt{\lambda_2} + C_{3w}\lambda_2\right]^{(1+\alpha)/2} \left[C_{11}/\lambda_2 + C_{12}/\sqrt{\lambda_2} + C_{13}\right]^{(1-\alpha)/2} \le \\
\le M_2 \lambda_2^{1/2+\alpha/2} + \varphi_2(\lambda_2), \quad |\varphi_2(\lambda_2)| \le M_{\varphi 2} \lambda_2^{\alpha/2}.
\end{gathered}$$

**3. Uniqueness of solution.** By means of estimates (5) some non local conditions of uniqueness of solutions to equation (3) can be established. This conditions are valid when $\|\mathbf{f}(\lambda_1, \lambda_2)\|_{\mathbf{H}}$ has not very high order of increasing by the parameter $\lambda_2$. During the proof of the uniqueness it will be need to estimate from above the norm $\|B(\mathbf{u}, \mathbf{v})\|$ through the norms of $\mathbf{u}$, $\mathbf{v}$. The intrinsic supposition in this case will be the next

$$|\langle \mathbf{u},\ B(\mathbf{v}, \mathbf{w})\rangle| \le M_B \|\mathbf{v}\|_{\mathcal{H}} \|\mathbf{u}\|_{E_\mu} \|\mathbf{w}\|_{E_\nu} \tag{**}$$

with arbitrary $0 < \mu, \nu < 1$, $\mu + \nu = 1/2$.

THEOREM 3. *Let operators $A, B, C, T$ map the spaces $E_\alpha$, $\alpha > 0$ and $\mathbf{H}$ into itself and $B, C, T$ are completely continuous in the topologies of these spaces. Let for any elements $\mathbf{H} \ni \mathbf{v}$, $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, $\mathbf{u}_i \in \mathbf{H}_i$, conditions (*), (**) are satisfied and*

$$\|\mathbf{f}(\lambda_1, \lambda_2)\|_{\mathbf{H}} \le C_1 + C_2\sqrt{\lambda_2}.$$

*Let values of parameter $\lambda_2 > \lambda_{21}(\lambda_1)$ where*

$$\begin{aligned}
\lambda_{21}^2 > \tfrac{\lambda_1}{2m_T} \Bigg\{ \tfrac{3}{4} M_B \|\mathbf{w}_0\|_{E_{0.25}} \Bigg\{ \beta_{0.25} \left[M_1(\lambda_1) + M_2(\lambda_1)\varphi(\lambda_2)\right]^{5/3} \left[3\lambda_1 \tfrac{M_B}{m_A} \beta_{0.25}\right]^{5/3} + \\
+ \tfrac{1}{2} \left[M_1(\lambda_1) + M_2(\lambda_1)\varphi(\lambda_2)\right]^{13/3} \left[3.25\lambda_1 \tfrac{M_B}{m_A} \eta_{0.25}\right]^{13/3} \Bigg\} + \\
+ (1-\alpha) M_C \left\{ \beta_\alpha \left[2\lambda_1 \tfrac{M_C}{m_A}(1+\alpha)\beta_\alpha\right]^{\frac{1+\alpha}{1-\alpha}} + \tfrac{1}{2}\eta_\alpha \left[\lambda_1 \tfrac{M_C}{m_A}(3+\alpha)\eta_\alpha\right]^{\frac{3+\alpha}{1-\alpha}} \right\} \Bigg\}.
\end{aligned} \tag{7}$$

*Then equation* (3) *has a unique solution in* $\mathbf{H}$.

*Proof.* Let $\mathbf{w}$ and $\mathbf{w}_0$ - two different solutions of equation (3) and $\mathbf{u} = \mathbf{w} - \mathbf{w}_0$. Then $\mathbf{u}$ satisfies to equation

$$A\mathbf{u} = \lambda_1 \left[B(\mathbf{w}_0, \mathbf{u}) + B(\mathbf{u}, \mathbf{w}_0) + B(\mathbf{u}, \mathbf{u}) + C\mathbf{u}\right] - \lambda_2^2 T\mathbf{u}.$$

As well as in previous theorem from this equation follows next "energy" inequality

$$m_A \|\mathbf{u}\|_{\mathcal{H}}^2 + m_T \lambda_2^2 \|\mathbf{u}_1\|_{\mathbf{H}}^2 \le \lambda_1 M_B \|\mathbf{u}\|_{\mathcal{H}} \|\mathbf{u}\|_{E_\mu} \|\mathbf{w}_0\|_{E_\nu} + \lambda_1 M_C \|\mathbf{u}\|_{\mathcal{H}} \|\mathbf{u}\|_{E_\alpha}. \tag{8}$$

For the proof of the theorem it is need to estimate terms in the right part of inequality (8). For estimation of the first term it is need to establish estimates of $\|\mathbf{u}\|_{\mathcal{H}} \|\mathbf{u}\|_{E_\alpha}$ and

$\|\mathbf{w}_0\|_{E_\alpha}$. Product of norms is estimated by means of multiplicative and Jung's inequalities:

$$\begin{aligned}
&\|\mathbf{u}\|_{E_\mu}\|\mathbf{u}\|_{\mathcal{H}} \le \left(\|\mathbf{u}_1\|_{E_\mu} + \|\mathbf{u}_2\|_{E_\mu}\right)\|\mathbf{u}\|_{\mathcal{H}} \le \\
&\le \left(\beta_\mu\|\mathbf{u}_1\|_{\mathbf{H}}^{1-\mu}\|\mathbf{u}\|_{\mathcal{H}}^{\mu} + \eta_\mu\|\mathbf{u}_1\|_{\mathbf{H}}^{(1-\mu)/2}\|\mathbf{u}\|_{\mathcal{H}}^{(1+\mu)/2}\right)\|\mathbf{u}\|_{\mathcal{H}} = \\
&= \beta_\mu\|\mathbf{u}\|_{\mathcal{H}}^{1+\mu}\|\mathbf{u}_1\|_{\mathbf{H}}^{1-\mu} + \eta_\mu\|\mathbf{u}\|_{\mathcal{H}}^{(3+\mu)/2}\|\mathbf{u}_1\|_{\mathbf{H}}^{(1-\mu)/2} \le \\
&\le \tfrac{1}{2}\beta_\mu\left[(1+\mu)\,\varepsilon_1^{2/(1+\mu)}\|\mathbf{u}\|_{\mathcal{H}}^2 + (1-\mu)\,\varepsilon_1^{-2/(1-\mu)}\|\mathbf{u}_1\|_{\mathbf{H}}^2\right] + \\
&+\tfrac{1}{4}\eta_\mu\left[\|\mathbf{u}\|_{\mathcal{H}}^2 + (1-\mu)\,\varepsilon_2^{-4/(1-\mu)}\|\mathbf{u}_1\|_{\mathbf{H}}^2\right] = \\
&= \tfrac{1}{2}\left[\beta_\mu(1+\mu)\,\varepsilon_1^{2/(1+\mu)} + \tfrac{1}{2}(3+\mu)\,\varepsilon_2^{4/(3+\mu)}\right]\|\mathbf{u}\|_{\mathcal{H}}^2 + \\
&+\tfrac{1}{2}(1-\mu)\left[\beta_\mu\varepsilon_1^{-2/(1-\mu)} + \tfrac{1}{2}\varepsilon_2^{-4/(1-\mu)}\right]\|\mathbf{u}_1\|_{\mathbf{H}}^2.
\end{aligned}$$

The last term is estimated as well as in the theorem 1:

$$\begin{aligned}
&\|\mathbf{u}\|_{\mathcal{H}}\|\mathbf{u}\|_{E_\alpha} \le \tfrac{1}{2}\beta_\alpha\left[(1+\alpha)\,\varepsilon_3^{2/(1+\alpha)}\|\mathbf{u}\|_{\mathcal{H}}^2 + (1-\alpha)\,\varepsilon_3^{-2/(1-\alpha)}\|\mathbf{u}_1\|_{\mathbf{H}}^2\right] + \\
&+\tfrac{1}{4}\eta_\alpha\left[(3+\alpha)\,\varepsilon_4^{4/(3+\alpha)}\|\mathbf{u}\|_{\mathcal{H}}^2 + (1-\alpha)\,\varepsilon_4^{-4/(1-\alpha)}\|\mathbf{u}_1\|_{\mathbf{H}}^2\right] \le \\
&\le \tfrac{1}{2}\left[\beta_\alpha(1+\alpha)\,\varepsilon_3^{2/(1+\alpha)} + \tfrac{1}{2}\eta_\alpha(3+\alpha)\,\varepsilon_4^{4/(3+\alpha)}\right]\|\mathbf{u}\|_{\mathcal{H}}^2 + \\
&+\tfrac{1}{2}(1-\alpha)\left[\beta_\alpha\varepsilon_3^{-2/(1-\alpha)} + \tfrac{1}{2}\eta_\alpha\varepsilon_4^{-4/(1-\alpha)}\right]\|\mathbf{u}_1\|_{\mathbf{H}}^2.
\end{aligned}$$

The established estimates are substituted into the inequality (8):

$$\begin{aligned}
&m_A\|\mathbf{u}\|_{\mathcal{H}}^2 + m_T\lambda_2^2\|\mathbf{u}_1\|_{\mathbf{H}}^2 \le \\
&\le \tfrac{1}{2}\lambda_1 M_B\|\mathbf{w}_0\|_{E_\nu}\left\{\left[\beta_\mu(1+\mu)\,\varepsilon_1^{2/(1+\mu)} + \tfrac{1}{2}(3+\mu)\,\eta_\mu\varepsilon_2^{4/(3+\mu)}\right]\|\mathbf{u}\|_{\mathcal{H}}^2 + \right. \\
&\left. + (1-\mu)\left[\beta_\mu\varepsilon_1^{-2/(1-\mu)} + \tfrac{1}{2}\eta_\mu\varepsilon_2^{-4/(1-\mu)}\right]\|\mathbf{u}_1\|_{\mathbf{H}}^2\right\} + \\
&+\tfrac{1}{2}\lambda_1 M_C\left\{\left[\beta_\alpha(1+\alpha)\,\varepsilon_3^{2/(1+\alpha)} + \tfrac{1}{2}\eta_\alpha(3+\alpha)\,\varepsilon_4^{4/(3+\alpha)}\right]\|\mathbf{u}\|_{\mathcal{H}}^2 + \right. \\
&\left. + (1-\alpha)\left[\beta_\alpha\varepsilon_3^{-2/(1-\alpha)} + \tfrac{1}{2}\eta_\alpha\varepsilon_4^{-4/(1-\alpha)}\right]\|\mathbf{u}_1\|_{\mathbf{H}}^2\right\}.
\end{aligned}$$

After groping of terms of the same type next main inequality is obtained:

$$\begin{aligned}
&\left\{m_A - \tfrac{1}{2}\lambda_1\left\{M_B\|\mathbf{w}_0\|_{E_\nu}\left[\beta_\mu(1+\mu)\,\varepsilon_1^{2/(1+\mu)} + \tfrac{1}{2}\eta_\mu(3+\mu)\,\varepsilon_2^{4/(3+\mu)}\right] + \right.\right. \\
&\left.\left. - M_C\left[\beta_\alpha(1+\alpha)\,\varepsilon_3^{2/(1+\alpha)} + \tfrac{1}{2}\eta_\alpha(3+\alpha)\,\varepsilon_4^{4/(3+\alpha)}\right]\right\}\right\}\|\mathbf{u}\|_{\mathbf{H}}^2 + \\
&+\left\{\lambda_2^2 m_T - \tfrac{1}{2}\lambda_1\left\{M_B\|\mathbf{w}_0\|_{E_\nu}(1-\mu)\left[\beta_\mu\varepsilon_1^{-2/(1-\mu)} + \tfrac{1}{2}\varepsilon_2^{-4/(1-\mu)}\right] + \right.\right. \\
&\left.\left. + (1-\alpha)\,M_C\left[\beta_\alpha\varepsilon_3^{-2/(1-\alpha)} + \tfrac{1}{2}\eta_\alpha\varepsilon_4^{-4/(1-\alpha)}\right]\right\}\right\}\|u_1\|_{\mathbf{H}}^2 \le 0.
\end{aligned}$$

Values of numbers $\varepsilon_i$ $(i = 1, 2, 3, 4)$ are chosen from the condition of non negativeness of multiplier at $\|\mathbf{u}\|_{\mathcal{H}}^2$. For example it can be next:

$$\begin{aligned}
\varepsilon_1 &= \left[2\lambda_1(1+\mu)\,\beta_\mu\tfrac{M_B}{m_A}\|\mathbf{w}_0\|_{E_\nu}\right]^{-\frac{1+\mu}{2}}, \quad \varepsilon_2 = \left[\lambda_1(3+\mu)\,\eta_\mu\tfrac{M_B}{m_A}\|\mathbf{w}_0\|_{E_\nu}\right]^{-\frac{3+\mu}{4}}, \\
\varepsilon_3 &= \left[2\lambda_1(1+\alpha)\,\beta_\alpha\tfrac{M_C}{m_A}\right]^{-\frac{1+\alpha}{2}}, \quad \varepsilon_4 = \left[\lambda_1(3+\alpha)\,\eta_\alpha\tfrac{M_C}{m_A}\right]^{-\frac{3-\alpha}{4}}.
\end{aligned}$$

This values of $\varepsilon_i$ are substituted to the multiplier at $\|\mathbf{u}_1\|^2_{\mathbf{H}}$. The uniqueness conditions are established from the positiveness of this multiplier:

$$\lambda_2^2 > \frac{\lambda_1}{2m_T}\left\{M_B\|\mathbf{w}_0\|_{E_\nu}(1-\mu)\left\{\beta_\mu\|\mathbf{w}_0\|_{E_\nu}^{\frac{1+\mu}{1-\mu}}\left[2\lambda_1\frac{M_B}{m_A}\beta_\mu(1+\mu)\right]^{\frac{1+\mu}{1-\mu}}+\right.\right.$$
$$\left.+\frac{1}{2}\|\mathbf{w}_0\|_{E_\nu}^{\frac{3+\mu}{1-\mu}}\left[\lambda_1\frac{M_B}{m_A}\eta_\mu(3+\mu)\right]^{\frac{3+\mu}{1-\mu}}\right\}+ \tag{9}$$
$$\left.+(1-\alpha)M_C\left\{\beta_\alpha\left[2\lambda_1\frac{M_C}{m_A}(1+\alpha)\beta_\alpha\right]^{\frac{1+\alpha}{1-\alpha}}+\frac{1}{2}\eta_\alpha\left[\lambda_1\frac{M_C}{m_A}(3+\alpha)\eta_\alpha\right]^{\frac{3+\alpha}{1-\alpha}}\right\}\right\}.$$

From the estimates (5) follow

$$\begin{aligned}
\|\mathbf{w}_0\|_{E_\nu} &\le \beta_\nu\|\mathbf{w}_{01}\|_{\mathbf{H}}^{1-\nu}\|\mathbf{w}\|_{\mathcal{H}}^{\nu}+\eta_\nu\|\mathbf{w}_{01}\|_{\mathbf{H}}^{(1-\nu)/2}\|\mathbf{w}\|_{\mathcal{H}}^{(1+\nu)/2}\le\\
&\le \beta_\nu\left[C_{1w}+C_{2w}\sqrt{\lambda_2}\right]^{\nu}\left[C_{11}/\lambda_2+C_{12}/\sqrt{\lambda_2}\right]^{1-\nu}+\\
&+\eta_\nu\left[C_{1w}+C_{2w}\sqrt{\lambda_2}\right]^{(1+\nu)/2}\left[C_{11}/\lambda_2+C_{12}/\sqrt{\lambda_2}\right]^{(1-\nu)/2}\le\\
&\le \beta_\nu\left[C_{11}/\lambda_2+C_{12}/\sqrt{\lambda_2}\right]\left[\frac{C_{1w}+C_{2w}\sqrt{\lambda_2}}{C_{11}/\lambda_2+C_{12}/\sqrt{\lambda_2}}\right]^{\nu}+\\
&+\eta_\nu\left[C_{1w}+C_{2w}\sqrt{\lambda_2}\right]\left[C_{11}/\lambda_2+C_{12}/\sqrt{\lambda_2}\right]\left[\frac{C_{1w}+C_{2w}\sqrt{\lambda_2}}{C_{11}/\lambda_2+C_{12}/\sqrt{\lambda_2}}\right]^{\nu}=\\
&=\left[C_{11}/\lambda_2+C_{12}/\sqrt{\lambda_2}\right]\left[\lambda_2\frac{C_{1w}+C_{2w}\sqrt{\lambda_2}}{C_{11}+C_{12}\sqrt{\lambda_2}}\right]^{\nu}\left\{\beta_\nu+\eta_\nu\left[C_{1w}+C_{2w}\sqrt{\lambda_2}\right]\right\}\le\\
&\le M_1(\lambda_1)\lambda_2^{\nu-1/2}+M_2(\lambda_1)\varphi(\lambda_2),\qquad |\varphi(\lambda_2)|\le M_\varphi\lambda_2^{\nu-1}.
\end{aligned}$$

Let $\mu=\nu=1/4$. Then substitution of this estimate into inequality (9) results to completion of the proof.

**4. Approximated solutions.** The properties of operator $T$ make possible to construct a simple linearization of the equation (3). Let $\lambda_1=0$. Then equation (3) becomes to the form

$$A\mathbf{w}+\lambda_2^2T\mathbf{w}=\mathbf{f}(\lambda_1,\lambda_2). \tag{11}$$

Solution of equation (11) will be named a $\lambda_2$-approximation of the solutions of equation (3). Equation (11) has a unique solution $\mathbf{w}^{(0)}$.

THEOREM 4. *Let operators $A,B,C,T$ map the spaces $E_\alpha$, $\alpha>0$ and $\mathbf{H}$ into itself and $B,C,T$ are completely continuous in the topologies of these spaces. Let for any elements $\mathbf{H}\ni\mathbf{v},\mathbf{u}=\mathbf{u}_1+\mathbf{u}_2$, $\mathbf{u}_i\in\mathbf{H}_i$, conditions (*), (**) are satisfied and*

$$\|\mathbf{f}(\lambda_1,\lambda_2)\|_{\mathbf{H}}\le C_1+C_2\sqrt{\lambda_2}.$$

*Let values of parameter $\lambda_2>\lambda_{24}(\lambda_1)$, where*

$$\lambda_{24}^2>\lambda_1\left\{\frac{M_B}{m_T}\left\{\beta_\mu\left[M_1(\lambda_1)\lambda_2^{\nu-1/2}+M_2(\lambda_1)\varphi(\lambda_2)\right]^{2/\nu}\left[4\lambda_1\frac{M_B}{m_A}\beta_\mu(1+\mu)\right]^{\frac{2-\nu}{\nu}}+\right.\right.$$
$$\left.+\frac{1}{2}\eta_\mu\left[M_1(\lambda_1)\lambda_2^{\nu-1/2}+M_2(\lambda_1)\varphi(\lambda_2)\right]^{4/\nu}\left[2\lambda_1\frac{M_B}{m_A}\eta_\mu(3+\mu)\right]^{\frac{4-\nu}{\nu}}\right\}(1-\mu)+$$
$$+\frac{M_C}{m_T}(1-\alpha)\left\{\beta_\alpha\left[M_1(\lambda_1)\lambda_2^{\nu-1/2}+M_2(\lambda_1)\varphi(\lambda_2)\right]\left[4\frac{M_C}{m_A}\lambda_1(1+\alpha)\beta_\alpha\right]^{\frac{1+\alpha}{1-\alpha}}+\right.$$
$$\left.\left.+\frac{1}{2}\eta_\alpha\left[M_1(\lambda_1)\lambda_2^{\nu-1/2}+M_2(\lambda_1)\varphi(\lambda_2)\right]\left[2\frac{M_C}{m_A}\lambda_1(3+\alpha)\eta_\alpha\right]^{\frac{3+\alpha}{1-\alpha}}\right\}\right\},$$
$$0<\alpha<1,\quad \nu<1/2.$$

*Then for norm of $\delta\mathbf{w}$ - a residual of the exact and approximated solutions of the equation* (3) *next estimates are valid*

$$\begin{gathered}
\|\delta\mathbf{w}\|_{\mathcal{H}} \leq \frac{2}{m_{A_1}}\left[M_B M_1^2(\lambda_1) + \varphi_\lambda(\lambda_2)\right] \\
\|\delta\mathbf{w}_1\|_{\mathbf{H}} \leq \frac{2\lambda_2^{-1}}{(m_A m_T)^{1/2}}\left[M_B M_1^2(\lambda_1) + \varphi_\lambda(\lambda_2)\right]; \\
\|\delta\mathbf{w}\|_{\mathbf{H}} \leq \frac{2\lambda_2^{-1/2}}{m_A^{3/4} m_T^{1/4}}\left[M_B M_1^2(\lambda_1) + M_\lambda \varphi_\lambda(\lambda_2)\right]; \\
|\varphi_\lambda(\lambda_2)| < M_\lambda \lambda_2^{\alpha-1/2}.
\end{gathered} \tag{12}$$

*Proof.* Let $\delta\mathbf{w} = \mathbf{w} - \mathbf{w}^{(0)}$ be a residual of the exact and approximated solutions of the equation (3). Then $\delta\mathbf{w}$ satisfies to the equation

$$A\delta\mathbf{w} = \lambda_1\left[B(\delta\mathbf{w} + \mathbf{w}^{(0)}, \delta\mathbf{w} + \mathbf{w}^{(0)}) + C\delta\mathbf{w} + C\mathbf{w}^{(0)}\right] - \lambda_2^2 T\delta\mathbf{w}.$$

Main energetic inequality has the form

$$\begin{gathered}
m_A\|\delta\mathbf{w}\|_{\mathcal{H}}^2 + \lambda_2^2 m_T\|\delta\mathbf{w}_1\|_{\mathbf{H}}^2 \leq \lambda_1\left(M_B\|\delta\mathbf{w}\|_{E_\mu}\left\|\mathbf{w}^{(0)}\right\|_{E_\nu} + M_C\|\delta\mathbf{w}\|_{E_\alpha}\right)\|\delta\mathbf{w}\|_{\mathcal{H}} + \\
+\lambda_1\left(M_B\left\|\mathbf{w}^{(0)}\right\|_{E_\nu}\left\|\mathbf{w}^{(0)}\right\|_{\mathcal{H}}\|\delta\mathbf{w}\|_{E_\mu} + M_C\left\|\mathbf{w}^{(0)}\right\|_{E_\alpha}\|\delta\mathbf{w}\|_{\mathcal{H}}\right).
\end{gathered}$$

Estimates of terms in the right part of this inequality are established in the previous statements. Then the main inequality has the form

$$\begin{gathered}
\left\{m_A - \lambda_1\left\{\frac{M_H}{2}\left\|\mathbf{w}^{(0)}\right\|_{E_\nu}\left[\beta_\mu(1+\mu)\varepsilon_1^{2/(1+\mu)} + \frac{1}{2}\eta_\mu(3+\mu)\varepsilon_2^{4/(3+\mu)}\right] + \right.\right. \\
\left.\left. +\frac{M_C}{2}\left[\beta_\alpha(1+\alpha)\varepsilon_3^{2/(1+\alpha)} + \frac{1}{2}\eta_\alpha(3+\alpha)\varepsilon_4^{4/(3+\alpha)}\right]\right\}\right\}\|\delta\mathbf{w}\|_{\mathcal{H}}^2 + \\
+\lambda_2^2\left\{m_T - \lambda_1\lambda_2^{-2}\left\{\frac{M_B}{2}(1-\mu)\left\|\mathbf{w}^{(0)}\right\|_{E_\nu}\left[\beta_\mu\varepsilon_1^{-2/(1-\mu)} + \frac{1}{2}\eta_\mu\varepsilon_2^{-4/(1-\mu)}\right] + \right.\right. \\
\left.\left. +\frac{M_C}{2}(1-\alpha)\left[\beta_\alpha\varepsilon_3^{-2/(1-\alpha)} + \frac{1}{2}\eta_\alpha\varepsilon_4^{-4/(1-\alpha)}\right]\right\}\right\}\|\delta\mathbf{w}_1\|_{\mathbf{H}}^2 \leq \\
\leq \lambda_1\left[M_B\left\|\mathbf{w}^{(0)}\right\|_{E_\mu}\left\|\mathbf{w}^{(0)}\right\|_{E_\nu} + M_C\left\|\mathbf{w}^{(0)}\right\|_{E_\alpha}\right]\|\delta\mathbf{w}\|_{\mathcal{H}},
\end{gathered} \tag{13}$$

where

$$\begin{gathered}
\varepsilon_1 = \left[4\lambda_1(1+\mu)\beta_\mu\frac{M_B}{m_A}\left\|\mathbf{w}^{(0)}\right\|_{E_\nu}\right]^{-\frac{1+\mu}{2}}, \quad \varepsilon_2 = \left[2\lambda_1(3+\mu)\eta_\mu\frac{M_B}{m_A}\left\|\mathbf{w}^{(0)}\right\|_{E_\nu}\right]^{-\frac{3+\mu}{4}}, \\
\varepsilon_3 = \left[4\lambda_1(1+\alpha)\beta_\alpha\frac{M_C}{m_A}\right]^{-\frac{1+\alpha}{2}}, \quad \varepsilon_4 = \left[2\lambda_1(3+\alpha)\eta_\alpha\frac{M_C}{m_A}\right]^{-\frac{3+\alpha}{4}}.
\end{gathered}$$

In the inequality (13) it is need to estimate $\left\|\mathbf{w}^{(0)}\right\|_{E_\mu}\left\|\mathbf{w}^{(0)}\right\|_{E_\nu}$ and $\left\|\mathbf{w}^{(0)}\right\|_{E_\alpha}$. From estimates (5) follow

$$\begin{gathered}
M_B\left\|\mathbf{w}^{(0)}\right\|_{E_\mu}\left\|\mathbf{w}^{(0)}\right\|_{E_\nu} + M_C\left\|\mathbf{w}^{(0)}\right\|_{E_\alpha} \leq \\
\leq M_B\left[M_1(\lambda_1)\lambda_2^{1/2-\nu} + M_2(\lambda_1)\varphi(\lambda_2)\right]\left[M_1(\lambda_1)\lambda_2^{\nu-1/2} + M_2(\lambda_1)\varphi(\lambda_2)\right] + \\
+M_C\left[M_1(\lambda_1)\lambda_2^{\alpha-1/2} + M_2(\lambda_1)\varphi(\lambda_2)\right] \leq M_B M_1^2(\lambda_1) + \varphi_\lambda(\lambda_2).
\end{gathered}$$

From this estimate and multiplicative inequalities follow estimates (12). Theorem is proven.

The main results of this paper can be applicated to the theory boundary value problems of magnetic hydrodynamics.

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