SPECTRAL PROBLEM FOR A CLASS OF POLYNOMIAL OPERATOR PENCILS

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Abstract. We consider the spectral problem for the class of polynomial operator pencils, defined recursively by $P_n(\lambda) = a_n(A - \lambda B) - \lambda^2 a_{n-1}C$, $a_0 = a_1 = 1$, $a_{n+1} = a_n - \lambda^2 a_{n-1}$, $n = 1, 2, \ldots$, where A, B and C are symmetrizable (in general, unbounded and nonsymmetric) operators in a separable complex Hilbert space H. A method for generating two-sided bounds for the eigenvalues of $P(\lambda)$ is developed and sufficient conditions for the convergence of the method are obtained. The theory is illustrated with a numerical example.

1. Introduction

Spectral problems for polynomial operator pencils of the form $L_\lambda = A_0 - \lambda A_1 - ... - \lambda^n A_n$, where λ is the spectral parameter and $A_0, A_1, ..., A_n$ are linear operators in a Hilbert or Banach space, arise in many areas of application (control theory, wave propagation, hydrodynamics, elasticity theory)and have been the subject of investigation by a number of authors, under various conditions on the operators A_n, $n \geq 1$ (see, e.g. [1,2]). In [3] the quadratic eigenvalue problem $L_\lambda u = 0$, $L_\lambda = A - \lambda B - \lambda^2 C$, was studied in the case when the operators A, B, C are linear, unbounded and, in general, nonsymmetric. Results concerning the existence and approximation of the eigenvalues were derived under aditional conditions of K-symmetry and K-positivity of the operators A, B, C. In this paper we extend these results to spectral problems involving polynomial operator pencils $P_n(\lambda)$ and develop a method for generating two-sided improvable bounds for the eigenvalues.

2. The Setting

Let H be a separable complex Hilbert space with the norm

$$||x|| = (x, x)^{1/2} , \quad (x \in H) \tag{1}$$

Let us define the sequence

$$a_0 = 1,\ a_1 = 1,\ a_2 = a_1 - \lambda^2 a_0, ...,\ a_{n+2} = a_{n+1} - \lambda^2 a_n \tag{2}$$

and consider in H the nonlinear eigenvalue problem

$$a_n(Ax - \lambda Bx) - \lambda^2 a_{n-1} Cx = 0 \ n = 1, 2, ... \tag{3}$$

where A and C are K-p.d. operators with domains $D_C \supseteq D_A$ dense in H, and B is K-symmetric operator with $D_B \supseteq D_C$. By definition of A, B and C (see,e.g.[4,5]) there exists a closable operator K with $D_K \supseteq D_C$ mapping D_A onto a dense subset KD_A of H, and positive constants α_1, α_2, β_1, β_2 such that

$$(Ax, Kx) \geq \alpha_1 ||x||^2 \ , \ ||Kx||^2 \leq \alpha_2(Ax, Kx) \ , \quad (x \in D_A) \tag{4}$$

$$(Cx, Kx) \geq \beta_1 ||x||^2 \ , \ ||Kx||^2 \leq \beta_2(Cx, Kx) \ , \quad (x \in D_C) \tag{5}$$

$$(Bx, Ky) = (Kx, By) \ , (x, y \in D_B) \tag{6}$$

Let H_A be the completion of D_A in the metric (4)

$$(x, y)_A = (Ax, Ky), \quad ||x||_A^2 = (x, x)_A, \quad (x, y \in D_A), \tag{7}$$

and define H_C to be the completion of D_C in the metric (5).

$$(x, y)_C = (Cx, Ky), \quad ||x||_C^2 = (x, x)_C, \quad (x, y \in D_C) \tag{8}$$

Let $H_n = H \times \Pi_{i=1}^n H_C$ be the Cartesian product space of $n+1$ Hilbert spaces, with the norm and inner product defined by

$$(u, v)_n = (x, p) + \sum_{i=1}^{n} (y_i, q_i)_C, \quad u = (x, y_1, ..., y_n)^T \text{ and } v = (p, q_1, ..., q_n)^T \in H_n \tag{9}$$

$$||u||_n = (u, u)_n^{1/2} = \left(||x||^2 + \sum_{i=1}^{n} ||y_i||_C^2 \right)^{1/2} . \tag{10}$$

and define the operator $T : D_T \subseteq H_n \to H_n$, $D_T = D_A \times \Pi_{i=1}^n D_C$, as follows:

$$T = \begin{pmatrix} A & 0 & 0 & \dots & 0 \\ 0 & I & 0 & \dots & 0 \\ . & . & . & & . \\ 0 & 0 & 0 & \dots & I \end{pmatrix}, \quad T \begin{pmatrix} x \\ y_1 \\ \dots \\ y_n \end{pmatrix} = \begin{pmatrix} Ax \\ y_1 \\ \dots \\ y_n \end{pmatrix}, \tag{11}$$

Let $D_S = D_B \times \Pi_{i=1}^n D_C$, $S : D_S \subseteq H_n \to H_n$, $D_{\hat{K}} = D_K \times \Pi_{i=1}^n D_C$, $\hat{K} : D_{\hat{K}} \subseteq H_n \to H_n$ be the operator matrices

$$S = \begin{pmatrix} B & C & 0 & 0 & \dots & 0 & 0 \\ I & 0 & I & 0 & \dots & 0 & 0 \\ 0 & I & 0 & I & \dots & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & \dots & I & 0 \end{pmatrix}, \tag{12}$$

$$\hat{K} = \begin{pmatrix} K & 0 & 0 & \dots & 0 \\ 0 & I & 0 & \dots & 0 \\ . & . & . & & . \\ 0 & 0 & 0 & \dots & I \end{pmatrix}, \tag{13}$$

3. The equivalent linear problem

Our original nonlinear eigenproblem (3) is equivalent to the system

$$\begin{aligned} Ax - \lambda Bx - \lambda Cy_1 &= 0 \\ y_1 - \lambda x - \lambda y_2 &= 0 \\ y_2 - \lambda y_1 - \lambda y_3 &= 0 \\ &\dots \\ y_n - \lambda y_{n-1} &= 0 \end{aligned} \tag{14}$$

which, in view of (11) and (12), is equivalent to the linear equation

$$Tu - \lambda Su = 0 \tag{15}$$

in the sense that if x_i is a solution of (3) corresponding to $\lambda = \lambda_i$, then $u_i = (x_i, y_i^1, \dots, y_i^n)^T$ is a solution of (15) and, conversely, if u_i is a solution of (15) corresponding to $\lambda = \lambda_i$, then (x_i, λ_i) is a solution of (3). The following propositions are based on the corresponding results obtained in [3,4], and can be proved similarly.

Proposition 1. *The operator T defined by (11) is $\hat{K}$-p.d. in the space $H_n = H \times \Pi_{i=1}^n H_C$; i.e., T satisfies the following conditions:*

(a) D_T is dense in H_n.

(b) $D_{\hat{K}} \supseteq D_T$ and $\hat{K} D_T$ is dense in H_n.

(c) $\hat{K}$ is closable in H_n.

(d) There exist positive constants γ_1, γ_2 such that

$$(Tu, \hat{K}u)_n \geq \gamma_1 ||u||_n^2 \ , \ ||\hat{K}u||_n^2 \leq \gamma_2 (Tu, \hat{K}u)_n \ , \quad (u \in D_T). \tag{16}$$

Let us introduce in D_T a new inner product and norm

$$(u, v)_N = (Tu, \hat{K}v)_n = (x, p)_A + \sum_{i=1}^n (y_i, q_i)_C \ , \ ||u||_N{}^2 = ||x||_A^2 + \sum_{i=1}^n ||y_i||_C^2. \tag{17}$$

where $u = (x, y_1, \dots, y_n)^T$, $v = (p, q_1, \dots, q_n)^T$.

From (16) we have the inequalities

$$||u||_N \geq \sqrt{\gamma_1} ||u||_n \ , \ \ ||\hat{K}u||_n \leq \sqrt{\gamma_2} ||u||_N \ , \quad (u \in D_T) \tag{18}$$

Let us denote by H_N the completion of D_T in the metric (17).

Proposition 2. *The Hilbert space H_N has the properties*

(a) $H_N = H_A \times \Pi_{i=1}^n H_C$.

(b) H_N is contained in H_n in the sense of identifying uniquely the elements from H_N with certain elements in H_n.

(c) $\hat{K}$ can be extended to a bounded operator $\hat{K}_0$ mapping all of H_N to H_n such that $\hat{K} \subset \hat{K}_0 \subset \overline{\hat{K}}$, where $\overline{\hat{K}}$ denotes the closure of $\hat{K}$ in H_n.

(d) T has a unique closed $\hat{K}_0$-p.d. extension T_0 such that $T_0 \supseteq T$, T_0 has a bounded inverse T_0^{-1} defined on all of $H_n = R_{T_0}$, and the inequalities (17) remain valid in H_N in the form

$$||u||_N \geq \sqrt{\gamma_1} ||u||_n \ , \ \ ||\hat{K}_0 u||_n \leq \sqrt{\gamma_2} ||u||_N \ , \quad (u \in H_N). \tag{19}$$

In the sequel we shall assume, unless otherwise stated, that the operators $\hat{K}$ and T have already been extended and the notation T_0 and $\hat{K}_0$ will not be used.

PROPOSITION 3. *The equivalent linear problem (15) $Tu - \lambda Su = 0$ with T, S, and $\hat{K}$ defined as in (11)-(13) has the property that T is $\hat{K}$-p.d. and S is $\hat{K}$-symmetric on D_T. Thus by definition [3], problem (15) is $\hat{K}$-real.*

Proof. In view of Proposition 1, only the $\hat{K}$-symmetry of S needs to be verified. To this end let $u = (x, y_1, ..., y_n)^T$ and $v = (p, q_1, ..., q_n)^T$ be elements in $D_T \subseteq H_n$ and by using K-symmetry of the operators B and C on $D_A \subseteq H$, it follows that

$$(Su, \hat{K}v)_n = (\hat{K}u, Sv)_n\,, \quad (u, v \in D_T) \tag{20}$$

It is known [3,4] that $\hat{K}$-real eigenvalue problems have the following properties. In particular, the eigenvalues of problem (15) are real, and the eigenfunctions u_1, u_2 corresponding to distinct eigenvalues λ_1, λ_2 are orthogonal in the sense $(Tu_1, \hat{K}u_2)_n = 0$. Since H_n is separable, the point spectrum of problem (15), i.e. $p\sigma(15)$ is countable, and from the equivalence of problems (15) and (3)it follows that $p\sigma(15) = p\sigma(3)$.

Let $\{u_i \,:\, i = 1, 2, ...\}$ be the set of eigenfunctions, orthonormal in H_n, of the $\hat{K}$-real eigenproblem (15) $Tu - \lambda Su = 0$ in H_n, which is equivalent to the problem (3) in H. Using the methods developed in the theory of K-real eigenvalue problems, we may now derive theorems 1 and 2, which extend to the problems (3) the corresponding results obtained in [3].

THEOREM 1. *Suppose the operators K and $L_\lambda = a_n(Ax - \lambda Bx) - \lambda^2 a_{n-1}Cx$ defined in (3) are closed, with $D_K = D_C$ and $L_\lambda : D_A \subseteq H \to H$ is a bijection for all λ, except possibly for a discrete set of eigenvalues of the problem (3). Then the equivalent $\hat{K}$-real eigenproblem (15) $Tu - \lambda Su = 0$ has the following properties:*

(a) The eigenvalues and eigenfunctions satisfy the variational principle

$$\frac{1}{|\lambda_m|} = \sup_{u \in D_T} \{ \frac{|(Su, \hat{K}u)_n|}{(Tu, \hat{K}u)_n} \,:\, (Tu, \hat{K}u_i)_n = 0,\; 1 \le i \le m-1 \} = \frac{|(Su_m, \hat{K}u_m)_n|}{(Tu, \hat{K}u_m)_n} \tag{21}$$

Moreover, the eigenvalues determined by (21) exhaust entirely the set $p\sigma(15)$.

(b) If $u \in D_T$, then $T^{-1}Su$ has the expansion (convergent in the H_n and H_N norm):

$$T^{-1}Su = \sum_{i=1}^{\infty} (Su, \hat{K}u_i)_n u_i \tag{22}$$

4. EIGENVALUE APPROXIMATION METHOD

Let $f_0 = (x_0, y_0^1, ..., y_0^n)^T$ be an element in D_T such that $f_0 \notin N(S)$ (the null space of S), and denote by $f_k = (x_k, y_k^1, ..., y_k^n)^T$ the iterant at the kth step of our process. Then the succeeding iterant f_{k+1} is obtained by solving the equation $Tf_{k+1} = Sf_k$, i.e.

$$\begin{pmatrix} A & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & I & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & I & 0 & ... & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & ... & 0 & I \end{pmatrix} \begin{pmatrix} x_{k+1} \\ y_{k+1}^1 \\ y_{k+1}^2 \\ ... \\ y_{k+1}^n \end{pmatrix} = \begin{pmatrix} B & C & 0 & 0 & ... & 0 & 0 \\ I & 0 & I & 0 & ... & 0 & 0 \\ 0 & I & 0 & I & ... & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & ... & I & 0 \end{pmatrix} \begin{pmatrix} x_k \\ y_k^1 \\ y_k^2 \\ ... \\ y_k^n \end{pmatrix}. \tag{23}$$

Now, let us determine the constants

$b_k = (Sf_{k-i}, \hat{K}f_i)_n = (Bx_{k-i}, Kx_i) + (Cy_{k-i}^1, Kx_i) + (Cx_{k-i}, Ky_i^1) +$

$$\sum_{j=2}^{n}[(Cy_{k-i}^{j-1}, Ky_i^j) + (Cy_i^j, Ky_{k-i}^{j-1})] \; (0 \le i \le k, \; k = 1, 2, \dots). \tag{24}$$

Let H_N^i be the space spanned by the eigenfunction u_i, $(H_N^i)^\perp$ be the orthogonal complement of H_N^i in H_N, and let $w_k = b_{2k-1}/b_{2k+1}$.

THEOREM 2. *Assume the hypothesis of Theorem 1 and suppose that $|\lambda_r| < |\lambda_{r+1}|$ for some positive integer r. If f_0 is chosen from the space*

$$f_0 \in D_T \,\cap\, [\cap_{i=1}^{r-1}(H_N^i)^\perp], \; f_0 \notin (H_N^r)^\perp, \quad r \ge 1. \tag{25}$$

then the following statements are true:

(a) the sequence $\{\sqrt{w_k}\}$ converges monotonicly from above to $|\lambda_r|$,

(b)If l_{r+1} is a lower bound f or $|\lambda_{r+1}|$ such that for some integer M, $\sqrt{w_M} \le l_{r+1} \le |\lambda_{r+1}|$, then

$$|\lambda_r| \ge d_k = \{(l_{r+1}^2 - w_k)w_{k+1}/(l_{r+1}^2 - w_{k+1})\}^{1/2}$$

for $k \ge M$. Moreover, the sequence of lower bounds converges to $|\lambda_r|$.

5. NUMERICAL EXAMPLE

Consider the nonlinear eigenvalue problem

$$(1 - \lambda^2)(x''' - \lambda p(t)x') - \lambda^2 q(t)x' = 0 \tag{26}$$

where $p(t)$, $q(t)$ are polynomials in t. Let us define the operators A, B and C in $H = L_2(0,1)$ as follows:

$$Ax = -x''', D_A = \{x \in C'''(0,1) : x(0) = x'(0) = x''(1) = 0\} \tag{27}$$

$$Bx = p(t)x', D_B = \{x \in C'(0,1) : x(0) = 0\} \tag{28}$$

$$Kx = x', D_K = \{x \in C'(0,1) : x(0) = 0\} \tag{29}$$

It is easy to see that this eigenproblem, expressed in the form $a_2(Ax - \lambda Bx) - \lambda^2 a_1 Cx = 0$, is K-real and satisfies the conditions of Theorem 1.

First, we shall test our method on the problem (26) in the case $p(t) = q(t) = 1$, (when the value of $|\lambda_1|$ can be determined exactly) and compare our numerical approximation with the exact value.

Applying our iterative method and proceeding fifty iterations, we get the decreasing upper bounds for the eigenvalue $|\lambda_1|$ (we show the first two and the last two iterations):

$$w_1 = 0.810169, w_2 = 0.794488, w_{49} = 0.79139675, w_{50} = 0.79139672.$$

The exact value of $|\lambda_1|$ is given by the solution of the equation $-\lambda^3 + \lambda + (1+\pi^2)/4\lambda^2 - \pi^2/4 = 0$ and is equal to 0.79139658388 (within an accuracy of 10^{-7}).

Now, let us consider problem (26) with polynomials $p(t) = 0.4 + t^2$, $q(t) = 1 + t$. The first fifty iterations give us the following approximations of the eigenvalue $|\lambda_1|$ from above (we show only the first and the last two iterations) :

$$w_1 = 0.776500, w_2 = 0.718666, w_{49} = 0.715846033, w_{50} = 0.715846030.$$

Although the exact solution of this problem is unknown, we can approximate the eigenvalue $|\lambda_1|$ from below using Theorem 2. With l_2 as 0.8, we get the following approximations converging to $|\lambda_1|$ from below (we list the first two and the last two of fifty iterations):

$$d_1 = 0.393587, d_2 = 0.709282, d_{49} = 0.7158460183, d_{50} = 0.7158460187,$$

We see that the approximations to $|\lambda_1|$ from below and above bracket the eigenvalue within an accuracy of 10^{-7} after fifty iterations.

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