 G.A. Sviridyuk

ONE CLASS OF SEMILINEAR SOBOLEV TYPE EQUATIONS AND PHASE SPACES

The solvability of the Cauchy problem $u(0) = u_0$ of an semilinear differential operator equation $L\dot{u} = Mu + N(u)$ is under consideration. The abstract results are illustrated by the Cauchy – Dirichlet problem for the Hoff equation and for the Oskolkov equations.

Introduction.

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, and let operators $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ (i. e. linear and continuous) and $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{F})$ (i. e. linear, closed and densely defined). We shall study the Cauchy problem

$$u(0) = u_0 \tag{0.1}$$

for the differential operator equation

$$L\dot{u} = Mu + N(u), \tag{0.2}$$

where $\ker L \neq \{0\}$, and $N : \mathrm{dom}N \subset \mathfrak{U} \to \mathfrak{F}$ is generally speaking nonlinear operator. We shall call the equation (0.2) a *semilinear Sobolev type equation*, in contrast to *linear Sobolev type equation*

$$L\dot{u} = Mu. \tag{0.3}$$

Abstract results on problems (0.1), (0.2) and (0.1), (0.3) will be illustrated by specific examples having applied relevance. If $\mathfrak{U} = \mathfrak{F} = \mathbb{R}^n$, and operators L, M are matrixes of order n, then the degenerate system of ordinary differential equations (0.3) is the generalization of the Leontief system of input-output economics [1]. Another concrete example of the problem (0.1), (0.3) is the initial-boundary value problem on the cylinder $\Omega \times \mathbb{R}$ for the Batenblatt – Zheltov – Kochina equation

$$(\lambda - \Delta)p_t = \alpha\Delta p \tag{0.4}$$

modeling the pressure dynamics of the fluid filtered in a fissured porous medium [2]. In addition, the Eq. (0.4) describes moisture transfer in soil [3] and the process of "two-temperatures"heat conductivity [4].

The famous concrete example of the abstract problem (1), (2) is the initial-boundary value problems on $\Omega \times \mathbb{R}$ for the Hoff equation

$$(\lambda + \Delta)h_t = \alpha h + \beta h^3,$$

which for $n = 1$ models the dynamics of bulging of an I-beam [5]. The unknown function $h = h(x,t)$ has the physical meaning of deviation of the beam from the vertical under constant load; the numerical parameters $\lambda \in \mathbb{R}_+$ and $\alpha, \beta \in \mathbb{R}$ characterize the magnitude of the load and the properties of the material, where the positivity of λ is essential.

Next concrete example of the abstract semilinear problem (0.1), (0.2) is the Cauchy – Dirichlet problem on $\Omega \times \mathbb{R}$ for the Oskolkov equations

$$(\lambda - \nabla^2)v_t = \nu\nabla^2 v - (v\cdot\nabla)v - \nabla p, \quad \nabla\cdot v = 0,$$

which generalizes the well-khown Navier-Stokes equations to the case of a non-Newtonian fluid, the velocity of which is the absence of stress decays exponentially [6]. Here the vector function $v = (v_1, \ldots, v_n)$, $v_k = v_k(x,t)$ has the physical sense of the velocity of the fluid and the function $p = p(x,t)$ corresponds to the fluid pressure. The parameter $\nu \in \mathbb{R}_+$ characterizes the viscous properties of the fluid, and the parameters $\lambda \in \mathbb{R}$ the elastic ones, where the possibility $\lambda \in \mathbb{R}_-$ is confirmed experimentally.

One of the first explorer of the Cauchy problem for the equation $\Delta u_{tt} + \omega^2 u_z z = 0$ modeling small oscillations of the roating fluid was S.L. Sobolev [7]. Since equations and system of the equations in partial derivatives unsolved with respect to the highest derivative used to be called *Sobolev type equations* [8, 9]. We are going to use this term for classification of abstract equations of the forms (0.2) and (0.3) refusing from the ferms "pseudoparabolic equations"[10] and "equations non Cauchy – Kovalevsky type" [11].

M.I. Vishik [12] and S.G. Krein and his students [13] were the first who began to study abstract equations of the form (0.3). Nowdays the theory of the Sobolev type equations exists in two paradigms. To the first paradigm the works in which equations and systems of the equations in partial derivatives are explored should be concerved. This direction is exact continuation and development of the Sobolev's results. One can find modern results of this paradigm in [14]. And to the second paradigm the works in which the object of an investigation is the abstract equations (0.2), (0.3) should be concerned. Concrete initial-boundary value problems for the equations and systems of the equations in partial derivatives unsolved with respect to highest derivative are showh as the illustrations of the abstract results. Many interesting results of this paradigm are contained in [15, 16, 17].

In contrast with all these results our approach bases on the concept of the phase space. The idea of the method consists in reducing Eq. (0.2) or Eq. (0.3) to

$$\dot{u} = Su + F(u) \tag{0.5}$$

or

$$\dot{u} = Su \tag{0.6}$$

respectively, given, however, not all of $\mathfrak{U}$, but on (smooth, Banach) manifold imbedded in $\mathfrak{U}$. This manifold contains all initial values u_0, for which problems (0.1), (0.2) or (0.1), (0.3) are well-possed. There exist many results devoted to "phase space method" now. The must important results were obtaind by the autor and his students.

The paper consists of four sections, except of Introduction. The first section is of propaedeutic character. It contains already known results [18], that are presented in our arrangement. The main goal of this section is to show the construction of resolving semigroups of the Eq. (0.3). The second section contains some applications of the described abstract results.

In the third section we carry out abstract discussions, consisting in the application of the modified Lyapunov – Schmidt method to studying of the problem (0.1), (0.2). Remark that the Cauchy problem $(\xi(0), \varphi(0)) = (0,0)$ for equations

$$0 = \eta - \xi^2, \quad \dot{\eta} = \xi \tag{0.7}$$

has two solutions stationary $(0,0)$ or nonstationary $(t/2, t^2/4)$, but the same problem for equations

$$0 = \eta - \xi^2, \quad \dot{\eta} = \xi + 1 \tag{0.8}$$

has not solution. Since Eq. (0.7) and Eq. (0.8) are simplest examples of the Eq. (0.2) then the problem (0.1), (0.2) is not well-posed in general. This simple observation shows the necessity of the restriction of the notion of the solution to the problem (0.1), (0.2).

The fourth section contains some examples arised in applications. We apply obtained abstract results to the Cauchy – Dirichlet problem for the Hoff equations and for the Oskolkov equations. The main goal of this section is to study the *morphology* (i. e. structure, lattice, organization) of the phase space of a concrete interpretation of the problem (0.1), (0.2).

In conclusion let us agree to all arguments that are carried out in real Banach spaces, but when "spectral" questions are considered, the natural complexification is introduced; all contours are oriented by "counterclockwise" motion and bound domains that lying on the "left hand" side under such motion; symbols I and $\mathbb{O}$ denote the "unique" and "null" operators respectively whose domains of definition are clear from context.

1. Relatively σ-bounded operators and degenerate analytic groups.

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, operators $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ and $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{F})$. Assuming the existence of the operator $L^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})$ the (0.1), (0.3) will be replaced by a pair of equivalent problems

$$\dot{u} = Su, \qquad u(0) = u_0; \qquad \dot{f} = Tf, \qquad f(0) = f_0, \tag{1.1}$$

which will be considered in the context of the problem

$$\dot{v} = Av, \qquad v(0) = v_0. \tag{1.2}$$

If the operator $A \in \mathcal{L}(\mathfrak{V})$, where $\mathfrak{V}$ is a Banach space, then, as is well known, there exists a unique solution $v = v(t)$ of problem (1.2) for every $v_0 \in \mathfrak{V}$ of the form $v(t) = V^t v_0$, where $\{V^t : t \in \mathbb{R}\}$ is a group of solving operators of Eq. (1.2) of the following form

$$V^t = \frac{1}{2\pi i}\int_\Gamma (\mu I - A)^{-1} e^{\mu t}\, d\mu. \tag{1.3}$$

Here, $\Gamma = \{\mu \in \mathbb{C} : |\mu| = r\}$ is the contour bounding the domain containing the spectrum $\sigma(A)$ of the operator A. The group (1.3) in the obvious way can be analytically extended to the whole complex plane.

Since operators S and T in problems (1.1) are similar, the operator $S \in \mathcal{L}(\mathfrak{U})$ precisely when the operator $T \in \mathcal{L}(\mathfrak{F})$. In this case, therefore, the pair of operators (L, M) generates a pair $(\{U^t\}, \{F^t\})$ of analytical groups of the form

$$U^t = \frac{1}{2\pi i}\int_\Gamma (\mu L - M)^{-1} L e^{\mu t}\, d\mu, \quad F^t = \frac{1}{2\pi i}\int_\Gamma L(\mu L - M)^{-1} e^{\mu t} d\mu,$$

where the contour Γ is the same as in (1.3). In this case the group $\{U^t : t \in \mathbb{R}\}$ contains solving operators of Eq. (**??**).

The main purpose of the present paragraph is to generalise these classical results to the case of an uninvertible operator L, in particular, when the kernel $\ker L \neq \{0\}$. The most important results in this area were obtained by G.A. Sviridyuk. A significant contribution to this theory has been made by T.G. Sukatheva and L.L. Dudko [18].

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, operator $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, and operator $M : \mathrm{dom} M \subset \mathfrak{U} \to \mathfrak{F}$ be linear and closed.

DEFINITION 1.1. *A set $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})\}$ is called a resolvent set of an operator M with respect to an operator L (or, briefly, L-resolvent set of an operator M). The set $\sigma^L(M) = \mathbb{C} \backslash \rho^L(M)$ is called spectrum of an operator M with respect to an operator L (or, briefly, L-spectrum of an operator M).*

REMARK 1.1. *When there exists an operator $L^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})$ L-resolvent set and L-spectrum of the operator M coincide with the resolvent set and the spectrum of the operator $L^{-1}M$ (or the operator ML^{-1}).*

REMARK 1.2. *The L-resolvent set of the operator M is always open, and, consequently, the L-spectrum of the operator M is always closed.*

DEFINITION 1.2. *Operator functions $(\mu L - M)^{-1}$, $R^L_\mu(M) = (\mu L - M)^{-1}L$, $L^L_\mu(M) = L(\mu L - M)^{-1}$ are called respectively a resolvent, right resolvent, and left resolvent of an operator M with respect to the operator L (or, briefly, L-resolvent, right L-resolvent, and left L-resolvent of the operator M).*

REMARK 1.3. *When there exists an operator $L^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})$ the right (left) L-resolvent of the operator M coincides with the resolvent of the operator $L^{-1}M$ (ML^{-1}).*

LEMMA 1.1. *If $\rho^L(M) \neq \emptyset$, then L-resolvent, right and left L-resolvents of the operator M are continuous on $\rho^L(M)$.*

THEOREM 1.1. *If $\rho^L(M) \neq \emptyset$, then the L-resolvent, right and left L-resolvents of the operator M are analytic in $\rho^L(M)$.*

DEFINITION 1.3. *An operator M is called spectrally bounded with respect to the operator L (or briefly, (L, σ)-bounded), if*

$$\exists a \in \mathbb{R}_+ \quad \forall \mu \in \mathbb{C} \quad (|\mu| > a) \Rightarrow (\mu \in \rho^L(M)) .$$

REMARK 1.4. *Let $\operatorname{dom} M = \mathfrak{U}$ and let there exist an operator $L^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})$. The operator M is (L, σ)-bounded precisely when the operator $L^{-1}M$ (or ML^{-1}) is bounded.*

REMARK 1.5. *Let an operator $L \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ be compact. Then either $\rho^L(M) = \emptyset$, or the L-spectrum $\sigma^L(M)$ of the operator M is discrete, of finite multiplicity and is condensed only to the point ∞. Indeed, let $\alpha \in \rho^L(M)$, then*

$$(\mu L - M)^{-1} = (\lambda K + I)^{-1} (\alpha L - M)^{-1},$$

where $\lambda = \mu - \alpha$, and the operator $K = (\alpha L - M)^{-1}L$ is compact. Therefore, in this case the operator M is not (L, σ)-bounded.

Let the operator M be (L, σ)-bounded, and the contour $\Gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$. Let us consider integrals *of F. Riss type*

$$P = \frac{1}{2\pi i} \int_\Gamma R^L_\mu(M)\, d\mu, \qquad Q = \frac{1}{2\pi i} \int_\Gamma L^L_\mu(M)\, d\mu .$$

LEMMA 1.2. *Let the operator M be (L, σ)-bounded. Then operators $P : \mathfrak{U} \to \mathfrak{U}$ and $Q : \mathfrak{F} \to \mathfrak{F}$ are projectors.*

Let us assume $\mathfrak{U}^0 = \ker P$, $\mathfrak{F}^0 = \ker Q$; $\mathfrak{U}^1 = \mathrm{im}P$, $\mathfrak{F}^1 = \mathrm{im}Q$. Let the restriction of the operator L (M) to $\mathfrak{U}^k$ ($\mathrm{dom}M \cap \mathfrak{U}^k$), $k = 0, 1$ be denoted by L_k (M_k).

THEOREM 1.2. *Let an operator M be (L, σ)-bounded. Then*

(i) the actings of operators $L_k : \mathfrak{U}^k \to \mathfrak{F}^k$, $M_k : \mathrm{dom}M \cap \mathfrak{U}^k \to \mathfrak{F}^k$, $k = 0, 1$ are decomposed;

(ii) there exist operators $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0; \mathfrak{U}^0)$ and $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1; \mathfrak{U}^1)$;

(iii) the operator $M_1 \in \mathcal{L}(\mathfrak{U}^1; \mathfrak{F}^1)$.

If the set $\rho^L(M) \neq \emptyset$, we can substitute the equation $L\dot{u} = Mu$ for a pair of equations equivalent to it

$$R^L_\alpha(M)\dot{u} = (\alpha L - M)^{-1} M u \,, \tag{1.4}$$

$$L^L_\alpha(M)\dot{f} = M(\alpha L - M)^{-1} f \,, \tag{1.5}$$

which will be regarded as specific interpretations of the equation

$$A\dot{v} = Bv \,, \tag{1.6}$$

where operators $A, B \in \mathcal{L}(\mathfrak{V})$, and $\mathfrak{V}$ is some Banach space. The *solution of the Eq.* (1.6) is then a vector function $v \in C^1(\mathbb{R}; \mathfrak{V})$ satisfying this equation.

DEFINITION 1.4. *The mapping $V^\cdot \in C^1(\mathbb{R}; \mathcal{L}(\mathfrak{V}))$ is called a group of solving operators of Eq. (1.6), if*

(i) $V^s V^t = V^{s+t} \quad \forall s, t \in \mathbb{R}$;

(ii) for every $v_0 \in \mathfrak{V}$ vector function $v(t) = V^t v_0$ is the solutions of Eq. (1.6).

Let us identify the group with its set of values $\{V^t : t \in \mathbb{R}\}$. The group $\{V^t : t \in \mathbb{R}\}$ will be called *analytical*, if it can be analytically extended to the whole complex plane retaining its properties (i) and (ii) from Definition 1.4.

THEOREM 1.3. *Let an operator M be (L, σ)-bounded. Then there exists an analytic solving group of Eq.* (1.4) (*Eq.* (1.5)).

Let the contour $\Gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$, then the required group is the integral of *Danford—Taylor type*

$$U^t = \frac{1}{2\pi i} \int_\Gamma R^L_\mu(M) e^{\mu t} \, d\mu, \quad t \in \mathbb{R},$$

$$\left(F^t = \frac{1}{2\pi i} \int_\Gamma L^L_\mu(M) e^{\mu t} \, d\mu, \quad t \in \mathbb{R} \right).$$

REMARK 1.6.*Projectors P and Q are obviously the identities of solving groups $\{U^t : t \in \mathbb{R}\}$ and $\{F^t : t \in \mathbb{R}\}$ respectively. Therefore,*

$$U^t = U^t P = \frac{1}{2\pi i} \int_\Gamma (\mu L_1 - M_1)^{-1} L_1 P e^{\mu t} = e^{tS} P,$$

where the operator $S = L_1^{-1}M_1 \in \mathcal{L}(\mathfrak{U}^1)$. *If the operator* $T = M_1L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1)$, *then* $F^t = F^tQ = e^{tT}Q$.

2. Phase spaces of linear Sobolev-type equations.

Let us consider the linear equation

$$A\dot{v} = Bv, \tag{2.1}$$

where operators $A, B \in \mathcal{L}(\mathfrak{V})$, $\mathfrak{V}$ is Banach space, and $\{V^t : t \in \mathbb{R}\}$ is solving analytical group of the Eq. (2.1).

DEFINITION 2.1. *A set* $\ker V^\cdot = \{v \in \mathfrak{V} : V^t = 0 \quad \forall t \in \mathbb{R}\}$ *is called a kernel, and a set* $\text{im}V^\cdot = \{v \in \mathfrak{V} : v = V^0 v\}$ *is called an image of the analytical group* $\{V^t : t \in \mathbb{R}\}$.

Obviously $\ker V^\cdot = \ker V^t$, $\quad \text{im}V^\cdot = \text{im}V^t \quad \forall t \in \mathbb{R}$, therefore Definition 2.1 is correct.

DEFINITION 2.2. *A set* $\mathfrak{P} \subset \mathfrak{V}$ *is called a phase space of Eq. (2.1), if*

(i) any solution $v = v(t)$ *to Eq. (2.1) lies in* $\mathfrak{P}$, *i. e.,* $v(t) \in \mathfrak{P}\ \forall t \in \mathbb{R}$;

(ii) for every $v_0 \in \mathfrak{P}$ *there exists a unique solution* $v \in C^1(\mathbb{R}; \mathfrak{V})$ *of Cauchy problem* $v(0) = v_0$ *for Eq. (2.1).*

Return to operators $L \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ and $M \in Cl(\mathfrak{U}; \mathfrak{F})$.

THEOREM 2.1. *If an operator* M *be* (L, σ)*-bounded, then*

$$(\mu L - M)^{-1} = -\sum_{k=0}^{\infty} \mu^k H^k M_0^{-1}(I - Q) + \sum_{k=1}^{\infty} \mu^{-k} S^{k-1} Q,$$

where operators $H = M_0^{-1}L_0$, $S = L_1^{-1}M_1$, *and* $|\mu| > a$.

DEFINITION 2.3. *The point* ∞ *is called removable singular point, pole of the order* $p \in \mathbb{N}$, *essential singular point of the* L*-resolvent* $(\mu L - M)^{-1}$ *of an operator* M, *if respectively* $H \equiv \mathbb{O}$, $H^p \not\equiv \mathbb{O}$ *and* $H^{p+1} \equiv \mathbb{O}$, $H^k \not\equiv \mathbb{O}\ \forall k \in \{0\} \cup \mathbb{N}$.

Set $\alpha \in \rho^L(M)$, and let us consider the parr of equations

$$R_\alpha^L(M)\dot{u} = (\alpha L - M)^{-1}Mu, \tag{2.2}$$

$$L_\alpha^L(M)\dot{f} = M(\alpha L - M)^{-1}f, \tag{2.3}$$

which are regarded as concrete interpretations of the Eq. (2.1).

THEOREM 2.2. *Let an operator* M *be* (L, σ)*-bounded, whereas* ∞ *is a removable singularity or a pole of order* $p \in \mathbb{N}$ *of the* L*-resolvent of the operator* M. *Then the phase space of Eq.* (2.2) *(Eq.* (2.3)*) coincides with the image of the solving group* $\{V^t : t \in \mathbb{R}\}$ *(solving group* $\{F^t : t \in \mathbb{R}\}$*).*

REMARK 2.1. *If* ∞ *is essential singularity, then the Theorem 2.2 is false (see the counter-example in [18]).*

REMARK 2.2. *If is easy to see that the phase space of the Eq. (2.2) coincides with the phase space of the equation (0.3).*

EXAMPLE 2.1. *Degenerate system of ordinary differential equations.*

Let L and M be square matrices of the order n, $\det L = 0$. Let us consider the (0.1), (0.3).

LEMMA 2.1. *If there exist a point $\alpha \in \mathbb{C}$ such that $\det(\alpha L - M) \neq 0$, then the operator M is (L,σ)-bounded, while ∞ is a nonessential singularity of the L-resolvent of the operator M.*

THEOREM 2.3. *If the condition of Lemma 2.1 is satisfied, then for every initial value $u_0 \in \mathfrak{M} = \{u \in \mathfrak{U} : (I-Q)Mu = 0\}$ $(\equiv \mathfrak{U}^1)$ there exists a unique solution $u \in C^1(\mathbb{R}, \mathfrak{M})$ that can be represented by the formula*

$$u(t) = \frac{1}{2\pi i}\int_\Gamma R^L_\mu(M)e^{\mu t}u_0 d\mu,$$

where Γ is a countour as above.

REMARK 2.3. *It is obvious that the set $\mathfrak{M}$ $(\equiv \mathfrak{U}^1)$ is the phase space of the Eq. (0.3).*

EXAMPLE 2.2. *The Barenblatt—Zheltov—Kochina equation.*

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a boundary $\partial\Omega$ of the class C^∞. Let us seek a function $u = u(x,t)$ satisfying in the cylinder $\Omega \times \mathbb{R}$ the equation

$$(\lambda - \Delta)u_t = \alpha\Delta u \tag{2.4}$$

and the Cauchy—Dirichlet conditions

$$u(x,0) = u_0(x), \quad x \in \Omega; \qquad u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}. \tag{2.5}$$

The problem (2.4), (2.6) is reduced to the problem (0.2), (0.3) by taking as spaces $\mathfrak{U}$ and $\mathfrak{F}$ either Sobolev spaces $\mathfrak{U} = \{u \in W^{k+2}_p(\Omega) : u(x) = 0,\ x \in \partial\Omega\}$, $\mathfrak{F} = W^k_p(\Omega)$, or Holder spaces $\mathfrak{U} = \{u \in C^{k+2+\mu}(\Omega) : u(x) = 0,\ x \in \partial\Omega\}$, $\mathfrak{F} = C^{k+\mu}(\Omega)$, where $1 < p < \infty$, $0 < \mu < 1$, $k = 0, 1, \ldots$ Then the operators $L = \lambda - \Delta$: $\mathfrak{U} \to \mathfrak{F}$, $M = \alpha\Delta$: $\mathfrak{U} \to \mathfrak{F}$ will be linear continuous and Fredholm (i. e., ind$L$ = ind = 0).

LEMMA 2.2. *The operator M is (L,σ)-bounded, while ∞ is a removable singularity of the L-resolvent of the operator M.*

THEOREM 2.4. *(i) if $\lambda \notin \sigma(\Delta)$, then for every $u_0 \in \mathfrak{U}$ and $f \in \mathfrak{F}$ there exists a unique solution $u \in C^1(\mathbb{R}; \mathfrak{U})$ of the problem (2.4), (2.5), which can be represented as*

$$u(t) = \sum_{k=1}^{\infty} e^{\alpha\lambda_k t/(\lambda - \lambda_k)}(u_0, \varphi_k)\varphi_k.$$

(ii) if $\lambda \in \sigma(\Delta)$, then for every $u_0 \in \mathfrak{M}_f = \{u \in \mathfrak{U} : (u, \varphi_k) = 0,\ \lambda_k = \lambda\}$ there exists a unique solution $u \in C^1(\mathbb{R};\ \mathfrak{M}_f)$ of the problem (2.4), (2.5), which can be represented in the following form:

$$u(t) = {\sum_{k=1}^{\infty}}' e^{\alpha\lambda_k t/(\lambda - \lambda_k)}(u_0, \varphi_k)\varphi_k.$$

Here, $\{\varphi_k\}$ and $\{\lambda_k\}$ are sets of orthonormalised eigenfunctions and their respective eigenvalues of the homogeneous Dirichlet problem for the Laplace operator in the domain Ω, numbered with respect to nonascending of the eigenvalues allowing for their multiplicity. The primed summation symbol denotes the absence of terms with numbers k such that $\lambda = \lambda_k$.

3. Quasistationary trajectories.

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, operators $L, M \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ and $N \in C^\infty(\mathfrak{U};\mathfrak{F})$. Consider the Cauchy problem for semilinear Sobolev-type equation

$$L\dot{u} = Mu + N(u), \quad u(0) = u_0. \tag{3.1}$$

If on operator L is continuously invertible, then the problem (3.1) is reduced trivially to the problem

$$\dot{u} = Su + F(u), \quad u(0) = u_0, \tag{3.2}$$

with an operator $S \in \mathcal{L}(\mathfrak{U})$ and a nonlinear operator $F \in C^\infty(\mathfrak{U})$. The existence of the unique solution $u \in C^\infty((-t_0, t_0);\mathfrak{U})$ to the problem (3.2) for some $t_0 \in \mathbb{R}_+$ is the classical Cauchy problem.

Let us consider the problem (3.1), where the operator L is uninvertible, more precisely, $\ker L \neq \{0\}$. Let us suppose in addition that the operator M is (L,σ)-bounded. Then we can reduce the Eq. (3.1) to the equivalent system

$$\begin{aligned} H\dot{u}^0 &= u^0 + M_0^{-1}(I-Q)N(u), \\ \dot{u}^1 &= Su^1 + L_1^{-1}QN(u), \end{aligned} \tag{3.3}$$

where $u^1 = Pu$, $u^0 = u - u^1$, the operator $P(Q)$ is the projectior onto $\mathfrak{U}^1$ $(\mathfrak{F}^1)$ along $\mathfrak{U}^0$ $(\mathfrak{F}^0)$, and operators $H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{U}^0)$, $S = L_1^{-1}M_1 \in \mathcal{L}(\mathfrak{U}^1)$. The simplest examples (0.7), (0.8) convice us that the Cauchy problem for Eq. (3.3) is not well-posed in general.

DEFINITION 3.1. *A solution $u = u(t)$ to the problem (3.1) is called a quasistationary trajectory of the equation (3.1) passing through the point u_0 if $H\dot{u}^0(t) = 0$ for every $t \in (-t_0, t_0)$.*

Recall that a stationary solution of the Eq. (3.1) is a quasistationary trajectory, but the converse is false. In the above mentioned example (0.7) the quasistationary trajectory coincide with the stationary one, i. e. with the point $(0,0)$. Another system (0.8) has not stationary solution, and therefore it has not quesistationary trajectory.

To find qusistationary semitrajectories of the Eq. (3.1) we introduce in consideration a set $\mathfrak{M} = \{u \in \mathfrak{U}; (I-Q)(Mu+N(u)) = 0\}$. It is obvious that if $u = u(t)$ is a quasistationary trajectory, then it lies in $\mathfrak{M}$ (i. e. $u(t) \in \mathfrak{M}$ for every $t \in (-t_0, t_0)$). Let a point $u_0 \in \mathfrak{M}$. Set $u_0^1 = Pu_0$ and by $O_0^1 \subset \mathfrak{U}^1$ define a neighborhood of the point $u_0^1 \in \mathfrak{U}^1$. If there exists a C^∞-diffeomorphism $\delta : O_0^1 \to \mathfrak{M}$ such that $\delta^{-1} = P$, then we shall call the set $\mathfrak{M}$ a *Banach C^∞-manifold at the point* u_0. If the set $\mathfrak{M}$ is a Banach C^∞-manifold at every point $u_0 \in \mathfrak{M}$, then we shall call the set $\mathfrak{M}$ a *Banach C^∞-manifold modeling by the subspace* $\mathfrak{U}^1$. Connected Banach C^∞-manifold is called a *simple Banach C^∞-manifold* if every its atlas is equivalent to the atlas containing only a map.

THEOREM 3.1. *Let an operator M be (L,σ)-bounded, moreover, ∞ be a removable singular point or a pole of the order $p \in \mathbb{N}$. Let an operator $N \in C^\infty(\mathfrak{U};\mathfrak{F})$, and the set $\mathfrak{M}$ be a Banach C^∞-manifold at the point u_0. Then for $t_0 \in \mathbb{R}_+$ there exists a unique quasistationary trajectory $u = u(t)$, $t \in (-t_0, t_0)$, of the Eq. (3.1) passing through the point u_0.*

4. Phase spaces.

Let us return to the Eq. (0.2).

DEFINITION 4.1. *A set $\mathfrak{P} \subset \mathfrak{U}$ is called a phase space of the Eq. (0.2), if*

(i) every solution $u = u(t)$ of the Eq. (0.2) lies in $\mathfrak{P}$, i.e. $v(t) \in \mathfrak{P} \forall t \in (-t_0, t_0)$;

(ii) for every $u_0 \in \mathfrak{P}$ there exists a unique solution to the problem (0.1), (0.2).

In this section we shall consider such examples, in which the phase space is simple Banach C^∞-manifold and coincides with the set $\mathfrak{M}$.

EXAMPLE 4.1. *The Hoff equation.*

Let $\Omega \subset \mathbb{R}^n$ be bounded domain with C^∞ boundary $\partial\Omega$. The Hoff equation

$$(\lambda + \Delta)u_t = -\alpha u - \beta u^3 \tag{4.1}$$

for $n = 1$ models the H-beam buckling dynamics. The initial-boundary value problem

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad u(x, t) = 0, \quad (x, t) \in \partial\Omega \times \mathbb{R}, \tag{4.2}$$

for Eq. (4.1) was first studied in [19], where it was also indicated that problem (4.1), (4.2) is not solvable in principle for arbitrary initial conditions. The study of the set of feasible initial values, treated as the phase space of problem (4.1), (4.2), was initiated in [20, 21], but only partial results were obtained in both papers. A complete description of the phase space of problem (4.1), (4.2) is contained in [22].

To reduce problem (4.1), (4.2) to problem (0.1), (0.2), we set $\mathfrak{U} = L_4$ and $\mathfrak{F} = W_2^{-1}$ (all spaces are defined on the domain Ω). We define the operators L, M and F by the formulas

$$\langle Lu, v\rangle = \int_\Omega (\lambda uv - u_{x_k} v_{x_k})dx \qquad \forall u, v \in \overset{\circ}{W}{}_2^1,$$

$$\langle Mu, v\rangle = -\alpha \int_\Omega uvdx, \quad \langle N(u), v\rangle = -\beta \int_\Omega u^3 vdx, \quad \forall u, v \in L_4,$$

where $\langle \cdot, \cdot\rangle$ is the L_2 inner product. The embedding $\overset{\circ}{W}{}_2^1 \hookrightarrow L_4$ is dense and continuous for $n \leq 4$, and so $L \in \mathcal{C}l(\mathfrak{U}; \mathfrak{F})$. It is obvious that $N \in C^\infty(\mathfrak{U}; \mathfrak{F})$ by virtue of the embedding $(L_4)^* \cong L_{4/3} \hookrightarrow W_2^{-1}$.

By construction, the operator L is Fredholm of index zero (i.e. $\mathrm{ind} L = 0$. Moreover, the spectrum $\sigma(L)$ of L is real and discrete, is of finite multiplicity, and accumulates only at the point $-\infty$. Let $0 \notin \sigma(L)$; then the projections P and Q satisfy $P = Q = \mathbb{O}$. Thus, all assumptions of Theorem 3.1 are obviously satisfied, and hence the following theorem holds.

THEOREM 4.1. *Let $0 \notin \sigma(L)$ and $n \leq 4$. Then for each $u_0 \in L_4$ and for some $t_0 = t_0(u_0)$ there exists a unique solution $u \in C^\infty((-t_0, t_0); L_4)$ of problem (4.1), (4.2).*

REMARK 4.1. *One can readily see that under the assumptions of Theorem 4.1 the phase space of the Hoff equation is the entire space L_4.*

Now let $0 \in \sigma(L)$. We choose an L_2-orthonormal basis $\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$ in the kernel $\ker L$. It readily follows that the phase space of the Hoff equation contains the points $u \in L_4$ such that

$$\int_\Omega (\alpha + \beta u^2) u \varphi_l dx = 0, \quad \varphi_l \in \ker L$$

and moreover, one has the nonzero determinant

$$\left| \int_\Omega (\alpha + 3\beta u^2) \varphi_k \varphi_l dx \right| \neq 0,$$

where $k, l = 1, 2, \ldots, m$. Let $\mathfrak{M}$ be the set of solutions of Eq. (4.1) in L_4. We introduce the set $\mathfrak{L} = \{u \in L_4 : \langle u, \varphi_l \rangle = 0, \quad l = 1, 2, \ldots, m\}$.

LEMMA 4.1. *Let $0 \in \sigma(L)$ and $n \leq 4$. Then for each $v \in \mathfrak{L}$ there exists a unique vector $\psi \in \ker L$ such that $u = \psi + v \in \mathfrak{M}$.*

THEOREM 4.2. *Let $0 \in \sigma(L)$ and $n \leq 4$. Then for each $u_0 \in \mathfrak{M}$ and for some $t_0 = t_0(u_0) \in \mathbb{R}_+$ there exists a unique solution $u \in C^\infty((-t_0, t_0); \mathfrak{M})$ of problem (4.1), (4.2).*

REMARK 4.2. *By virtue of Lemma 4.1 and Theorem 4.2, the phase space of the Hoff equation is a simple C^∞ manifold modeled on the space $\mathfrak{L}$.*

EXAMPLE 4.2. *The Oskolkov equations.*

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the boundary $\partial\Omega$ of the class C^∞. We will seek a couple of functions $(v(x,t), p(x,t))$ satisfying the following system of equations in the cylinder $\Omega \times \mathbb{R}$:

$$(1 - \xi\nabla^2) v_t = \nu \nabla^2 v - (v \cdot \nabla) v - \nabla p, \qquad 0 = -\nabla(\nabla \cdot v) \tag{4.3}$$

and the Cauchy—Dirichlet conditions

$$v(x, 0) = v_0(x), \quad x \in \Omega; \qquad v(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}. \tag{4.4}$$

As in the previous case, we reduce problem (4.3), (4.4) to the problem (0.1), (0.2). For this purpose similar to [23] let a set $\mathfrak{U} = \mathbb{H}_\sigma^2 \times \mathbb{H}_\pi^2 \times \mathbb{H}_p$, $\mathfrak{F} = \mathbb{H}_\sigma \times \mathbb{H}_\pi \times \mathbb{H}_p$, $\mathbb{H}_p = \mathbb{H}_\pi$. The element $u \in \mathfrak{U}$ is of the form $u = (u_\sigma, u_\pi, u_p)$, $u_\sigma = \Sigma u$, $u_\pi = \Pi u$, $u_p = \nabla p$; and the element $f \in \mathfrak{F}$ has the form $f = (f_\sigma, f_\pi, f_p)$, $f_\sigma = \Sigma f$, $f_\pi = \Pi f$.

LEMMA 4.2. *(i) Formula*

$$L = \begin{pmatrix} A_{\lambda\sigma} & \Sigma A_\lambda \Pi & \mathbb{O} \\ \Pi A_\lambda \Sigma & \Pi A_\lambda \Pi & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \end{pmatrix}$$

specifies the linear continuous operator $L : \mathfrak{U} \to \mathfrak{F}$. If $\lambda^{-1} \notin \sigma(A)$, then $\ker L = \{0\} \times \{0\} \times \mathbb{H}_p$, $\operatorname{im} L = \mathbb{H}_\sigma \times \mathbb{H}_\pi \times \{0\}$.

(ii) Formula

$$M = \begin{pmatrix} \Sigma B & \Sigma B & \mathbb{O} \\ \Pi B & \Pi B & -I \\ \mathbb{O} & C & \mathbb{O} \end{pmatrix}$$

specifies the linear continuous operator $M : \mathfrak{U} \to \mathfrak{F}$.

LEMMA 4.3. *The operator, given by formula*

$$N : u \longrightarrow \begin{pmatrix} -\Sigma(v\cdot\nabla)v \\ -\Pi(v\cdot\nabla)v \\ \mathbb{O} \end{pmatrix}, \quad v = u_\sigma + u_\pi$$

lies in $C^\infty(\overset{\circ}{\mathbb{H}}{}^1; \mathbb{L}^2)$, *if* $n = 2, 3$.

The projectors P and Q have the forms

$$P = \begin{pmatrix} \Sigma & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} \Sigma & \mathbb{O} & -\nu\Sigma A B_\pi^{-1} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \end{pmatrix},$$

where B_π is the restriction of the operator B on $\mathbb{H}^2_\pi$.

Using the projectors P and Q we construct the set $\mathfrak{M} = \{u \in \mathfrak{U} : u_\pi = 0,\ u_p = -\Pi(u_\sigma\cdot\nabla)u_\sigma)\}$.

THEOREM 4.3. *Suppose* $\nu \in \mathbb{R}_+$, $n = 2, 3$, $f \in \mathfrak{F}$, $f = (f_\sigma, f_\pi, 0)$, *then for every* $u_0 \in \mathfrak{M}$ *there exists a unique solution* $u \in C^\infty((-t_0; t_0); \mathfrak{M})$ *for some* $t_0 \in \mathbb{R}_+$ *of the problem (4.3), (4.4). The set* $\mathfrak{M}$ *is the simple Banach* C^∞*-manifold modeled by the subspace* $\overset{\circ}{\mathbb{H}}{}^1_\sigma \times \{0\} \times \{0\}$.

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Chelyabinsk State University,
129 Kashirin Brothers street ,
454021 Chelyabinsk, Russia
ridyu@csu.ru

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