ATTRACTORS OF MULTIVALUED DYNAMICAL PROCESSES IN TOPOLOGICAL SPACE WITH INVERSE TIME

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Abstract. In this paper we define and study multivalued dynamical processes with inverse time in Hausdorff topological spaces. Existence theorems for attractors of such processes are proved, their topological properties are studied.

In the last years deep results on global attractors for mathematical physics evolution equations [2,7,13] were extended to autonomous evolution equations without uniqueness of solution [3,4] and differential-operator inclusions [5,6,10]. For want of it one essentially used apparatus of multivalued semiflows which are multivalued analogs of one-parameter operators semigroups. In [2] a theory of abstract processes and semiprocesses was developed and authors also gave the applications to nonautonomous evolution equations in partial derivatives. In [8,9,11,12] this theory was generalized on multivalued case and was applied to reaction-diffusion equation without uniqueness and nonautonomous evolution inclusions. In [14] authors considered random dynamical processes, generated by the evolution inclusions with random influence.

In this paper abstract results from [8,9] are extended to multivalued processes with inverse time in a Hausdorff topological spaces.

Let X be a metric (with metric ρ) or to pological vector space, $P(X)$ ($\beta(X)$, $K(X)$) be the set of all nonempty (nonempty bounded, nonempty compact) subsets of the space X, $D \subseteq X$ be a Hausdorff topological space, $\Sigma = \Sigma_1 \times \Sigma_2$ be some set, $R_d = \{(t,s) \in R^2 | t \geq s\}$, $R(\tau) = \{t \in R | t \geq \tau\}$.

Definition 1. The map $U : R_d \times X \mapsto P(X)$ is called a multivalued dynamical process (MDP) on X if:

$$a) U(s,s,\cdot) = I_X \text{ is the identity map on } X \; \forall s \in R;$$
$$b) U(t,s,x) \subset U(t,\tau,U(\tau,s,x)) \; \forall t \geq \tau \geq s, \; \forall x \in X.$$

MDP is called strict if $U(t,s,x) = U(t,\tau,U(\tau,s,x))$.

Statement 1. *Let $\{U_\sigma | \sigma \in \Sigma\}$ be an arbitrary family of MDP. Then the map $U_\Sigma : R_d \times X \mapsto P(X)$, defined by*

$$U_\Sigma(t,s,x) = \bigcup_{\sigma \in \Sigma} U_\sigma(t,s,x)$$

also is MDP.

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Let $\sigma = (\sigma_1;\sigma_2) \in \Sigma = \Sigma_1 \times \Sigma_2$ and for any $\sigma_2 \in \Sigma_2$ consider the map $U_{\Sigma_1\sigma_2} : R_d \times X \mapsto P(X)$, where $U_{\Sigma_1\sigma_2}(t,s,x) = \bigcup_{\sigma_1 \in \Sigma_1} U_{(\sigma_1;\sigma_2)}(t,s,x)$.

DEFINITION 2. Let $t \in R$, $\sigma_2 \in \Sigma_2$. The set $A = A(t,\sigma_2) \subset D$ Σ_1-uniformly $(X{:}\,D)$-attracts the set $B \in \beta(X)$ in the moment t for σ_2, if for any neighbourhood $N(A)$ of the set A in D exists $T = T(t,\sigma_2,B) \leq t$ such, that

$$U_{\Sigma_1\sigma_2}(t,s,B) \cap D \subset N(A) \quad \forall\, s \leq T.$$

This fact we will note

$$U_{\Sigma_1\sigma_2}(t,s,B) \cap D \to A(t,\sigma_2), \quad s \to -\infty. \tag{1}$$

If condition (1) takes place for any $B \in \beta(X)$, then the set $A(t,\sigma_2)$ is called Σ_1-uniformly $(X;D)$-attracting set in the moment t for σ_2.
For $B \subset X$, $\sigma_2 \in \Sigma_2$ and $t \in R$ let us define

$$\gamma^s_{\Sigma_1}(t,\sigma_2,B) = \bigcap_{\tau \leq s} U_{\Sigma_1\sigma_2}(t,\tau,B),$$
$$\omega_{\Sigma_1}(t,\sigma_2,B) = \bigcap_{s \leq t} cl_D(\gamma^s_{\Sigma_1}(t,\sigma_2,B) \cap D),$$

where cl_D is closure in space D. The set $\omega_{\Sigma_1}(t,\sigma_2,B)$ is called Σ_1-uniform ω-limit set of the set $B \subset X$ in the moment $t \in R$ for $\sigma_2 \in \Sigma_2$.

LEMMA 1. *The following statements are equivalent:*

1) $y \in \omega_{\Sigma_1}(t,\sigma_2,B)$;

2) there exists a directedness $\xi_\alpha \in U_{\Sigma_1\sigma_2}(t,\tau_\alpha,B) \cap D$ *such that* $\xi_\alpha \to y$ *in* D *as* $\tau_\alpha \to -\infty$.

THEOREM 1. *Let us suppose that the following condition is true: for any* $t \in R$, $\sigma_2 \in \Sigma_2$ *and* $B \in \beta(X)$ $\exists$ $A(t,\sigma_2,B) \in K(D) \cap \beta(X)$ *such that*

$$U_{\Sigma_1\sigma_2}(t,s,B) \to A(t,\sigma_2,B), \quad s \to -\infty \tag{2}$$

Then $\omega_{\Sigma_1}(t,\sigma_2,B) \neq \emptyset$, $\omega_{\Sigma_1}(t,\sigma_2,B) = cl_D\omega_{\Sigma_1}(t,\sigma_2,B) \subset A(t,\sigma_2,B)$ *and, furthermore,*

$$U_{\Sigma_1\sigma_2}(t,s,B) \cap D \to \omega_{\Sigma_1}(t,\sigma_2,B), \quad s \to -\infty \tag{3}$$

If, moreover, D *is regular space, then* $\omega_{\Sigma_1}(t,\sigma_2,B)$ *is minimal closed set, possessing property (3).*

DEFINITION 3. The family of MDP $\{U_\sigma | \sigma \in \Sigma\}$ is called Σ_1-uniformly asymptotically upper semicompact if for any $t \in R$, $\sigma_2 \in \Sigma_2$, $B \in \beta(X)$ $\exists$ $T = T(t,\sigma_2,B) < t$ such that $\gamma^T_{\Sigma_1}(t,\sigma_2,B) \in \beta(X)$ and any directedness $\{\xi_\alpha\}$, $\xi_\alpha \in U_{\Sigma_1\sigma_2}(t,s_\alpha,B) \cap D$, $s_\alpha \to -\infty$, is precompact in D.

LEMMA 2. *Let for any* $B \in \beta(X)$, $\sigma_2 \in \Sigma_2$, $t \in R$ *exists* $T = T(t,\sigma_2,B) < t$ *such that* $\gamma^T_{\Sigma_1}(t,\sigma_2,B) \in \beta(X)$. *Then the family of MDP* $\{U_\sigma|\sigma \in \Sigma\}$ *is* Σ_1*-uniformly asymptotically upper semicompact if and only if for any* $t \in R$, $\sigma_2 \in \Sigma_2$, $B \in \beta(X)$ $\exists$ $A(t,\sigma_2,B) \in K(D) \cap \beta(X)$ *such that condition (2) is true.*

DEFINITION 4. The set $\Theta_{\Sigma_1}(t,\sigma_2)$ is called Σ_1-uniform global (X,D)-attractor of the family MDP $\{U_\sigma|\sigma \in \Sigma\}$ in the moment t for $\sigma_2 \in \Sigma_2$ if:

1) $\Theta_{\Sigma_1}(t,\sigma_2)$ is Σ_1-uniformly (X,D)-attracting set;

2) $\Theta_{\Sigma_1}(t,\sigma_2) \subset U_{\Sigma_1\sigma_2}(t,s,\Theta_{\Sigma_1}(s,\sigma_2)) \quad \forall\, (t,s) \in R_d$, $\sigma_2 \in \Sigma_2$;

3) for any closed in D Σ_1-uniformly (X, D)-attracting set Y in the moment $t \in R$ for $\sigma_2 \in \Sigma_2$, $\Theta_{\Sigma_1}(t, \sigma_2) \subset Y$.

THEOREM 2. *Let X be a complete metric space (or topological vector space), in which each compact is anywhere undense set, and the family of MDP $\{U_\sigma | \sigma \in \Sigma\}$ is Σ_1-uniformly asymptotically upper semicompact. Then the following statements hold:*

1. If the space D is regular and for any $(t, s) \in R_d$, $\sigma_2 \in \Sigma_2$ graph of the map $D \ni x \mapsto U_{\Sigma_1\sigma_2}(t, s, x) \cap D \in P(D)$ is closed in $D \times D$, then there exists Σ_1-uniform global $(X; D)$-attractor $\Theta_{\Sigma_1}(t, \sigma_2)$,

$$\Theta_{\Sigma_1}(t, \sigma_2) = \bigcup_{B \in \beta(X)} \omega_{\Sigma_1}(t, \sigma_2, B) \neq X$$

and it is Lindelöf in D and locally compact space in sum topology $\tau_\oplus$ on X;

2. If Σ_1 is topological connected space and the map $\Sigma_1 \times X \ni (\sigma_1, x) \mapsto U_{\sigma_1\sigma_2}(t, \tau, x) \cap D \in P(D)$ is upper semicontinuous and has connected values $\forall\ (t, \tau) \in R_d$, $\sigma_2 \in \Sigma_2$, $\Theta_{\Sigma_1}(t, \sigma_2) \subset B_1(t, \sigma_2)$, where $B_1(t, \sigma_2)$ is connected in X, and $\bigcup_{\tau \le t} B_1(\tau, \sigma_2) \in \beta(X)$, then $\forall\ (t, \sigma_2) \in R \times \Sigma_2$ attractor $\Theta_{\Sigma_1}(t, \sigma_2)$ is connected set.

COROLLARY. *If in Theorem 2 for any $(t, \sigma_2) \in R \times \Sigma_2$ $\exists$ $A(t, \sigma_2) \in K(D) \cap \beta(X)$, which is Σ_1-uniformly attracting set, then $\Theta_{\Sigma_1}(t, \sigma_2)$ is compact in D.*

STATEMENT 2. *Let Σ_1 be a compact metric space and for $(t, \tau) \in R_d$, $\sigma_2 \in \Sigma_2$ the map $\Sigma_1 \times D \ni (\sigma_1, x) \mapsto U_{\sigma_1\sigma_2}(t, \tau, x) \cap D \in P(D)$ is closed. Then the map $D \ni x \mapsto U_{\Sigma_1\sigma_2}(t, \tau, x) \cap D \in P(D)$ is closed too.*

Now let us suppose, that one-parameter group $T(h) : \Sigma \mapsto \Sigma$, $h \in R$ is defined, where $\Sigma = \Sigma_1 \times \Sigma_2$, $T(h) = (T_1(h), T_2(h))$, $T_i(h) : \Sigma_i \mapsto \Sigma_i$, $i = 1, 2$, and $\forall\ (t, s) \in R_d$, $x \in X$, $h \in R$ the following property

$$U_{\sigma_1\sigma_2}(t, s, x) \subset U_{T_1(h)\sigma_1 T_2(h)\sigma_2}(t - h, , s - h, x) \tag{4}$$

is fulfilled.

LEMMA 3. *Under conditions, given above, the following equality holds*

$$U_{\sigma_1\sigma_2}(t, s, x) = U_{T_1(h)\sigma_1 T_2(h)\sigma_2}(t - h, s - h, x)$$

Moreover, let the condition

$$T(h)\Sigma \subset \Sigma \quad \forall\ h \in R \tag{5}$$

be held.

DEFINITION 5. Let conditions (4),(5) be held. The family of sets $\{\Theta_{\Sigma_1}(\sigma_2)\}_{\sigma_2 \in \Sigma_2}$ is called the family of Σ_1-uniform global $(X; D)$-attractors of MDP $\{U_\sigma | \sigma \in \Sigma\}$, if:

1. $\Theta_{\Sigma_1}(\sigma_2)$ is Σ_1-uniformly $(X; D)$-attracting set $\forall\ \sigma_2 \in \Sigma_2$ in the moment $t = 0$;
2. For arbitrary $(t, s) \in R_d$, $\sigma_2 \in \Sigma_2$ $\Theta_{\Sigma_1}(T_2(t)\sigma_2) \subset U_{\Sigma_1\sigma_2}(t, s, \Theta_{\Sigma_1}(T_2(s)\sigma_2))$;
3. For any $\sigma_2 \in \Sigma_2$ $\Theta_{\Sigma_1}(\sigma_2)$ is minimal Σ_1-uniformly $(X; D)$-attracting set.

THEOREM 3. *Let conditions (4),(5) be he ld, the space X is from Theorem 2 and $\forall\ B \in \beta(X)$, $\sigma_2 \in \Sigma_2$ $\exists$ $A(\sigma_2, B) \in K(D) \cap \beta(X)$ such that*

$$U_{\Sigma_1\sigma_2}(0, s, B) \cap D \to A(\sigma_2, B), \quad s \to -\infty \tag{6}$$

Then the following statements hold:

1. If the space D is regular and $\forall\ \tau \leq 0,\ \sigma_2 \in \Sigma_2$ graph of the map $X \ni x \mapsto U_{\Sigma_1\sigma_2}(0,\tau,x) \cap D \in P(D)$ is closed in $D \times D$, then there exists the family $\{\Theta_{\Sigma_1}(\sigma_2)\}_{\sigma_2\in\Sigma_2}$ of Σ_1-uniform global attractors and $\forall\ \sigma_2 \in \Sigma_2$

$$\Theta_{\Sigma_1}(\sigma_2) = \bigcup_{B\in\beta(X)} \omega_{\Sigma_1}(0,\sigma_2,B) = \bigcup_{t\in R}\bigcup_{B\in\beta(X)} \omega_{\Sigma_1}(t,\sigma_2,B) \neq X$$

This set is Lindelöf in D and locally compact in the sum topology $\tau_{\oplus}$ on X;
2. If Σ_1 is metric compact connected space and $\forall\ \tau \leq 0,\ \sigma_2 \in \Sigma_2$ the map $\Sigma_1 \times X \ni (\sigma_1, x) \mapsto U_{\sigma_1\sigma_2}(0,\tau,x) \cap D \in P(D)$ is upper semicontinuous, has connected values and

$$\Theta_{\Sigma_1}(T_2(\tau)\sigma_2) \subset B_1(\sigma_2)\ \forall\ \tau \leq 0,$$

where $B_1(\sigma_2)\ \forall\ \sigma_2 \in \Sigma_2$ is connected and bounded set in X, then $\Theta_{\Sigma_1}(\sigma_2)$ is connected set $\forall\ \sigma_2 \in \Sigma_2$.

REMARK. If in condition (6) the sets $A(\sigma_2, B)$ do not depend on $B \in \beta(X)$, then $\Theta_{\Sigma_1}(\sigma_2)$ is compact in D.

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