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ON THE APPOXIMATE EIGENVECTORS OF QUASILINEAR OPERATORS

We study a convergence of finite-dimensional aproximate eigenvectors of quasi-linear operators in Hilbert space.

1. The main results.

Let H be the real separable Hilbert space with a scalar product $\langle\cdot,\cdot\rangle$; $u \in H$, $S = \{u \in H : ||u|| = 1\}$. Let $B : H \to H$ a nonlinear completely continuous operator (i.e., B is continuous and its image is compact). For nonlinear equation $B(u) = f$ the convergence of Bubnov-Galerkin-Petrov projective method is investigated in works of M.A.Krasnoselskii and G.M.Vainikko (see the reference in [1]). At second, by variational Rayleigh-Ritz method the finite-dimensional approximations of eigenvectors of a linear self-adjoint compact operator $\mathbf{A}$ are investigated [1]. I.e. approximate solutions (λ, u) of problem

$$\mathbf{A}u = \lambda u, \quad u \in S, \quad \lambda \in \mathbf{R}. \tag{1}$$

are investigated. Finally, L.A.Lusternik and L.G.Shnirelman employed the finite-dimensional approximations for the investigation of eigenvectors of non-linear completely continuous potential operators [2]. Thus they investigated the problem

$$Bu = \lambda u, \quad u \in S, \quad \lambda \in \mathbf{R}, \tag{2}$$

where B is completely continuous potential operator. However we do not know works corresponding to research of approximate eigenvectors of nonlinear non-potential operators. Apparently, the reason is that the method to distinguish eigenvalues by their numbers (as this happens to be in linear and variational theories) is absent.

Here we consider the special class of completely continuous operators. Let $L(H)$ be the Banach space of linear self-adjoint compact operators. Let a mapping $A : S \to L(H)$ be completely continuous, i. e., A is continuous mapping and its image $Im(A)$ is compact in $L(H)$. Next assertion is true.

LEMMA 1. ([3, p. 605]). *The mapping $B : H \to H$, where $B(u) := A(u)u$ is completely continuous.*

The mapping $B(u) := A(u)u$ is called *quasilinear*.

We shall need the vector space $CC = CC(S, L(H))$ of all completely continuous mappings $A : S \to L(H)$. The space CC is Banach space supplied with the norm $||A||_C = \sup_{u\in S} ||A(u)||$ [4].

Consider the quasilinear eigenvector problem

$$A(u)u = \lambda u, \quad u \in S, \quad \lambda \in \mathbf{R}, \quad \lambda > 0. \tag{3}$$

By Lemma 1, problem (3) is analogous to problem (2). Note that problem (3) is the generalization of linear self-adjoint eigenvector problem (1); for linear problem (1) the mapping $A(u) \equiv const = \mathbf{A} \in L(H)$ is constant.

The pair (λ^*, u^*) is called the *solution* if it satifaies problem (3). At the same time with nonlinear problem (3) consider associated linear self-adjoint eigenvector problem (1) where

$$\mathbf{A} = A(u^*) \in L(H). \tag{4}$$

Obviously, λ^* is an eigenvalue of associated linear problem (1), (4) and among eigenvectors corresponding to λ^* the vector u^* is present. Thus every solution of nonlinear problem (3) is the solution of associated linear problem (1), (4).

DEFINITION 1. *A solution (λ^*, u^*) of problem (3) and its elements are called simple or m-multiple if λ^* is simple or multiple respectively as eigenvalue of linear problem (1), (4).*

We order all the positive eigenvalues of linear operator (4) in decreasing order, counting multiplicity. The eigenvalue λ^* receives some number (or finitely many numbers)

DEFINITION 2. *We assign the same number (numbers) to the solution (λ^*, u^*) of problem (3) and its elements.*

We note that the pair (λ^*, u^*) itself does not contain an information about number and multiplicity. It receives these characteristics as a solution of concrete linear problem (1), (4).

In order to investigate the quasilinear eigenvalue problem, we shall need some geometrical objects:

1) the subset of pairs

$$P := \{p = (\mathbf{A}, u) \in L(H) \times S : \textit{ there exists } \lambda > 0, \textit{ such that } \mathbf{A}u = \lambda u\},$$

2) the mapping "graph A"

$$GrA : S \to L(H) \times S, \quad GrA(u) := (A(u), u).$$

We know [3] that the subset P is C^∞-submanifold.

DEFINITION 3. *Let the mapping A is smooth. Let (λ^*, u^*) is a simple solution of quasilinear problem (3). The solution (λ^*, u^*) is called stable if the mapping GrA is transverse to the manifold P at the point u^*.*

Let $\{e_i\}_{i=1}^{\infty}$ be some orthonormal basis in H. By $\nu^{(k)}$ denote the orthoprojective operator onto the subspace $\mathbf{R}^k$, which is generated by the basic vectors $e_1, ...e_k$. Let $S^{k-1} = S \cap \mathbf{R}^k$. Together with infinite-dimensional problem (3) let us consider the approximate finite-dimensional (k-dimensional) problem

$$(\nu^{(k)} A(u) \nu^{(k)})u = \lambda u, \quad \lambda > 0, \quad u \in S^{k-1}. \tag{3_k}$$

In matrix natation problem (3) is the following form

$$\begin{pmatrix} a_{11}(u) & a_{12}(u) & ... & a_{1k}(u) & a_{1k+1}(u) & ... \\ a_{21}(u) & a_{22}(u) & ... & a_{2k}(u) & a_{2k+1}(u) & ... \\ ... & ... & ... & ... & ... & \\ a_{k1}(u) & a_{k2}(u) & ... & a_{kk}(u) & a_{kk+1}(u) & ... \\ ... & ... & ... & ... & ... & \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ ... \\ u_k \\ ... \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_2 \\ ... \\ u_k \\ ... \end{pmatrix}. \tag{5}$$

In order to obtain k-dimensional problem (3, k) it is necessary in problem (5) to keep the quadratic matrix that its size is equal to k and the k-dimensional vector $(u_1, ..., u_k)$. For finite-dimensional problem (3,k) the notions of number and multiplicity remain unchanged.

Let us formulate the main statements.

THEOREM 1. *For each stable solution (λ^*, u^*) of problem (3) there is a sequence (λ_k, u_k) $(k \to \infty)$ of solutions of problems (3,k) that converges to (λ^*, u^*) .*

THEOREM 2. *Let* (λ_k, u_k) $(k \to \infty)$ *be a sequence of solutions of problems (3,k) and there is* $lim_{k\to\infty}(\lambda_k, u_k) = (\lambda^*, u^*)$, *where* $\lambda^* > 0$. *Then the following assertions are true.*

1) The pair (λ^*, u^*) *is the solution of problem (3).*

2) There exists the subsequence $(\lambda_{k_i}, u_{k_i}) \subset (\lambda_k, u_k)$ *such that all elements have at least one common number. The quantity of the common numbers is finite. The solution* (λ^*, u^*) *has of all common numbers.*

Due to quasilinear form of problems (3) and (3,k), we may distinguish the solutions with the common number and to obtain the limit solution of problem (3) with same number.

THEOREM 3. *Let each operator* $\mathbf{A} \in Im(A) \subset L(H)$ *has a positive eigenvalue with number* n. *Let* (λ_k, u_k) $(k \to \infty)$ *be a sequence of solutions of problems (3,k) such that all of solutions have the common number* n. *Then the following assertions are true.*

1) This sequence has at least one limit point.

2) Each of limit point is a solution of problem (3) having the number n.

REMARK 1. *Theorems 1 and 2 are analogous to assertions of Theorem 18.1 (see [1, p. 258]) about convergence of approximate eigenvalues for linear problem.*

REMARK 2. *For linear eigenvalue problem first it prove a convergence of sequence of eigenvalues, after which it prove a convergence of sequence of eigenvectors. For nonlinear problem we consider the pair (eigenvalue, eigenvector) at the same time.*

2. The proofs.

A mapping A is called *image-k-dimensional (image-finite-dimensional)* if for $u \in S$ the following condition is true: $A(u) \equiv \nu^{(k)}A(u)\nu^{(k)}$, i. e.,

$$A(u) = \begin{pmatrix} A^{(k)}(u) & 0 \\ 0 & 0 \end{pmatrix}, \tag{6}$$

where $A^{(k)}(u) = (\langle A(u)e_i, e_j\rangle)$ $(i, j = 1, \ldots, k)$ is a symmetric k-dimensional matrix. By $A^{\{k\}}(u) := \nu^{(k)}A(u)\nu^{(k)}$ $(u \in S,\ k = 0, 1, \ldots)$ we denote *the image-k-dimensional approximation* of an arbitrary mapping A.

LEMMA 2. ([3, p. 612]).

In the space CC *image-finite-dimensional mappings of form (6) are dense. For any choice of an orthnormal basis* $\{e_i\}_{i=1}^{\infty}$ *the convergence of image-k-dimensional approximation takes place:* $A^{\{k\}} \to A$ *in* CC *as* $k \to \infty$.

LEMMA 3. *The stable solution is isolated and smoothly depends on a small perturbation of the mapping* A.

Proof is by implicit mapping theorem.

By TS denote the tangential bundle over S. Using the condition of normalization $u \in S$, we exclude the eigenvalue from equation (3) and obtain the equation with an unknown $u \in S$ only and with the functional parameter $A \in CC$:

$$\Xi(u, A) = 0, \ \textit{where}\ \Xi : S \times CC(S, L(H)) \to TS, \quad \Xi(u, A) = A(u)u - \langle A(u)u, u\rangle u.$$

The mapping Ξ is the section of the tangential bundle TS depending on the functional parameter $A \in CC$. Let u_0 be the eigenvector; in another way, u_0 is the singular point of the section Ξ. The operator of partial derivative is

$$D_1\Xi(u_0, A) : T_{u_0}S \to T_{(u_0,0)}(TS),$$

$$D_1\Xi(A,u_0)(\Delta u) = A(u_0)\Delta u - \langle A(u)u, u\rangle\Delta u +$$

$$(DA(u_0)\Delta u)u_0 - \langle (DA(u_0)\Delta u)u_0, u_0\rangle u_0 - 2\langle A(u_0)u_0, \Delta u\rangle u_0.$$

Since u_0 is the eigenvector and the tangential space $T_{u_0}S$ is orthogonal to u_0, we have $\langle A(u_0)u_0, \Delta u\rangle = 0$. Thus, the operator $D_1\Xi(A,u_0)$ takes the tangential space to itself. By the definitions of the manifold P and the stable eigenvector, we have that the operator $D_1\Xi(A,u_0) : T_{u_0}S \to T_{u_0}S$ is a linear isomorphism. Consequently for the mapping Ξ implicit mapping theorem is applicable: there exists the local smooth mapping $u = u(A)$. This completes the proof. □

Proof of Theorem 1 is by lemmas 2, 3. □

Proof of Theorem 2 is by Lemma 2 and Definition 2. □

By $\Lambda_n(A)$ denote the set of all eigenvalues of problem (3) that have number n, and by $U_n(A)$ denote the set of all eigenvectors of problem (3) that have number n. We need the lemma about *a priori* estimate.

LEMMA 4. *Let a subset $T \subset CC$ be a compact subspace. Let each operator $\mathbf{A} \in \cup_{A\in T} Im(A) \subset L(H)$ has a positive eigenvalue with number n. Then:*

1) the following lower and upper estimates take place

$$0 < \inf_{A\in T} \Lambda_n(A), \ \sup_{A\in T} \Lambda_n(A) < \infty,$$

2) the subset $\cup_{A\in T} U_n(A) \subset S$ is a compact subspace.

Proof. The assertion is a direct consequence of the continuous dependence of eigenvalues under perturbations of operators $\mathbf{A} \in L(H)$ and the compactness of images of these operators. □

Proof. of Theorem 3 is by Lemma 4, Definition 2 and Theorem 2. □

3. The quasilinear eigenfunctions problem.

Let $\Omega \subset \mathbf{R}^m$ be a bounded domain with a C^2-smooth bound $\partial\Omega$, $x \in \overline{\Omega}$; $W_2^i(\Omega)$ the Hilbert separable space (the Sobolev space) of functions with distributional derivatives up through order i, which are 2-integrable (we recall that $W_2^0(\Omega) = L_2(\Omega) = H$ is the space of functions, which are 2-integrable). Let $0 < \zeta_1 \le \zeta_2$ be fixed numbers. Let $\mathbf{L} : W_2^2 \to L_2$, be an elliptic self-adjoint operator:

$$\mathbf{L}y = -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}(a_{i,j}(x)\frac{\partial y}{\partial x_j}) + a(x)y,$$

where $a \in C^0(\overline{\Omega})$, $a_{i,j} = a_{j,i} \in C^1(\overline{\Omega})$ and elliptic conditions

$$\zeta_1|\xi|^2 < a_{i,j}(x)\xi_i\xi_j < \zeta_2|\xi|^2, \ \ \xi = (\xi_1, ..., \xi_m), \ \ |\xi|^2 = \xi_1^2 + ... + \xi_m^2$$

are fulfilled on $\overline{\Omega}$. Consider following quasilinear Dirichlet problem: find an eigenfunction $y \in W_2^2(\Omega)$ and an eigenvalue $\lambda \in \mathbf{R}$ of

$$\mathbf{L}y + p(y, \nabla y, x)y = \gamma y, \ \ y|_{\partial\Omega} = 0, \ \ \int_\Omega y^2 dx = 1, \tag{7}$$

The pair (γ, y) that satisfies the problem (7) is called the solution. Let a function $p(a, b, x)$ be continuous on the domain $\mathbf{R} \times \mathbf{R}^m \times \overline{\Omega}$ and following lower and upper estimates take place:

$$0 < \sigma \leqslant p(a, b, x) \leqslant \Sigma \qquad (8)$$

where σ, Σ are some constants.

For solutions of problem (7) the definitions 1 and 2 remain true.

We shall shown that problem (7) may be investigated by *a sequence* of problems of the form (3,k) in the space $H = L_2(\Omega)$.

At first, we shall formulate the lemma about *a priori* estimates. Introduce following notations: $\Phi(\sigma, \Sigma)$ is the set of continuous functions that satisfy estimates (8); $\{(\lambda, u)\}_p^n$ is the set of solutions of problem (7), which have the number n; $\{(\lambda, u)\}_\Phi^n = \cup_{p\in\Phi}\{(\lambda, u)\}_p^n$, where $\Phi = \Phi(\sigma, \Sigma)$.

LEMMA 5. ([5, 6].) *For any fixed n the set $\{(\lambda, u)\}_\Phi^n$ is bounded in the space $\mathbf{R} \times W_2^2(\Omega)$ and in the space $\mathbf{R} \times C^1(\Omega)$ by some constant $T = T(n, \Phi)$.*

Let $\{y_i\}_{i=1}^\infty$ be the system of orthonormal eigenfunctions of the linear operator $\mathbf{L}$ under Dirichlet condition. Denote by Pr_l the operator of the orthogonal projection

$$Pr_l(y) = \sum_{i=1}^{l} \left(\int_\Omega y y_i dx \right) y_i.$$

At the same time with problem (7) let us consider the sequence of auxiliary boundary problems:

$$\mathbf{L}y + p(Pr_l(y), \nabla(Pr_l(y)), x)y = \gamma y, \quad y|_{\partial\Omega} = 0, \quad \int_\Omega y^2 dx = 1, \qquad (7, l)$$

where $l = 0, 1, \ldots$.

REMARK 3. *For solutions of the problem (7, l) the definitions 1, 2 and (if l is sufficiently large) the assertions of Lemma 5 remain true.*

Reduce each of problems (7, l) to the problem of the form (3). Consider the next mappings.

1) Denote by $C_+^0(\overline{\Omega}) \subset C^0(\overline{\Omega})$ the open set of positive continuous functions. Consider the completely continuous mapping

$$\widehat{p}_l : H \to C_+^0(\overline{\Omega}), \quad \widehat{p}_l(y) = p(Pr_l(y), \nabla Pr_l(y), x),$$

which generated by the l-dimensional projection operator and the function p.

2) Denote by $L(W_2^2(\Omega), H)$ the Banach space of continuous linear operators and by $L_{is}(W_2^2(\Omega), H) \subset L(W_2^2(\Omega), H)$ the open subset of linear isomorphisms. Consider the smooth mapping

$$D : C_+^0(\overline{\Omega}) \to L_{is}(W_2^2(\Omega), H), \quad D(q) = \mathbf{L} + q,$$

which takes a function q to the continuous positive linear differential operator (positivity means that for any $y \in H$ the inequality $\int_\Omega [D(q)y(x)]y(x)dx > 0$ is true.)

3) Consider the smooth mapping

$$inv : L_{is}(W_2^2(\Omega), H) \to L_{is}(H, W_2^2(\Omega)), \quad inv(F) = F^{-1},$$

which takes a continuous linear isomorphism F to the inverse isomorphism.

4) Denote by $j : W_2^2(\Omega) \to H$ the imbedding operator. The operator j is Hilbert-Schmidt one [7]. Consider the continuous linear operator h that takes each linear isomorphism $\mathbf{C} \in L_{is}(H, W_2^2(\Omega))$ to the linear Hilbert-Schmidt operator by $h(\mathbf{C}) = j\mathbf{C}$.

Now, consider the mapping product $A_l = h \cdot inv \cdot D \cdot \widehat{p}_l : H \to L$. By the complete continuity of $\widehat{p}_l$, the mapping A_l is completely continuous. By the positivity of $D(q)$, for any $y \in H$ the operator $A_l(y)$ is positive; in particularly, one is self-adjoint. Thus, $A_l \in CC$. From the given in this item definitions of the space H and the mapping A_l it follows that problem (7, l) is identified to problem (3), where $A = A_l$ and $\lambda = 1/\gamma$.

By $k(l)$ $(l \to \infty)$ denote a sequence of numbers $k \in \mathbf{Z}$ such that $k(l) \to \infty$ as $l \to \infty$. Let (3, k(l)) be the problem (3, k), where $A = A_l$ and $\lambda = 1/\gamma$. Using Remark 3, we obtain that for solutions of problems (7) and (3, k(l)) the assertions of theorems 2, 3 remain true in the space $H = L_2(\Omega)$. Since $inv \cdot D \cdot \widehat{p}_l(y) \in W_2^2(\Omega)$, the assertions of theorems 2, 3 remain true in the space $W_2^2(\Omega)$.

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