EXISTENCE OF A GLOBAL CLASSICAL SOLUTION IN A PROBLEM DESCRIBING THE COMBUSTION PROCESS

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Abstract. In this work we prove the existence of the global classical solution in multidimensional two phase nonstationary problem modeling the combustion process. These problem are different from the Stefan problem:it is nonlinear not only because of the free boundary but also because of the nonlinearity of boundary conditions

$$u^+ = u^- = 0, \ \ (u_\nu^+)^2 - (u_\nu^-)^2 = Q^2(x).$$

Here ν is the spacial unit vector normal to the free boundary and directed towards increasing of $u(x,t)$, $u^+ = \max(u,0)$, $u^- = \max(-u,0)$. We prove that the free boundary is given by the graph of a function from the $H^{2+\alpha,1+\frac{\alpha}{2}}, \alpha \in (0,1)$ class.

0. Introduction. In this paper we study a two–phase free boundary problem arising in description of the combustion process. The problem is to find a function satisfying the heat equation

$$\Delta u - u_t = 0, \quad \text{in} \quad \Omega_T \cup G_T, \quad D_T = D \times (0,T), \tag{0.1}$$

where D is a bounded domain in $\mathbb{R}^3$, T is a given positive number,

$$\Omega_T = \{(x,t) \in D_T : u(x,t) < 0\}, G_T = \{(x,t) \in D_T : u(x,t) > 0\}.$$

On the free boundary $\gamma_T = \partial\Omega_T \cap D_T = \partial G_T \cap D_T$ the following coditions hold

$$u^+ = u^- = 0, \quad (u_\nu^+)^2 - (u_\nu^-)^2 = Q^2(x). \tag{0.2}$$

Here ν is the spacial unit vector normal to the free boundary γ_T and directed towards increasing of $u(x,t)$, $u^+ = \max(u,0)$, $u^- = \max(-u,0)$. On the known part of the boundary

$$u(x,t) = \varphi(x,t) \quad \text{on} \quad (\partial\Omega_T \cap \partial D_T) \cup (\partial G_T \cap \partial D_T),$$

$$\varphi(x,t) < 0 \quad \text{on} \quad \partial\Omega_T \cap \partial D_T, \quad \varphi(x,t) > 0 \quad \text{on} \quad \partial G_T \cap \partial D_T. \tag{0.3}$$

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Initial conditions are as follows.

$$u(x,0)=\psi(x) \quad \text{in} \quad D, \quad \varphi(x,0)=\psi(x) \qquad \partial D, \tag{0.4}$$

$$\psi(x)>0 \quad \text{on} \quad \partial G_0\cap\partial D, \;\; \psi(x)<0 \quad \text{on} \quad \partial\Omega_0\cap\partial D,$$

$$\Omega_0=\{x\in D:\psi(x)<0\}, \;\; G_0=\{x\in D:\psi(x)>0\}, \gamma_0=\{x\in D:\psi(x)=0.\}$$

Similar problems have been extensively studied. Existence of classical solutions was proved in for the multi–dimensional problem under the assumption of radial spatial symmetry [1], existence of a weak solution was proved in [2], [3]. In this paper we use the method, suggested in [4]. This paper is a natural continuation of the investigation started of the author in [5]. We prove the existence of a global classical solution in the problem (0.1)-(0.4) with minimal necessary restrictions on initial and boundary conditions. The method consists of the following: first we construct a parabolic difference–differential approximation of the problem, then we prove cetain uniform estimates and pass to the limit.

For the sake of convenience we consider the 3–dimensional space $\mathbb{R}^3$, all our results also hold in any finite dimensional space $\mathbb{R}^n$, $n\geq 1$.

1. Aprroximation of the problem. Properties of the approximate solutions. Let us construct the approximating problems. Assume, for simplicity, that

$$D=\{x\in\mathbb{R}^3: R_1<|x|<R_2\}, \;\; \partial D_i=\{|x|=R_i, \;\; i=1,2\}, \partial D=\partial D_1\cup\partial D_2.$$

Cut the cylinder $D_T\times(0,T_1)$ by the planes $\tau=kh$, $hN=T_1$, $k=1,2,...,N$, N is a given positive integer. Define for all $\varepsilon>0$ a function $\chi_\varepsilon(x)\in C^k(\mathbb{R}^1)$, $k>2$ as follows

$$\chi_\varepsilon(x)=1 \;\; \forall x\leq 0, \;\; \chi_\varepsilon(x)=0 \;\; \forall x\geq\varepsilon, \chi'_\varepsilon(x)\leq 0.$$

Let us assume that the functions $\{u_k(x,t,h,\varepsilon)\}$, $\{F_k(x,t,h,\varepsilon)\}$, are solutions of the following problem

$$\Delta u_k-\frac{\partial u_k}{\partial t}-a\frac{u_k-u_{k-1}}{h}=-\lambda\frac{\chi_\varepsilon(u_k)-\chi_\varepsilon(u_0)}{h}-\frac{Q^2(x)}{2}\sum_{l=1}^{k-1}\chi'_\varepsilon(u_l)+\frac{a}{h}F_{k-1} \text{ in } D_T, \tag{1.1}$$

$$u_k(x,t,h,\varepsilon)=\varphi^i(x,t,kh)=\varphi^i_k(x,t,h) \quad \text{on} \quad \partial D_i\times[0,T],$$

$$u_k(x,0,h,\varepsilon)=\psi(x,kh)=\psi_k(x,h) \quad \text{in} \quad D, \;\; u_0=\omega(x,t) \quad \text{in} \quad D_T, \tag{1.2}$$

where $\varphi(x,t,\tau)$, $\psi(x,\tau)$, $\omega(x,t)$ are given functions

$$\Delta F_k-\frac{\partial F_k}{\partial t}-a\frac{F_k}{h}=-\lambda\frac{\chi_\varepsilon(u_k)-\chi_\varepsilon(u_0)}{h}-\frac{Q^2(x)}{2}\sum_{l=1}^{k-1}\chi'_\varepsilon(u_l) \quad \text{in} \quad D_T, \tag{1.3}$$

$$F_k(x,t,h,\varepsilon)=0 \text{ on } \partial D_i\times(0,T), \;\; F_k(x,0,h,\varepsilon)=0 \text{ in } D, \;\; F_0=0 \;\text{ in } D_T. \tag{1.4}$$

The problem (1.1)-(1.4) can be studied step by step, starting from $k=1$. First we find the function $F_{k-1}(x,t,h,\varepsilon)$ (note that $F_0=0$), then we substitute this function in the right–hand side of (1.1) and study the correpsonding initial boundary value problem for $u_k(x,t,h,\varepsilon)$. The function $u_k(x,t,h,\varepsilon)$ thus obtained we substitute in the right–hans side of the equation (1.3) and find the function $F_k(x,t,h,\varepsilon)$ and so on. Solvability for any of the problems mentioned above in Hölder spaces is well known ([6], ch. 5).

THEOREM 1.1. *Assume that the following conditions hold*

$$\psi_k(x,h) \in C^{2+\alpha}(\overline{D}),\ \varphi_k^i(x,t) \in H^{2+\alpha,1+\frac{\alpha}{2}}(\partial D_i \times [0,T]),$$

$$\omega(x,t) \in H^{2+\alpha,1+\alpha/2}(\overline{D}_T),\ \ Q(x) \in C^{\alpha}(\overline{D}),\ \alpha \in (0,1),$$

and for the functions $\psi_k(x,h,\varepsilon)$, $\varphi_k^i(x,t)$ *the corresponding compatibility conditions on* ∂D_i *at* $t=0$ *hold. Than for any* $h>0$, $\varepsilon>0$ *the problem* (1.1)–(1.4) *has a unique solution and*

$$u_k(x,t,h,\varepsilon), F_k(x,t,h,\varepsilon) \in H^{2+\alpha,1+\frac{\alpha}{2}}(\overline{D}_T).$$

Denote by

$$w_k(x,t,h,\varepsilon) = u_k(x,t,h,\varepsilon) - F_k(x,t,h,\varepsilon). \tag{1.5}$$

Subtract (1.1) from (1.3) taking into account (1.5). We get

$$\Delta w_k - \frac{\partial w_k}{\partial t} - a\frac{w_k - w_{k-1}}{h} = 0\ \ \forall (x,t) \in D_T, \tag{1.6}$$

$$w_k(x,t,h,\varepsilon) = \varphi_k^i(x,t,h)\ \text{ on }\ \partial D_i \times [0,T],\ \ w_k(x,0,h,\varepsilon) = \psi_k(x,h)\ \text{ in }\ D,$$

$$w_0 = \omega(x,t)\ \text{ in }\ D_T. \tag{1.7}$$

THEOREM 1.2. *Let the assumptions of Theorem* 1.1 *hold and let there exist positive constants* c_j, $i=1,...,5$ *such that* $\Delta\omega - \frac{\partial \omega}{\partial t} < -c_1$,

$$0 \le c_2 h \le \varphi_{k-1}^i(x,t,h) - \varphi_k^i(x,t,h) \le c_3 h \quad on \quad \partial D_i \times [0,T],$$

$$0 \le c_4 h \le \psi_{k-1}(x,h) - \psi_k(x,h) \le c_5 h \ in\ \overline{D}.$$

Then there exist positive constants c_6, c_7 *not depending on* h,ε,k *such that everywhere in* $\overline{D}_T$ *The following estimate holds*

$$c_6 h \le w_{k-1}(x,t,h,\varepsilon) - w_k(x,t,h,\varepsilon) \le c_7 h, \tag{1.8}$$

where the constants c_i *do not depend on* a,h,ε,k .

On $\partial D_i \times [0,T]$ the estimate (1.8) is evident. To prove it inside D_T, it suffices to write down the equation

$$\Delta(w_{k-1} - w_k) - \frac{\partial}{\partial t}(w_{k-1} - w_k) - \frac{a}{h}(w_{k-1} - w_k) = -\frac{a}{h}(w_{k-2} - w_{k-1})$$

and employ the fact that at the points of local maxima or minima the function $\Delta(w_{k-1} - w_k) - \frac{\partial}{\partial t}(w_{k-1} - w_k)$ is nonpositve and nonnegative, respectively.

COROLLARY 1. *Let the assumptions of Theorem 1.2 hold and*

$$0 \le -\Delta\omega + \frac{\partial \omega}{\partial t} \le c_1 a.$$

Then the following estimate holds

$$0 \le c_2 h \le w_{k-1}(x,t,h,\varepsilon) - w_k(x,t,h,\varepsilon) \le c_3 h,$$

where the constants c_i *do not depend on* a,h,ε,k .

This estimate follows from the previous theorem.

COROLLARY 2. *Let the assumptions of Theorem 1.2 hold. Then if*

$$\left\|\varphi_k^i(x,t,h)\right\|_{H^{2+\alpha,1+\frac{\alpha}{2}}(\partial D_i\times[0,T])}\leq c_1,\quad \alpha\in(0,1),$$

then

$$\|w_k(x,t,h,\varepsilon)\|_{H^{2-\alpha,1-\frac{\alpha}{2}}(\overline{D}_T)}\leq c_2, \tag{1.9}$$

where the constants c_i do not depend on h,ε,k.

This follows from the fact that

$$\Delta w_k-\frac{\partial w_k}{\partial t}=a\frac{w_k-w_{k-1}}{h}\in L_\infty(D_T),$$

smoothness of initial and boundary conditions, and the corresponding embedding theorem.

THEOREM 1.3. *Let the following conditions hold*

$$\Delta w_k-\frac{\partial w_k}{\partial t}=a\frac{w_k-w_{k-1}}{h}\in L_\infty(D_T),\Delta\omega_0-\frac{\partial\omega_0}{\partial t}\in H^{\alpha,\frac{\alpha}{2}}(\overline{D}_T),$$

$$\left\|\frac{\varphi_k^i(x,t,h)-\varphi_{k-1}^i(x,t,h)}{h}\right\|_{H^{\alpha,\frac{\alpha}{2}}(\overline{D}_T)}+\|\varphi_k^i(x,t,h)\|_{H^{2+\alpha,1+\frac{\alpha}{2}}(\overline{D}_T)}\leq c_1,$$

$$\left\|\frac{\psi_k-\psi_{k-1}}{h}\right\|_{C^\alpha(\overline{D})}+\|\psi_k\|_{C^{2+\alpha}(\overline{D})}\leq c_2$$

and for the functions ψ_k, φ_k^i compatibility conditions of the first order on ∂D_i at $t=0$ hold. Then

$$\left\|\frac{w_k-w_{k-1}}{h}\right\|_{H^{\alpha,\frac{\alpha}{2}}(\overline{D}_T)}+\|w_k\|_{H^{2+\alpha,1+\frac{\alpha}{2}}(\overline{D}_T)}\leq c_3, \tag{1.10}$$

where the constants c_i do not depend on k,h,ε.

Denote

$$v_k(x,t,h,\varepsilon)=w_k(x,t,h,\varepsilon)-w_k(x,0,h,\varepsilon)=w_k(x,t,h,\varepsilon)-\psi_k(x,h).$$

Then $\{v_k-v_{k-1}\}$ satisfy the equations

$$\Delta(v_k-v_{k-1})-\frac{\partial}{\partial t}(v_k-v_{k-1})-a\frac{(v_k-v_{k-1})}{h}+$$

$$+a\frac{(v_{k-1}-v_{k-2})}{h}=-(f_k-f_{k-1}),$$

where $f_k=\Delta\psi_k-a\frac{\psi_k-\psi_{k-1}}{h}$, and have zero initial conditions. Let $\zeta_k(x,t),\dots,\zeta_l(x,t)$ be nonnegative infinitely smooth compactly supported functions which provide a partition of unity in $\overline{D}_T$, i. e.,

$$\sum_{k=1}^{l}\zeta_k(x,t)=1\quad\forall(x,t)\in\overline{D}_T.$$

Let us represent the functions $v_k(x,t,h,\varepsilon)$ as

$$v_k(x,t,h,\varepsilon) = \sum_{s=1}^{l} v_k^s(x,t,h,\varepsilon), \quad k = 1,2,...,N,$$

where $v_k^s(x,t,h,\varepsilon) = \zeta_s(x,t)v_k(x,t,h,\varepsilon)$. If the support of $\zeta_s(x,t)$ lies inside D_T, then the functions $v_k^s(x,t,h,\varepsilon)$ can be viewed as compactly supported functions from $H^{2+\alpha,1+\frac{\alpha}{2}}(\mathbb{R}^4)$, satisfying the equations

$$\Delta(v_k^s - v_{k-1}^s) - \frac{\partial}{\partial t}(v_k^s - v_{k-1}^s) - a\frac{(v_k^s - v_{k-1}^s)}{h} + a\frac{(v_{k-1}^s - v_{k-2}^s)}{h} =$$

$$= -(f_k^s - f_{k-1}^s) - (\Phi_k^s - \Phi_{k-1}^s), \;\; \Phi_k^s = -v_k\Delta\zeta_s - 2\nabla v_k \nabla\zeta_s + v_k\frac{\partial\zeta_k}{\partial t}, \;\; f_k^s = \zeta_s f_k.$$

To prove the theorem we shall need integral representations of the solutions of these equations. If the support of $\zeta_s(x,t)$ is partially contained in D_T, i.e., it contains a part of the parabolic boundary of this domain, then by a regular transformation we can straighten out this part of the boundary. After that we can repeat, with minor modifications, previous considerations for the half-space obtained.

Let us find out under which restrictions on initial and boundary conditions of $\{\frac{w_k - w_{k-1}}{h}\}$ the estimate of the type (1.10) holds. Suppose that

$$\omega(x,t) \in H^{4,2}(\overline{D}_T), \;\; \Delta f - \frac{\partial f}{\partial t} - a\frac{f}{h} = -\frac{1}{h}L(L\omega) \text{ in } D_T,$$

$$L = \Delta - \frac{\partial}{\partial t}, hf = a\frac{\varphi_1^i - \varphi_0^i}{h} - L\omega = 0 \text{ on } \partial D_i \times (0,T),$$

$$hf|_{t=0} = -L\omega|_{t=0} + \frac{a}{h}(\psi_1 - \omega)|_{t=0},$$

$$\max_{1\le k\le N,(x,t)\in\overline{D}_T}\left|\frac{\varphi_k^i - \varphi_{k-1}^i}{h} - \frac{\varphi_{k-1}^i - \varphi_{k-2}^i}{h}\right| +$$

$$+ \max_{1\le k\le N, x\in\overline{D}}\left|\frac{\psi_k - \psi_{k-1}}{h} - \frac{\psi_{k-1} - \psi_{k-2}}{h}\right| \le c_1 h,$$

$$\left\|\frac{\varphi_k^i - \varphi_{k-1}^i}{h}\right\|_{H^{2+\alpha,1+\frac{\alpha}{2}}(\partial D_i\times[0,T])} \le c_2, \tag{1.11}$$

where the constants c_i do not depend on h.

THEOREM 1.4. *Let the assumptions of Theorem* 1.3 *and the conditions* (1.11) *hold. Then*

$$\max_{1\le k\le N,(x,t)\in\overline{D}_T}\left|\frac{w_k - w_{k-1}}{h} - \frac{w_{k-1} - w_{k-2}}{h}\right| \le c_3 h,$$

$$\left\|\frac{w_k - w_{k-1}}{h}\right\|_{H^{2-\alpha,1-\frac{\alpha}{2}}(\overline{D}_T)} \le c_4, \;\; \alpha \in (0,1).$$

If the norms

$$\left\| \frac{\psi_k^i - \psi_{k-1}^i}{h} \right\|_{C^{2+\alpha}(\overline{D})}, \quad \left\| \Delta\omega - \frac{\partial \omega}{\partial t} \right\|_{H^{2+\alpha,1+\frac{\alpha}{2}}(\overline{D}_T)}$$

are uniformly bounded, and the correponding compatibility conditions are satisfied, then

$$\left\| \frac{w_k - w_{k-1}}{h} \right\|_{H^{2+\alpha,1+\frac{\alpha}{2}}(\overline{D}_T)} \le c_5,$$

where the constants c_i do not depend on h, k, ε. Under appropriate assumptions a similar statement can be proved for the functions $\frac{1}{h}\left(\frac{w_k - w_{k-1}}{h} - \frac{w_{k-1} - w_{k-2}}{h}\right)$ in the space $H^{\alpha,\frac{\alpha}{2}}(\overline{D}_T)$.

The first claim is proved by the maximum principle with subsequent use of an embedding theorem. The other estimates can be proved in a way similar to the one used in Theorem 2.3.

2. Uniform estimates for $\{u_k(x,t,h,\varepsilon)\}$. Let us estimate the difference quotients $\{\frac{u_{k-1}(x,t,h,\varepsilon) - u_k(x,t,h,\varepsilon)}{h}\}$.

THEOREM 2.1. *Let the assumptions of Theorem 1.3 hold. Then if $\varepsilon^4 \ge \sqrt{h}$, $Q(x) \ne 0$, there exist positive constants c_1, c_2 not depending on h, k, ε, such that*

$$0 \le c_1 h \le u_{k-1}(x,t,h,\varepsilon) - u_k(x,t,h,\varepsilon) \le c_2 h \quad \forall (x,t) \in \overline{D}_T. \tag{2.1}$$

Represent the equation (1.1) as follows

$$\Delta(u_{k-1} - u_k) - \frac{\partial}{\partial t}(u_{k-1} - u_k) - \frac{a}{h}(u_{k-1} - u_k) = -\frac{a}{h}(w_{k-2} - w_{k-1}) +$$

$$+ \frac{\lambda}{h}\chi'_\varepsilon(u_{k-1}) \left[\frac{h}{2\lambda} Q^2(x) - (u_{k-1} - u_k) \right] + \frac{\lambda}{h}\chi''_\varepsilon(\xi_k) \frac{(u_{k-1} - u_k)^2}{2},$$

where ξ_k is an intermediate point arising in the Lagrange's form for the remainder of the Taylor series. To conclude the proof it suffices to assume that $\varepsilon^4 \ge \sqrt{h}$, and use the maximum principle.

LEMMA. *Let $f(x,t) \in H^{1,1}(\overline{D}_T)$, $u(x,0)$, $u(x,t) \in H^{3,2}(\overline{D}_T)$ and in D_T it satisfies the equation*

$$Lu - \frac{a}{h}u = -\frac{f}{h}.$$

Then for any point $(x_0, t_0) \in D_T : dist(x_0, \partial D) \ge h^\sigma$, $\sigma \in (0, \frac{1}{2})$ the following estimates hold

$$|u(x_0,t_0)| \le c h^{\sigma_1} \max_{(x,t) \in \overline{D}_T} |u(x,t)| + \frac{1}{a} \max_{(x,t) \in \overline{D}_T} |f(x,t)|, \tag{2.2}$$

$$|u_x(x_0,t_0)| + |u_t(x_0,t_0)| \le c h^{\sigma_2} \max_{(x,t) \in \overline{D}_T} |u(x,t)| +$$

$$+\frac{1}{a}\max_{(x,t)\in\overline{D}_T}(|f_x(x,t)|,|f_t(x,t)|), \tag{2.3}$$

where the positive constants c, σ_i do not depend on h.

Note that the estimate (2.3) holds, under appropriate assumptions, for derivatives of an arbitrary order.

Let us apply the estimates obtained above to the functions $\{F_k(x,t,h,\varepsilon)\}$. The relation (2.3) implies that in the domain D_T these functions satisfy the equation

$$\Delta(F_k - F_{k-1}) - \frac{\partial}{\partial t}(F_k - F_{k-1}) - \frac{a}{h}(F_k - F_{k-1}) =$$

$$= -\frac{\lambda}{h}[\chi_\varepsilon(u_k) - \chi_\varepsilon(u_{k-1})] - \frac{1}{2}Q^2(x)\chi'_\varepsilon(u_{k-1}),$$

and if $(x,t) \notin \omega_{k,k-1} = \{u_{k-1} > 1,\ u_k < 1+\varepsilon\}$, then the right–hand side of this equation vanishes. Thus, one can prove the following statement:

THEOREM 2.2. *Let the assumptions of Theorem* 1.3 *hold. Then*

$$\forall (x,t) \in D_T : dist[(x,t), \partial\{u_k(x,t,h,\varepsilon) > 1+\varepsilon\}] \geq h^\sigma$$

the following estimate holds

$$\|u_k(x,t,h,\varepsilon)\|_{H^{2+\alpha,1+\frac{\alpha}{2}}} \leq c,$$

where the constant c *does not depend on* h, ε, k. *If the assumptions of Theorem* 1.4 *hold then*

$$\forall (x,t) \in D_T : dist[(x,t), \partial(D_T \setminus \overline{\omega}_{k,k-1})] \geq h^\sigma$$

for the functions $u_k(x,t,h,\varepsilon)$ *all statements of Theorem* 1.4 *hold. Moreover, the following estimate holds*

$$\min_{1\leq k\leq N,(x,t)\in\overline{D}_T}(w_{k-1} - w_k) - c_1 h^{\sigma_1} \leq u_{k-1} - u_k \leq$$

$$\leq \max_{1\leq k\leq N,(x,t)\in\overline{D}_T}(w_{k-1} - w_k),$$

where the constants c_1, σ_1 *do not depend on* h, ε.

The proof of these statements follows from the estimates (2.2), (2.3).

3. Limit transition. Let $D_{T,T_1} = D_T \times (0, T_1)$, a function $\eta(x,t,\tau) \in C^{2,1,1}(\overline{D}_{T,T_1})$, vanishes on $\partial D \times (0,T]$, η_{x_i} vanishes on ∂D_T. Let us multiply the equation (1.1) by $h\eta(x,t,kh) = h\eta_k$, integrate the result over D_T and sum over k ranging from 1 to N. After simple transformations we get

$$h\sum_{k=1}^{N}\int_{D_T}\left\{-u_k\Delta\eta_k - u_k\frac{\partial\eta_k}{\partial t} + a\frac{u_k - u_{k-1}}{h}\eta_k\right\}dxdt - h\sum_{k=1}^{N}\int_D \psi_k(x,h)\eta_k(x,0)dx+$$

$$+h\sum_{k=1}^{N}\int_{D_T}\lambda\chi_\varepsilon(u_k)\frac{\eta_{k-1}-\eta_k}{h}dxdt-$$

$$-h\sum_{k=1}^{N}\int_{D_T}\frac{1}{2}Q^2(x)\chi'_\varepsilon(u_{k-1})\eta_k dxdt+h\sum_{k=1}^{N}\int_{D_T}(\Delta f_k+\frac{\partial f_k}{\partial t})\frac{F_{k-1}-F_k}{h}dxdt+$$

$$+\lambda\int_{D_T}\chi_\varepsilon(u_0)\eta_1 dxdt=0,\ f_k=-h\sum_{l=k}^{N}\eta_l. \tag{3.1}$$

Denote by $u(x,t,\tau,h,\varepsilon)$ the cubic interpolation of the functions u_k. Suppose that the assumptions of Theorem 2.1 hold. Then the families of functions $\{u(x,t,\tau,h,\varepsilon)\}$, $\{u_\tau(x,t,\tau,h,\varepsilon)\}$ are uniformly bounded, and Theorem 2.2 implies the boundedness from zero of $u_\tau(x,t,\tau,h,\varepsilon)$. Thus, the level surfaces $u(x,t,\tau,h,\varepsilon)=\varepsilon+h^\sigma$ can be given by equations $\tau=s(x,t,h,\varepsilon)$, where the functions $s(x,t,h,\varepsilon)$ belong to the class $H^{1+\alpha,1+\frac{\alpha}{2}}$, and their first order derivatives are uniformly bounded by constants not depending on h,ε.

Let $\varepsilon^4\geq\sqrt{h}$,

$$u(x,t,\tau)=\lim_{\varepsilon,h\to 0}u(x,t,\tau,h,\varepsilon),\ \ s(x,t)=\lim_{\varepsilon,h\to 0}s(x,t,h,\varepsilon).$$

We pass to the limit in the integral identity (3.1) as $h,\varepsilon\to 0$. The limit integral identity implies that the finction $u(x,t,\tau)$ on the surface given by the equation $\tau=s(x,t)$, satisfies the condition

$$|\nabla u^+|^2-|\nabla u^-|^2=\lambda(u_\tau^+ + u_\tau^-)+Q^2(x).$$

Hence, $|\nabla u^+|\neq 0$ for λ small enough. After that, taking into account the results if Corollary 1 of Theorem 1.2 and independence of the uniform estimates for $u(x,t,\tau)$ from a and λ, we can make one more limit transition $a,\lambda\to 0$. As the result we obtain

THEOREM 3.1. *Let the following assumptions hold:*

$$\varphi^i(x,t)\in H^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\partial D_T^i}),\ \psi(x)\in C^{2+\alpha}(\overline{D}),\ \ Q(x)\in C^{1+\alpha}(D),$$

$$Q(x)\neq 0,\ \ \varphi(x,t)<0\ \ on\ \overline{\partial D_1},\ \ \varphi(x,t)>0\ \ \ \overline{\partial D_2},$$

and the corresponding compatibility conditions hold.

Then for all $T>0$ *the problem* (0.1)–(0.4) *is solvable, and*

$$u(x,t)\in C(\overline{D}_T)\cap\{H^{2+\alpha,1+\frac{\alpha}{2}}(\Omega_T\cup\gamma_T\cup\partial D_1)\times$$

$$\times H^{2+\alpha,1+\frac{\alpha}{2}}(G_T\cup\gamma_T\cup\partial D_2)\},$$

γ_T *is a surface of the class* $H^{2+\alpha,1+\frac{\alpha}{2}}$.

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