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NONLINEAR WAVES IN TWO-DIMENSIONS GENERATED ON THE FREE SURFACE OF AN IDEAL HEAVY LIQUID BY A SORCE OF FINITE POWER

The development of nonlinear waves on the free surface of a heavy liquid initially at rest is treated analytically in cases where the external pressure force of limited power is distributed over a large area in the free surface. On this assumption, a small parameter is introduced and expansions in powers of the parameter are used.

Compared to the available literature the novelty of the formulation of the problem and the results obtained in the present study can be displayed as follows:

1. The equations of the free surface are sought in parametric form. The form is appropriate for the surfaces which may steepen sufficiently that overturning occurs.

2. Conditions at infinity on the free surface require the volume of the liquid transferred across the horizontal equilibrium plane to be finite at any given value of time, t. Though the luquid is non-dissipative, the conditions at infinity assure the decaying of initial transients.

3. A quite general mathematical model is introduced for the external pressure to the free surface.

4. For a numerable set of specific distributions of the external pressure corresponding numerable set of specific solutions (3.1) of the leading-order equations is obtained. According to the solution, the crests of the wave move faster than the troughs, so, when the wave gets away from the variable pressure zone, a chain of structures develops similar to so called Kelvin-Helmholtz billows.

5. The limits (as the time $t \to +\infty$) of the specific solutions are found.

6. The existence of nonlinear standing waves is discovered which have a finite number of nodes in the free surface infinite in extent

7. For any distribution of the external pressure the leading-order solution of the problem can be obtained as a superposition of the specific solutions. The reason is that in obtaining the parametric equations of the free surface a nonlinear transformation of the governing equations is employed. Though the waves are linear to the transformed equations, they are nonlinear to the original equations.

The paper follows very closely the book [1] and extends its results.

1.Introduction

Experimental studies involving various sources of disturbances to a liquid-liquid interface (in particular, to the free surface of a liquid) show that large-scale breaking of a gravity wave on the surface occurs some time after the primary billows became developed [2,3]. In contrast, in the exact (or asymptotically exact) theoretical studies (by Stokes, Nekrasov, Levi-Chivita, Struik and others) the irrotational nonlinear free surface waves examined are periodic along the free surface infinite in extent, and any source of disturbances to the free surface was not indicated in the studies. It was neither an external pressure (the pressure remains constant along the free surface) nor a moving solid body. It was not shown how the wave was created; the existence of the wave was taken as a given fact.

In the present paper it will be assumed that the source of disturbances to the free surface is an external pressure force of limited power distributed over a large area in the free surface.

Referring to Fig.1, consider two-dimensional motion of an ideal heavy uniform liquid of density $\gamma > 0$ in the (x, y) plane, with the x - axis oriented upward and the y - axis in the horizontal direction. The liquid is thought of as being contained between two parallel planes at unit distance apart; the planes are parallel to that of (x, y) . Let the curve Γ (in Fig.1)

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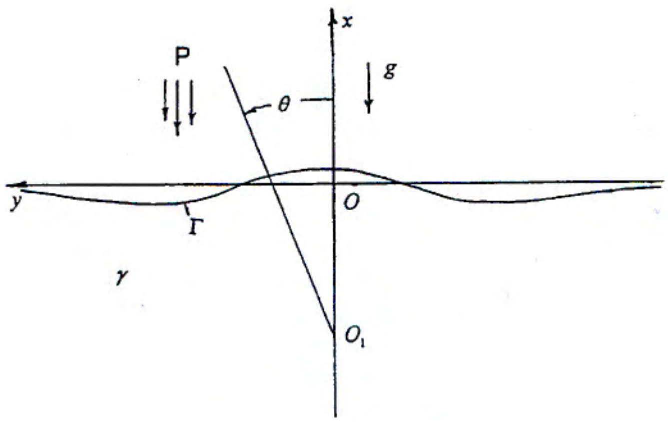

Figure 1: Flow diagram.

be the trace of the free surface S in the (x, y) plane, $x = f < 0$, $y = 0$ be the coordinates of the pole O_1 of the polar coordinate system in the (x, y) plane, θ be the polar angle measured from the positive x-axis in the counterclockwise direction, t be the time.

Initially, the liquid is at rest, the external pressure, $P = P_0$, is constant, and the free surface is a horizontal plane $x = 0$. Then the liquid is set in motion by a variable "gauge"pressure, P_*, on the free surface.

The well-known mathematical formulation of the problem includes (i) the Laplace's equation for the velocity potential, Φ, (ii) nonlinear kinematic condition which means that a liquid particle in the free surface can have no velocity relative to the surface in the direction of the normal, (iii) nonlinear condition of the pressure continuity across the free surface. (iiii) conditions at infinity.

The greatest mathematical difficulties in finding the solution of the problem arise from the two nonlinear conditions on the evolving free surface unknown in advance. To make the problem manageable, its linearised version was introduced in some papers, in which a small-amplitude departure of the free surface from its equilibrium position were considered. This allowed the authors to linearize the conditions on the free surface and to remove them from the unknown free surface to the equilibrium plane $x = 0$. But even for the linearized problem only a few analytical results were obtained, when the external pressure varies in accordance with the given law of a specific form [4,5]. Stoker [5] investigated the linearized problem and noted that the problem should be solved for the liquid starting from rest, and then the steady-state wave should be found as the limit of the solution of the initial-boundary value problem when time goes to infinity. Though the liquid is non-dissipative, Stoker believed the effect of the initial conditions would be a transient effect that would die out after a long time, leaving a final steady-state solution that is independent of the initial conditions, if harder (than for simple harmonic solutions of the linearised problem) conditions on the velocity potential at infinity would be placed. The Stoker's idea was illustrated in [5]. However, with time effects caused by nonlinearity and by dispersion respectively become comparable [6].

In the present paper the Stoker's concept is carried out for the nonlinear problem with the aid of some new analytical technique developed in [1].

2. Preliminary considerations.

In this section we will summarise very briefly the pertinent analytical results for the title problem, that we need.

All equations are written in dimensionless variables. Since the problem has no characteristic linear size, the dimensional unit of length, L, may be chosen arbitrarily, the dimensional unit of time, T, is defined by the relation $T^2 g = L$, where g is the acceleration of free fall. The dimensionless acceleration of free fall is equal to unity. Let γ_0 be some characteristic density and $P_0 \neq 0$ be some characteristic pressure (for example, the constant "atmospheric"pressure on the free surface in static equilibrium). All parameters, variables and equations are made dimensionless by the quantities L, T, γ_0, P_0.

In [1] the velocity potential, Φ, is sought in the form of a doublet distribution over the free surface (i.e. Φ is represented by a line integral), so the Laplace's equation is satisfied.

In [1] the equations of the free surface are sought in the parametric form

$$x = W(\theta, t), \quad y = (W - f) \tan\theta, \quad -\pi/2 < \theta < \pi/2, \tag{2.1}$$

where $W(\theta, t)$ is an unknown function that must be found while solving the problem. Formally, equations (2.1) for each specified function W describe a family of curves depending on f, t being considered as constant. The value of f may be assigned arbitrarily, however, the choice of the value determines the horizontal scale of the curve.

In the (x, y) plane curvilinear coordinates (σ, θ) are defined by the relations

$$x = \sigma + W(\theta, t), \quad y = (\sigma + W - f) \tan\theta$$

so the equation of the free surface takes the form $\sigma = 0$ (the heavy liquid occupies the half-space $\sigma < 0$).

In [1], the nonlinear conditions (ii) and (iii) on the free surface are written in the curvilinear coordinates (σ, θ), and the conditions at infinity are taken in the form

$$|W(\theta, t)| < C(t)\cos^2\theta, \quad \lim_{\cos\theta\to 0} \frac{\partial W}{\partial\theta} = 0, \quad |\nu| < C(t), \tag{2.2}$$

where $|\nu|$ is the density of the doublet distribution, $C(t)$ is a positive quantity which is independent of θ.

The line integral for velocity potential, conditions (ii) and (iii) constitute the basic system of nonlinear integrodifferential equations in two unknown functions, $W(\theta, t)$ and $\nu(\theta, t)$. The solutions of the system determine the velocity field which, in the region $\sigma < 0$, satisfy the Euler's equations.

The initial conditions of the form

$$t = 0 \quad \nu = 0 \quad W = 0$$

have to be appended to the basic system of equations.

It can be shown that under conditions (2.2), the line integral (for the velocity potential) along the curve infinite in extent converges and the liquid is at rest at infinity.

Compared to [5], conditions (2.2) require the volume of the liquid transferred (due to the external pressure) across the horizontal equilibrium plane to be finite at any given value of t.

It is supposed that the gauge pressure, P_*, on the free surface (between two parallel planes at unit distance apart) is due to a force of finite power, so the pressure P_* must vary inversely as f, and directly as the product of $\cos^2\theta$ and a function which remains bounded as $|f| \to +\infty$

It is convenient to take that the gauge pressure is given in the form

$$P_*(\theta, t; f) = \frac{a\gamma}{f}\cos^2\theta\, F(\theta, t; f), \tag{2.3}$$

$$F(\theta, t) = \sum_{k=1}^{+\infty} \alpha_k(t)\cos(2k\theta) + \beta_k(t)\sin(2k\theta), \quad \sum_{k=1}^{+\infty}\sqrt{{\alpha_k}^2 + {\beta_k}^2} < +\infty,$$

where $F(\theta, t; f)$ is a dimensionless function bounded uniformly with respect to θ, t, f. The factor $|a\gamma/f|$ may be thought of as an amplitude of the pressure P_*. Functions $\alpha_k(t)$, $\beta_k(t)$ are not specified but are required to be functions of interest in applications, for example,

$$\alpha_k = \sum_j A_{kj}\cos(\Omega_{kj}t) + B_{kj}\sin(\Omega_{kj}t).$$

When the pressure P_* is due to a pressure force of finite power independent of f, increasing the value of $-f$ does lessen the pressure P_* on the free surface by spreading the finite pressure's power out over a region of larger horizontal size, so the magnitude of W decreases as $-f$ increases. For large values of $-f$, parameter $\varepsilon = a/f^2$, proportional to the ratio of the free surface wave to the typical horizontal size of the disturbed zone in the surface, is introduced. Extended zone of variable pressure corresponds to small values of $|\varepsilon|$ [7].

For $|\varepsilon| << 1$, the solution of the initial-boundary value problem is sought in the form of series in powers of ε:

$$W(\theta, t) = \frac{a}{f}\sum_{k=0}^{+\infty}\varepsilon^k W_k(\theta, t), \quad \nu(\theta, t) = \sum_{k=0}^{+\infty}\varepsilon^k \nu_k(\theta, t). \tag{2.4}$$

The mathematical technique for constructing the series is described in some details in [1]. In [1] the leading-order terms of the series (2.4) have been found explicitly in the form of the trigonometric series similar to that in (2.3).

3. Leading-order theory: exact solution for the pressure with a point frequency spectrum.

Three new results obtained recently are presented below. The first, for the case of almost-periodic external pressure (2.3), convergence of the trigonometric series (for the leading-order terms) has been proved. The theorem states that for any finite time-interval the series converge uniformly with regard to θ and t.

The leading-order "response"$W_0(\theta, t)$ of the free surface to the gauge pressure depends on the factor $F(\theta, t; f)$ linearly. For the linearity, from now on we shall confine our attention to a particular case of the gauge pressure:

$$F(\theta, t) = \alpha_m(t)\cos(2m\theta), \quad \alpha_m = A_m\cos(\Omega t - \phi_m), \quad A_m > 0 \text{ for } m \geq 1.$$

For the case, (and this is the second) the sums of the trigonometric series have been obtained in closed form, so the following equations to the free surface hold:

$$x = \frac{a}{f}W_0(\theta, t), \quad y = (x - f)\tan\theta,$$

$$W_0(\theta,t) = (-1)^m A_m[R_m(\tau,\theta;\lambda)\cos(\tau\lambda-\phi_m) + Q_m(\tau,\theta;\lambda)\sin(\tau\lambda-\phi_m)], \tag{3.1}$$

$$t = \tau\sqrt{2|f|},\ \ \lambda = \Omega\sqrt{2|f|},$$

$$R_m(\tau,\theta;\lambda) = -\frac{1}{2m}\int_0^{+\infty} x^4 \exp\left(-\frac{1}{2}x^2\right) L^{(1)}_{m-1}(x^2) f_1 \cos\left(\frac{1}{2}x^2\tan\theta\right)dx, \tag{3.2}$$

$$Q_m(\tau,\theta;\lambda) = -\frac{1}{2m}\int_0^{+\infty} x^4 \exp\left(-\frac{1}{2}x^2\right) L^{(1)}_{m-1}(x^2) f_2 \cos\left(\frac{1}{2}x^2\tan\theta\right)dx, \tag{3.3}$$

$$f_1 = -\frac{1}{\lambda+x}\sin^2\left(\frac{\lambda+x}{2}\tau\right) + \frac{1}{\lambda-x}\sin^2\left(\frac{\lambda-x}{2}\tau\right),$$

$$f_2 = \frac{1}{2}\frac{1}{\lambda+x}\sin\left((\lambda+x)\tau\right) - \frac{1}{2}\frac{1}{\lambda-x}\sin\left((\lambda-x)\tau\right).$$

The right parts of eqns (3.2) and (3.3) involve the Laguerre polynomials, $L^{(1)}_k(u)$, defined by the recursion relation

$$L^{(1)}_0 = 1,\ \ L^{(1)}_1(u) = 2-u,\ \ kL^{(1)}_k(u) = (2k-u)L^{(1)}_{k-1}(u) - kL^{(1)}_{k-2}(u).$$

Three applications of the integration by parts formula show that corresponding to any preassigned value of τ there exists a quantity $c(\tau)$ such that

$$|\tan^3\theta R_m(\tau,\theta)| < c(\tau),\ \ |\tan^3\theta Q_m(\tau,\theta)| < c(\tau). \tag{3.4}$$

4. Steady-state free surface waves

By passing to the limit in the leading-order solution as the time goes to infinity the approximate equations of the steady-state wave have been found (and this is the third):

$$x = \frac{a}{f}W_0^*(\theta,\tau),\quad y = (x-f)\tan\theta, \tag{4.1}$$

$$W_0^*(\theta,\tau) = (-1)^m A_m[R^*_m(\theta;\lambda)\cos(\tau\lambda-\phi_m) + Q^*_m(\theta;\lambda)\sin(\tau\lambda-\phi_m)].$$

$$R^*_m(\theta;\lambda) = \lim_{\tau\to+\infty} R_m(\tau,\theta;\lambda),\quad Q^*_m(\theta;\lambda) = \lim_{\tau\to+\infty} Q_m(\tau,\theta;\lambda), \tag{4.2}$$

$$R^*_m(\theta;\lambda) = \frac{1}{4m}(I_1+I_2+I_3), \tag{4.3}$$

$$I_1 = \int_0^{+\infty}\frac{G(x)}{x+\lambda}dx,\ \ I_2 = \int_0^{\lambda}\frac{G(\lambda+s)-G(\lambda-s)}{s}ds,\ \ I_3 = \int_{2\lambda}^{+\infty}\frac{G(x)}{x-\lambda}dx, \tag{4.4}$$

$$Q^*_m(\theta;\lambda) = \pi G(\lambda), \tag{4.5}$$

$$G(x) = v_m(x)\cos\left(\frac{1}{2}x^2\tan\theta\right), \quad v_m(x) = \frac{1}{4m}x^4 L_{m-1}^{(1)}(x^2)e^{-x^2/2}. \tag{4.6}$$

For large values of $|\tan\theta|$ the following asymptotic formula holds

$$R_m^*(\theta;\lambda) = -\pi v_m(\lambda)\sin\left(\frac{1}{2}\lambda^2|\tan\theta|\right) + O\left(\frac{1}{|\tan\theta|}\right) \tag{4.7}$$

5. Hydrodynamical interpretation of the results obtained.

We consider the case of $L_{m-1}^{(1)}(\lambda^2) \neq 0$ unless indicated otherwise. Figures 2 - 4 offer comprehensive view of the theoretical steady-state wave (4.1). The wave is symmetrical about the vertical plane $y = 0$. Calculations were performed using eqns (4.1) - (4.6) with $\lambda = 2$, $m = 3$, $f = -10$, $aA_3 = -10$. The wave is symmetrical about the vertical plane $y = 0$.

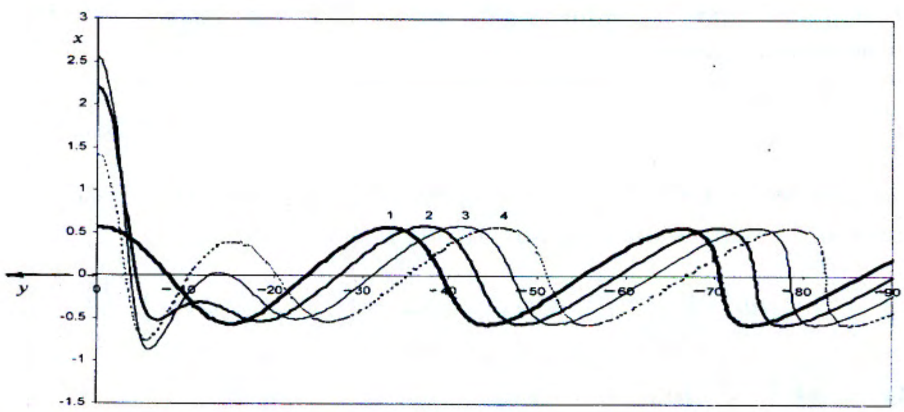

Figure 2: Successive profiles of the steady-state free surface wave (4.1) at four instants, a quarter of the period apart: 1 - for $\lambda\tau = 0$; 2 - for $\lambda\tau = \pi/2$; 3 - for $\lambda\tau = \pi$; 4 - for $\lambda\tau = 3\pi/2$; $|y| < 90$.

For large values of $|\tan\theta|$ (at points far from the origin) we obtain the following simple approximation to the equations of the theoretical steady-state wave:

$$x = l_m(\lambda)\sin\left(\lambda\tau - \phi_m - \Omega^2\frac{|y|}{1+\frac{x}{|f|}}\right). \tag{5.1}$$

These equations follow as an immediate consequence of equations (4.1), (4.5) - (4.7).

We conclude from equation (5.1) that (for large values of $|\tan\theta|$) a point of the free surface with given elevation $x = x_0 = const$, $y = y(t)$ moves horizontally with velocity

$$\frac{dy}{dt} = \left(1 + \frac{x_0}{|f|}\right)\frac{1}{\Omega}\frac{y(t)}{|y(t)|},$$

so the points of higher elevation travel faster. Since the crests of the wave move faster than the troughs, the horizontal distance between a crest and the adjacent trough increases when the wave runs away from the origin. This phenomenon is shown in figures 2 - 4.

Needless to say, the theoretical steady-state wave cannot be expected to exist in nature.

Inequalities (3.5) show that $R_m \to 0$ and $Q_m \to 0$ as $|\theta| \to \pi/2$, t being considered as constant. On the other hand, by equations (4.2), $R_m \to R_m^*$ and $Q_m \to Q_m^*$ as $t \to +\infty$, treating θ as a constant. Thus, for sufficiently large values of t there exists a region $|\theta| < |\theta_1|$ where the excited free surface wave is close to the theoretical steady-state one, and there exists a region $|\theta| > |\theta_2| > |\theta_1|$ where the free surface displacement from its equilibrium position is very small. In the course of time, the size of the first region increases gradually. Calculations show that the profiles of the free surface become close to that shown in figures 2 - 4 : in the region $|y| < 90$ at $\lambda\tau = 28\pi$, in the region $|y| < 230$ at $\lambda\tau = 74\pi$, in the region $|y| < 390$ at $\lambda\tau = 120\pi$, respectively.

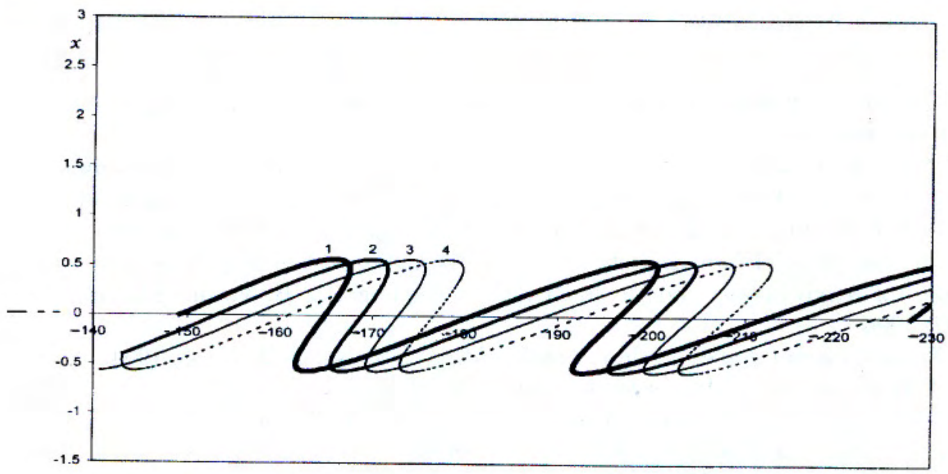

Figure 3: Same as Fig.2, but for $140 < |y| < 230$.

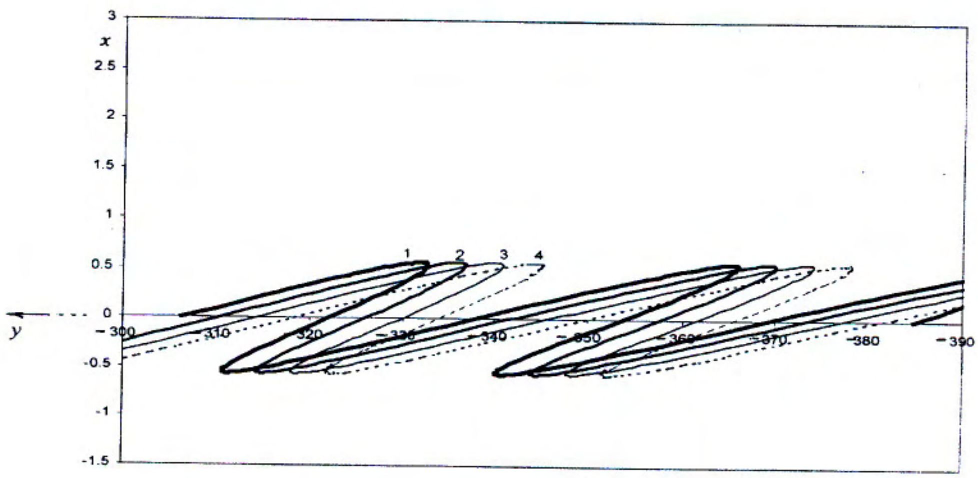

Figure 4: Same as Fig.2, but for $300 < |y| < 390$.

There is no doubt that the theoretical free surface wave is unstable. That is why in experimental studies only a finite set of billows was observed [2,3].

In the case of $L_{m-1}^{(1)}(\lambda^2) = 0$, identity $Q_m^*(\theta; \lambda) \equiv 0$ holds, and equation (4.1) reduces to

$$x = \frac{a}{f} W_0^*(\theta, \tau), \quad y = (x - f)\tan\theta, \tag{5.2}$$

$$W_0^*(\theta,\tau) = (-1)^m A_m R_m^*(\theta;\lambda)\cos(\tau\lambda - \phi_m).$$

In [1] the expression for power absorption functional is found, which relates the power absorbed by the liquid in the steady-state motion and the pressure's frequency-, amplitude-, and spatial distribution. In the case of steady-state wave (5.2) the absorption functional is equal to zero, which means that the wave does not transport energy. Only standing waves have this property. The nodes of the free-surface wave (5.2) occur in fixed positions in the rays $\theta = \theta_j$, $R_m^*(\theta_j;\lambda) = 0$ ($x = 0$, $y = -f\tan\theta_j$). Since $R_m^*(\theta)$ is a polynomial of $\cos\theta$ (the polynomial is obtained explicitly in [1]), this wave has a finite number of nodes in the free surface infinite in extent. In contrast with it, a classical standing wave has an infinite set of nodes.

Similar results for nonlinear axially symmetric standing waves were presented in [8].

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