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# ON SOLVABILITY FOR EVOLUTIONARY VARIATIONAL INEQUALITY WITH SET-VALUED GENERALIZED PSEUDOMONOTONE OPERATORS

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Abstract. The sufficient conditions for weak and strong generalized solvability of evolutionary variational inequalities with set-valued operators are proposed. We consider the operators of pseudomonotone type. We apply this theory to the study of variational inequalities which are perturbed by convex functionals.

The variational inequalities play an important role in the investigation of many systems in mechanics, physics, economics etc. in the case of one-side restrictions. For a single-valued operator the solvability of evolutionary variational inequalities was considered in [1]. In [2] these results were generalized to a wider class of operators. We obtain the sufficient conditions for weak and strong generalized solvability of evolutionary variational inequalities with set-valued operators. Two classes of operators are considered: the generalized pseudomonotone operator and the operators of $(X,W)$–semibounded variation. Such operators arise in control systems with indeterminacy or insufficient smoothness, see [3].

Let $V$ be a reflexive Banach space, $V^*$ its dual space with respect to some Hilbert space $H$, $T \in (0,+\infty)$, $p \in [2,\infty)$. Then $X = L_p(0,T;V)$ is reflexive Banach too, $X^*$ is dual for $X$, $L_2(0,T;H)$, $\langle\cdot,\cdot\rangle : X \times X^* \to \mathbb{R}$ is the duality, $Conv(X^*)$ is the totality of convex closed sets from $X^*$. Denote by $A : X \to Conv(X^*)$ a closed-convex-set-valued operator with $Dom(A) = X$. The upper and lower support functions and upper norm on $Conv(X^*)$ are defined by the formulae

$$[A(y),\xi]_+ = \sup_{d\in A(y)} \langle d,\xi\rangle, \qquad [A(y),\xi]_- = \inf_{d\in A(y)} \langle d,\xi\rangle,$$

and upper norm on $Conv(X^*)$ is defined by $\|A(y)\|_+ = \sup\limits_{d\in A(y)} \|d\|_{X^*}$.

Taking into account that the support functions define an operator to within a convex closure of values (see Lemma 1 [4]), this theory holds for any set-valued operator.

Let us define the reflexive Banach space $W$ with the graph norm of the operator $\partial_t$ by the formula

$$W = \{y \in X : \partial_t y \in X^*\} \quad \text{with} \quad \|y\|_W = \|y\|_X + \|\partial_t y\|_{X^*}.$$

*Key words and phrases.* evolutionary variational inequality, generalized pseudomonotone operator, operator of $(X,W)$–semibounded variation.

We also consider the semi-norm $\|\cdot\|'_W$. This semi-norm is defined for any $y \in X$, $\|\cdot\|'_W$ is continuous with respect to $\|\cdot\|_X$ and is compact with respect to $\|\cdot\|_W$.

Let $G$ be a semi-group generated by $\partial_t$, $\mathbb{K} \subset V$ be a closed convex set satisfying the agreement condition with respect to $G$

$$G(s)\mathbb{K} \subset \mathbb{K} \qquad \forall s \geq 0 \tag{1}$$

(see [1]). We consider the strong evolutionary variational inequality

$$\begin{gathered} \langle \partial_t y, \xi - y\rangle + [A(y), \xi - y]_+ \geq \langle f, \xi - y\rangle \qquad \forall \xi \in K, \\ \text{where} \quad K = \{\zeta \in X : \zeta(t) \in \mathbb{K}\,\text{a.e.}, \zeta_{|t=0} = \zeta^0\}, \\ \zeta^0 \in \mathbb{K} \quad \text{is an initial function.} \end{gathered} \tag{2}$$

By virtue of Theorem 2.9.1 [1], the condition (1) is equivalent to the following one: for any $y \in K$ there exists sequence $W \cap K \ni \zeta_i \to y$ in $X$ such that $\overline{\lim\limits_{i\to\infty}} \langle \partial_t \zeta_i, \zeta_i - y\rangle \leq 0$.

Note that in form (2) we require the additional smoothness: $\partial_t y \in X^*$. We can omit this restriction if we consider the weak variational inequality

$$\langle \partial_t \xi, \xi - y\rangle + [A(y), \xi - y]_+ \geq \langle f, \xi - y\rangle \qquad \forall \xi \in K \cap W. \tag{3}$$

Definition 1. A mapping $A : X \to Conv(X^*)$ is said to be *generalized pseudomonotone* if for arbitrary $\{(y_n, w_n)\} \subset graph(A)$ such that $y_n \to y$ weakly in $X$, $w_n \to w$ weakly in $X^*$ and $\overline{\lim\limits_{n\to\infty}} \langle w_n, y_n - y\rangle \leq 0$, we have $w \in A(y)$ and $\langle w_n, y_n\rangle \to \langle w, y\rangle$.

Definition 2. A mapping $A : X \to Conv(X^*)$ is said to be *monotone* if for any $\{(y_n, w_n)\} \subset graph(A)$ ($n = 1, 2$) we have that $\langle w_1 - w_2, y_1 - y_2\rangle \geq 0$. A monotone mapping is *maximal monotone* if its graph is not subset of some other monotone operator's graph.

Definition 3. $A : X \to Conv(X^*)$ is an operator *of* $(X, W)$ *–semibounded variation* if for any $R > 0$ and $\|y_i\|_X \leq R$ ($i = 1, 2$) there exists a continuous function $C : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$[A(y_1), y_1 - y_2]_- \geq [A(y_2), y_1 - y_2]_+ - C(R; \|y_1 - y_2\|'_W), \tag{4}$$

$\tau^{-1}C(R; \tau h) \to 0$ as $\tau \to +0$ for arbitrary $h, R > 0$, $\|\cdot\|'_W$ is a compact semi-norm with respect to $\|\cdot\|_W$ and is continuous with respect to $\|\cdot\|_X$.

Definition 4. A mapping $A : X \to Conv(X^*)$ is said to be *radially semicontinuous* if $\underline{\lim\limits_{\tau\to+0}} [A(y_0 + \tau\xi), h]_+ \geq [A(y_0), h]_-$ for any $y_0, \xi, h \in X$. A mapping is *demicontinuous* if it is continuous from strong topology of $X$ to weak topology of $X^*$.

Definition 5. A mapping $A : X \to Conv(X^*)$ is said to be *bounded (s–weakly locally bounded)* if an image of bounded set is bounded too (if for any $y_n \to y$ weakly in $X$ there exists the subsequence $\{y_{n_k}\}$ such that $\|A(y_{n_k})\|_+ \leq N$).

Definition 6. A mapping $A : X \to Conv(X^*)$ is said to be *coercive on* $K$ if there exist $y_0 \in K$ and $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$[A(y), y - y_0]_- \geq c(\|y\|_X)\|y - y_0\|_X, \quad c(\gamma) \to \infty \text{ as } \gamma \to \infty. \tag{5}$$

DEFINITION 7. A mapping $A : X \to Conv(X^*)$ is said to be — $\varphi$*-coercive on* $K$ if there exists $y_0 \in K$ and $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$[A(y), y - y_0]_- + \varphi(y) \geq c(\|y\|_X)\|y - y_0\|_X, \quad c(\gamma) \to \infty \text{ as } \gamma \to \infty .$$

REMARK. This coercivity condition can be modified (see §4.4 [3]).

DEFINITION 8. A functional $\varphi : X \to \overline{\mathbb{R}}$ is said to be *weakly lower semicontinuous* if $\varliminf_{n\to\infty} \varphi(y_n) \geq \varphi(y)$ as $y_n \to y$ weakly in $X$.

THEOREM 1. *Let* $X$ *be a reflexive Banach space;* $A : X \to Conv(X^*)$ *s–weakly locally bounded and generalized pseudomonotone;* $\mathbb{K}$ *convex and closed. Assume that the agreement condition (1) holds. Suppose also that* $\mathbb{K}$ *is bounded or* $A$ *is coercive on* $K \cap W$.

*Then variational inequality (3) has a nonempty weakly compact set of solutions. If* $A$ *is coercive, then any solution satisfies the following estimate*

$$c(\|y\|_X) \leq \|f\|_{X^*} + \|\partial_t y_0\|_{X^*}, \tag{6}$$

*where* $c, y_0$ *are defined in (5).*

*Proof.* We consider the additional inequality

$$\langle h^{-1}(I - G(h))y_h, \xi - y_h\rangle + [A(y_h), \xi - y_h]_+ \geq \langle f, \xi - y_h\rangle \qquad \forall \xi \in K. \tag{7}$$

We define $B = h^{-1}(I - G(h)) + A : X \to Conv(X^*)$. The map $h^{-1}(I - G(h))$ is linear, maximal monotone and bounded for any $h > 0$:

$$\langle h^{-1}(I - G(h))\varphi, \varphi\rangle \geq 0 \qquad \forall \varphi. \tag{8}$$

Hence $B$ is s–weakly locally bounded and generalized pseudomonotone too. If $K$ is bounded (for a finite $T$ the sets $K$ and $\mathbb{K}$ are bounded simultaneously), then variational inequality (7) is solvable (see Theorem 1 [4]) and the solution set $\{y_h\} \subset K$ is bounded in $X$. If $K$ is not bounded and $A$ is coercive, we have

$$[B(y), y - y_0]_- = \langle \frac{(I - G(h))}{h}(y - y_0), y - y_0\rangle + [A(y), y - y_0]_- +$$
$$+ \langle \frac{(I - G(h))}{h} y_0, y - y_0\rangle \geq \left( c(\|y\|_X) - \|\frac{(I - G(h))}{h} y_0\|_{X^*} \right) \|y - y_0\|_X.$$

This implies that $B$ is coercive and (7) is solvable (see Theorem 2 [4]). Moreover, from the last inequality we obtain

$$\|f\|_{X^*}\|y_h - y_0\|_X \geq \langle f, y_h - y_0\rangle \geq [B_h(y_h), y_h - y_0]_- \geq$$
$$\geq \left( c(\|y_h\|_X) - \|h^{-1}(I - G(h))y_0\|_{X^*} \right) \|y_h - y_0\|_X,$$

i.e.

$$c(\|y_h\|_X) \leq \|f\|_{X^*} + \|h^{-1}(I - G(h))y_0\|_{X^*}. \tag{9}$$

Hence the solutions set $\{y_h\}$ is bounded in $X$. Thus for any $h > 0$ there exists $w_h \in A(y_h)$ such that

$$\langle h^{-1}(I - G(h))y_h, \xi - y_h\rangle + \langle w_h, \xi - y_h\rangle \geq \langle f, \xi - y_h\rangle \qquad \forall \xi \in K. \tag{10}$$

Since $A$ is s–weakly locally bounded, we can consider the weakly convergent subsequences which we denote by the same symbols:

$$y_h \to y \quad \text{weakly in } X, \qquad\qquad w_h \to \chi \quad \text{weakly in } X^*.$$

Using (10) and (8) for $\varphi = \xi - y_h$, we have

$$\langle h^{-1}(I - G(h))\xi, \xi - y_h\rangle + \langle w_h, \xi - y_h\rangle \geq \langle f, \xi - y_h\rangle \quad \forall \xi \in K. \tag{11}$$

Substituting $\xi \in K \cap W$ in (11), we get

$$\overline{\lim_{h\to 0}}\langle w_h, y_h - y\rangle \leq \lim_{h\to 0}\langle h^{-1}(I - G(h))\xi, \xi - y_h\rangle + \lim_{h\to 0}\langle w_h, \xi\rangle -$$
$$- \lim_{h\to 0}\langle f, \xi - y_h\rangle = \langle \partial_t\xi, \xi - y\rangle + \langle \chi, \xi\rangle - \langle f, \xi - y\rangle.$$

But for any $y \in K$ there exists $\{y_i\} \subset K \cap W$ such that $\overline{\lim_{i\to\infty}}\langle \partial_t y_i, y_i - y\rangle \leq 0$ (see the agreement condition (1)). It follows that

$$\inf_{\xi\in K\cap W}\langle \chi - f + \partial_t\xi, \xi - y\rangle \leq \overline{\lim_{i\to\infty}}\langle \chi - f + \partial_t y_i, y_i - y\rangle \leq 0.$$

Consequently $\overline{\lim_{h\to 0}}\langle w_h, y_h - y\rangle \leq 0$. For generalized pseudomonotone operator $A$ this means that $\chi \in A(y)$ and $\langle w_h, y_h\rangle \to \langle \chi, y\rangle$. Thus,

$$\langle \partial_t\xi + \chi, \xi - y\rangle \geq \langle f, \xi - y\rangle \qquad \forall \xi \in K \cap W.$$

Moreover, from last result and from estimate (9) we obtain that (6) is true.

It remains to prove that the solution set is weakly compact. Let $y_n \in K$ be solutions of (3), $y_n \to y$ weakly in $X$. Then there exists $\zeta_i$ such that $(W \cap K) \ni \zeta_i \to y$ in $X$ and $\overline{\lim_{i\to\infty}}\langle \partial_t\zeta_i, \zeta_i - y\rangle \leq 0$ (see (1)). Hence,

$$\langle \partial_t\xi, \xi - y_n\rangle + [A(y_n), \xi - y_n]_+ = \langle \partial_t\xi + \chi_n, \xi - y_n\rangle \geq \langle f, \xi - y_n\rangle \quad \forall \xi \in K \cap W.$$

For s–weakly locally bounded mapping $A$ there exists the subsequences $\chi_n \to \chi$ which converges weakly in $X^*$. Thus,

$$\overline{\lim_{n\to\infty}}\langle \chi_n, y_n - y\rangle \leq \overline{\lim_{n\to\infty}}\ \overline{\lim_{\{\zeta_i\}}}\langle \chi_n, y_n - \zeta_i\rangle \leq$$
$$\leq \overline{\lim_{i\to\infty}}\ \overline{\lim_{\{\zeta_i\}}}\langle f - \partial_t\zeta_i, y_n - \zeta_i\rangle =$$
$$= \lim_{\{\zeta_i\}}\langle \partial_t\zeta_i, \zeta_i - y\rangle \leq \overline{\lim_{i\to\infty}}\langle \partial_t\zeta_i, \zeta_i - y\rangle \leq 0.$$

By assumption, the operator $A$ is generalized pseudomonotone. This proves that $\langle \chi_h, y_h \rangle \to \langle \chi, y \rangle$ and $y$ is a solution of variational inequality (3). ■

Let us define the additional condition of regularity

$$\text{for any } \varepsilon > 0 \text{ and some } g \in X^* \text{ the following inclusion is solvable}$$
$$\partial_t y_\varepsilon + A(y_\varepsilon) + \frac{1}{\varepsilon} J(y_\varepsilon - y) \ni g, \qquad y_{\varepsilon|t=0} = \zeta^0, \qquad y_\varepsilon \in K, \tag{12}$$
$$\text{where } J(y) = \{w \in X^* : \langle w, y \rangle = \|y\|_X^2 = \|w\|_{X^*}^2\}.$$

THEOREM 2. *Let $X$ be a reflexive Banach space; $A : X \to Conv(X^*)$ radially semicontinuous operator of $(X, W)$ -semibounded variation; $\mathbb{K}$ convex and closed; the agreement condition (1) holds; $K = \{\zeta \in X : \zeta(t) \in \mathbb{K}\, a.e., \zeta_{|t=0} = \zeta^0\}$. Moreover, $A$ and $K$ satisfy one of the following conditions: $\mathbb{K}$ is bounded or $A$ is coercive on $K$. And let $y$ be a solution of (3) which satisfies the additional condition of regularity (12). Then $y$ is a solution of (2).*

*Proof.* Let $y$ be a solution of (3), and for any $\varepsilon > 0$ there exists $y_\varepsilon \in K$ which satisfies (12). Since $A$ is operator of $(X, W)$ -semibounded variation, then

$$\begin{aligned}\langle \partial_t y_\varepsilon - f, y_\varepsilon - y \rangle + [A(y_\varepsilon), y_\varepsilon - y]_- = \langle \partial_t y_\varepsilon - f, y_\varepsilon - y \rangle + \\ + [A(y_\varepsilon), y_\varepsilon - y]_- - [A(y), y_\varepsilon - y]_+ + \\ + [A(y), y_\varepsilon - y]_+ \geq -C(R; \|y_\varepsilon - y\|'_W).\end{aligned}$$

On the other hand, $y_\varepsilon$ satisfies the inclusion (12)

$$\begin{aligned}\langle \partial_t y_\varepsilon - f, y_\varepsilon - y \rangle + [A(y_\varepsilon), y_\varepsilon - y]_- \leq \langle g - \varepsilon^{-1} J(y_\varepsilon - y) - f, y_\varepsilon - y \rangle = \\ = \langle g - f, y_\varepsilon - y \rangle - \varepsilon^{-1} \|y_\varepsilon - y\|_X^2.\end{aligned}$$

Thus $\quad \varepsilon^{-1} \|y_\varepsilon - y\|_X^2 \leq C(R; \|y_\varepsilon - y\|'_W) + \|g - f\|_{X^*} \|y_\varepsilon - y\|_X,$

$$\varepsilon^{-1} \|y_\varepsilon - y\|_X \leq \|y_\varepsilon - y\|_X^{-1} C(R; \|y_\varepsilon - y\|'_W) + \|g - f\|_{X^*}. \tag{13}$$

If $A$ is coercive then

$$\begin{aligned}[A(y_\varepsilon), y_\varepsilon - y_0]_- + \varepsilon^{-1} \langle J(y_\varepsilon - y), y_\varepsilon - y_0 \rangle \geq c(\|y_\varepsilon\|_X) \|y_\varepsilon - y_0\|_X + \\ + \varepsilon^{-1} \left( \|y_\varepsilon - y\|_X^2 - \|y_\varepsilon - y\|_X \|y_0 - y\|_X \right) \to \infty \quad \text{as } \|y_\varepsilon\|_X \to \infty.\end{aligned}$$

Hence the solution set $\{y_\varepsilon\}$ is bounded in $X$ and in topology which is indicated by $\|\cdot\|'_W$ (it is continuous with respect to topology of $X$ ). If $K$ is bounded this estimate is trivial. Using (13) and continuity of function $C$, we get $\varepsilon^{-1} \|y_\varepsilon - y\|_X \leq C_1 < \infty$. It follows that $\|y_\varepsilon - y\|_X \leq \varepsilon C_1 \to 0$ as $\varepsilon \to 0$. For s–weakly locally bounded operator $A$ there exists the subsequence $\{y_{\varepsilon_n}\} \subset \{y_\varepsilon\}$ such that $\{A(y_{\varepsilon_n})\}$ are bounded in totality, i.e. $w_{\varepsilon_n} \in A(y_{\varepsilon_n})$ (such that $\partial_t y_{\varepsilon n} + w_{\varepsilon n} + \varepsilon_n^{-1} J(y_{\varepsilon n} - y) = g$ ) are bounded in totality too. Thus,

$$\|\partial_t y_\varepsilon + w_\varepsilon\|_{X^*} \leq \|g\|_{X^*} + \varepsilon^{-1} \|y_\varepsilon - y\|_X. \tag{14}$$

From (14) $\{\partial_t y_{\varepsilon_n}\}$ are bounded too. We can choose the subsequence $\{y_{\varepsilon_n}\}$ such that $y_{\varepsilon_m} \to \zeta$ weakly in $W$. The weak limit is unique. Hence $\zeta \equiv y$. ■

COROLLARY 1. *Let $X$ be a reflexive Banach space; $A : X \to Conv(X^*)$ be maximal monotone. Let also $\mathbb{K}$ be convex and closed; the agreement condition (1) holds. Assume also that $\mathbb{K}$ is bounded or $A$ is coercive on $K \cap W$.*

*Then variational inequality (2) has a nonempty weakly compact set of solutions. If $A$ is coercive on $K \cap W$, then any solution satisfies estimate (6) where $c, y_0$ are defined in (5).*

*Proof.* A maximal monotone operator is radially semicontinuous, s–weakly locally bounded, generalized pseudomonotone and monotone (see [4,6,7]). A monotone operator is a mapping of $(X, W)$–semibounded variation. By Theorem 2.1 [5], the condition of regularity (12) holds. Thus the conditions of Theorems 1 and 2 hold too. ■

COROLLARY 2. *Let $X$ be a reflexive Banach space; $A : X \to X^*$ be s–weakly locally bounded and generalized pseudomonotone, $\varphi : X \to \overline{\mathbb{R}}$ be convex, weakly lower semicontinuous and strong; $\mathbb{K}$ be convex and closed; the agreement condition (1) holds. Assume also that $\mathbb{K}$ is bounded or $A$ is $\varphi$–coercive on $K \cap W$.*

*Then variational inequality*

$$\langle \partial_t \xi, \xi - y\rangle + \langle A(y), \xi - y\rangle + \varphi(\xi) - \varphi(y) \geq \langle f, \xi - y\rangle \qquad \forall \xi \in K$$

*has a nonempty weakly compact set of solutions.*

*If, additionally, $A$ is a mapping of $(X, W)$–semibounded variation and the condition of regularity (12) holds, then*

$$\langle \partial_t y, \xi - y\rangle + \langle A(y), \xi - y\rangle + \varphi(\xi) - \varphi(y) \geq \langle f, \xi - y\rangle \qquad \forall \xi \in K$$

*has a nonempty weakly compact set of solutions.*

*Proof.* The subdifferential $\partial\varphi : X \to Conv(X^*)$ is maximal monotone. Thus we can use Theorems 1 and 2 for variational inequality with operator $A + \partial\varphi$. ■

EXAMPLE. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with regular boundary $\partial\Omega$, $Q = [0,T] \times \Omega$. We consider problems in Sobolev space $X = \{y \in L_p(0,T; W_p^1(\Omega))\}$, where $p \in [2, \infty)$. Here $L_p(Q) = L_p(0,T; L_p(\Omega))$. We denote $1/q + 1/p = 1$. Then $W = \{y \in L_p(0,T; W_p^1(\Omega)) : \partial_t y \in L_q(0,T; W_q^{-1}(\Omega))\}$.

The variational inequality with convex, weakly lower semicontinuous, strong functional $\varphi : L_p(0,T; W_p^1(\Omega)) \to \overline{\mathbb{R}}$ has the form

$$\int_Q \partial_t y(\xi - y)\, dx\, dt + \sum_{i=1}^{n} \int_Q \left|\frac{\partial y}{\partial x_i}\right|^{p-2} \frac{\partial y}{\partial x_i} \frac{\partial(\xi - y)}{\partial x_i}\, dx\, dt +$$

$$+ \varphi(\xi) - \varphi(y) \geq \int_Q f(\xi - y)\, dx\, dt \qquad \forall \xi \in K, \tag{15}$$

$$K = \{\zeta \in L_p(0,T; W_p^1(\Omega)) : \zeta_{|t=0} = 0,\ \zeta_{|x \in \partial\Omega} \geq 0\}.$$

Here we have variational inequality (2) with operator $A = A_1 + A_2$, where

$$A_1(y) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \left| \frac{\partial y}{\partial x_i} \right|^{p-2} \frac{\partial y}{\partial x_i} \right), \ A_1 : L_p(0,T;W_p^1(\Omega)) \to L_q(0,T;W_q^{-1}(\Omega)),$$

and subdifferential $A_2 = \partial\varphi : L_p(0,T;W_p^1(\Omega)) \to Conv\big(L_q(0,T;W_q^{-1}(\Omega))\big)$ are maximal monotone (see [7]). Thus (15) is solvable.

## References


1. Lions J.-L., *Quelques Méthodes de Résolution de Problèmes aux Limities Non Linéaires*, Paris: Dunod (1969).
2. Solonukha O.V., *On solvability for parabolic variational inequalities with operator of monotone type*, Naukovi visti NTUU "KPI" (2001), no. 6, (in Ukrainian).
3. Mel'nik V.S., Zgurovskii M.Z., *Nonlinear Analysis and control of infinite dimensional systems*, Naukova dumka, Kiev (1999), (in Russian).
4. Mel'nik V.S., Solonoukha O.V., *On stationary variational inequality with multivalued operators*, Kibernetica i systemny analis (1997), no. 3, 74–89 (in Russian); *English translation in* Cybernetics and Systems Analysis, Vol. 33, No. 3, May-June, 1997, p.366-378.
5. Mel'nik V.S., Vakulenko A.N., *Topological method in the theory of operator inclusions with densely defined mappings in Banach spaces*, Naukovi visti NTUU "KPI" **3** (1999), 3105-112 (in Ukrainian).
6. Solonoukha O.V., *On the Stationary Variational Inequalities with the Generalized Pseudomonotone Operators*, Methods of Functional Analysis and Topology **3** (1997), no. 4, 81-95.
7. Barbu V., *Analysis and control of non-linear infinite dimensional systems*, Acad. Press, Inc. (1995).



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