©2004. **M. Borsuk and A. Zawadzka**

EXACT ESTIMATES OF SOLUTIONS TO THE ROBIN BOUNDARY VALUE PROBLEM FOR LINEAR ELLIPTIC NON DIVERGENCE SECOND ORDER EQUATIONS IN A NEIGHBORHOOD OF THE CONICAL POINT

We investigate the behavior of strong solutions to the Robin boundary value problem for linear elliptic non divergence second order equations in a neighborhood of the boundary conical point. We establish precise exponent of the solution decreasing rate.

1. Introduction.

Let $G \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain with boundary ∂G that is a smooth surface everywhere except at the origin $\mathcal{O} \in \partial G$ and near the point $\mathcal{O}$ it is a **convex** conical surface with vertex at $\mathcal{O}$. We consider the elliptic value problem

$$\begin{cases} a^{ij}(x)u_{x_i x_j} + a^i(x)u_{x_i} + a(x)u = f(x), \ x \in G, \\ \frac{\partial u}{\partial \vec{n}} + \frac{1}{|x|}\gamma(x)u = g(x), \ x \in \partial G \setminus \mathcal{O}. \end{cases} \qquad (L)$$

(summation over repeated indices from 1 to n is understood), $\vec{n}$ denotes the unite outward normal to $\partial G \setminus \mathcal{O}$. We obtain **best possible estimates** of the strong solutions of problem (L) near conical boundary point. Analogous results were established in [2, 1] for the Dirichlet problem. Many mathematicians have considered the third boundary value problem. For the first time, V.A.Kondrat'ev [8] studied general elliptic boundary value problems in domains with conical points. He proved the solvability of such problems with infinitely differentiable coefficients in weighted and usual L_2- Sobolev spaces. Later V.G. Maz'ya and B.A. Plamenevskiy extended the results of V.A. Kondrat'ev to the L_p- Sobolev weighted spaces (see e.g.[9, 10]). The oblique derivative problem for elliptic equations in non smooth domains investigated M.Faierman [4], G. Liberman [11]-[15], M.Garroni, V.A.Solonikov and M.Vivaldi [5], H.Reisman [16]. G.Lieberman [11, 14] proved local and global maximum principle for general second order linear and quasi-linear elliptic equations. He studied in [12]-[15] problems of existence and regularity of solutions in Lipschitz domains for equation with Hölder continuous coefficients. Completely recently G. Lieberman [14] proved Hölder continuity of strong solutions under weak hypotheses on the coefficients of (L). M.Faierman [4] investigated the regularity in Sobolev space $W^{2,p}(G)$ of generalized solution of the problem in a rectangle. H.Reisman [16] investigated such problem in weighted Sobolev spaces for domain with dihedral edges, and coefficients of equation are infinitely smooth. At last, authors of [5] considered the problem for the Poisson equation on the infinite angle. A principal new feature of this article is the consideration of our estimates for equations with coefficients that the smoothness are **the minimal possible!** Our examples demonstrate this fact.

We use the notations from [2]. Let $\mathcal{C}$ be the rotational cone $\{x_1 > r\cos\frac{\omega_0}{2}\}$. For a domain G which has a conical point at $\mathcal{O} \in \partial G$ we introduce the notations:

- $\Omega := \mathcal{C} \cap S^{n-1}$;
- $d\Omega :=$ area element of Ω;
- $G_a^b := G \cap \{(r,\omega) : 0 \leq a < r < b, \omega \in \Omega\}$– a layer in $\mathbb{R}^n$;

- $\Gamma_a^b := \partial G \cap \{(r,\omega) : 0 \le a < r < b, \omega \in \partial\Omega\}$ – the lateral surface of the layer G_a^b;
- $G_d := G \setminus G_0^d$;
- $\Gamma_d := \partial G \setminus \Gamma_0^d$;
- $\Omega_\rho := \overline{G_0^d} \cap \partial B_\varrho(0), \ \varrho \le d.$

We use the standard function spaces:

- $C^k(\overline{G}), C_0^k(G)$ with the norm $|u|_{k,G}$,
- Lebesgue space $L_p(G), p \ge 1$ with the norm $\|u\|_{p,G}$,
- the Sobolev space $W^{k,p}(G)$.

We define the weighted Sobolev spaces: $V_{p,\alpha}^k(G)$ for integer $k \ge 0$ and real α as the closure of $C_0^\infty(\overline{G} \setminus \mathcal{O})$ with respect to the norm

$$\|u\|_{V_{p,\alpha}^k(G)} = \left(\int_G \sum_{|\beta|=0}^k r^{\alpha+p(|\beta|-k)} |D^\beta u|^p \, dx \right)^{\frac{1}{p}}$$

and $V_{p,\alpha}^{k-\frac{1}{p}}(\partial G)$ as the space of functions φ, given on ∂G, with the norm

$$\|\varphi\|_{V_{p,\alpha}^{k-\frac{1}{p}}(G)} = \inf \|\Phi\|_{V_{p,\alpha}^k(G)},$$

where the infimum is taken over all functions Φ such that $\Phi\big|_{G} = \varphi$ in the sense of traces. We denote

$$W^k(G) \equiv W^{k,2}(G), \quad \mathring{W}_\alpha^k(G) \equiv V_{2,\alpha}^k(G),$$
$$\mathring{W}_\alpha^{k-\frac{1}{2}}(\partial G) \equiv V_{2,\alpha}^{k-\frac{1}{2}}(\partial G).$$

Definition 1.*A strong solution of problem (L) is a function $u(x) \in W_{loc}^{2,n}(G) \cap W^2(G_\varepsilon) \cap C^0(\overline{G})$, $\forall \varepsilon > 0$ which satisfies an equation for almost all $x \in G_\varepsilon$ and the boundary condition in the sense of traces on Γ_ε, $\forall \varepsilon > 0$. We assume that $M_0 = \max_{x \in \overline{G}} |u(x)|$ is known (see e.g. [11, 14]).*

We assume the existence $d > 0$ such that G_0^d is *the* **convex** *rotational cone* with the vertex at $\mathcal{O}$ and the aperture ω_0, thus

$$\Gamma_0^d = \left\{ (r,\omega) \Big| x_1^2 = \cot^2 \frac{\omega_0}{2} \sum_{i=2}^N x_i^2; \ |\omega_1| = \frac{\omega_0}{2}, \ \omega_0 \in (\frac{\pi}{2}, \pi) \right\}. \tag{1.1}$$

Regarding the equation we assume that the following **conditions** are satisfied:

(a) *the condition of uniform ellipticity:*

$$\nu\xi^2 \le a^{ij}(x)\xi_i\xi_j \le \mu\xi^2, \quad \forall x \in \overline{G}, \quad \forall \xi \in \mathbb{R}^n;$$
$$\nu, \mu = const > 0, \text{ and } a^{ij}(0) = \delta_i^j \text{ (the Kronecker symbol)};$$

(b) $a^{ij} \in C^0(\overline{G})$, $a^i \in L_p(G)$, $p > n$, $a, f \in L_n(G)$, $\gamma(x) \in C^1(\partial G)$; *for them the inequalities*

$$\left(\sum_{i,j=1}^{n} |a^{ij}(x) - a^{ij}(y)|^2\right)^{\frac{1}{2}} \le \mathcal{A}(|x-y|); \quad |x|\left(\sum_{i=1}^{n} |a^i(x)|^2\right)^{\frac{1}{2}} + |x|^2|a(x)| \le \mathcal{A}(|x|)$$

hold for $x, y \in \overline{G}$, *where* $\mathcal{A}(r)$ *is monotonically increasing, nonnegative function,* **continuous at zero**, $\mathcal{A}(0) = 0$;

(c) *there exist numbers* $f_1 \ge 0$, $g_1 \ge 0$, $\beta > -1$, $s > 1$, $\gamma_0 > \tan\frac{\omega_0}{2}$ *such that*

$$|f(x)| \le f_1|x|^\beta, \ |g(x)| \le g_1|x|^{s-1}, \ \gamma(x) \ge \gamma_0;$$

(d) $a(x) \le 0$ *in* G.

Let $\Omega \subset S^{n-1}$ be bounded domain with smooth boundary $\partial\Omega$. Let $\overrightarrow{\nu}$ be the exterior normal to $\partial\Omega$. Let $\gamma(x) \in C^0(\partial\Omega)$, $\gamma(x) \ge \gamma_0 > 0$. We consider the problem of the eigenvalues for the Laplace-Beltrami operator $\triangle_\omega$ on the unit sphere:

$$\begin{cases} \triangle_\omega u + \vartheta u = 0, \ \omega \in \Omega, \\ \frac{\partial u}{\partial \overrightarrow{\nu}} + \gamma(x)u\Big|_{\partial\Omega} = 0, \end{cases} \tag{EVP}$$

which consists of the determination of all values ϑ (eigenvalues) for which (EVP) has a non-zero weak solutions (eigenfunctions).

The eigenvalue problem (EVP) *was studied in Section VI [3] and in §2.5 [17] (see as well §2.4 [10]). We observe that* $\vartheta = 0$ *is not an eigenvalue of* (EVP).

We denote for the smallest positive eigenvalue ϑ

$$\lambda = \frac{2 - n + \sqrt{(n-2)^2 + 4\vartheta}}{2}. \tag{1.2}$$

Our main results are following Theorems.

THOREM 1.1. *Let* u *be a strong solution of problem* (L) *and assumptions (a) - (d) are satisfied with* $\mathcal{A}(r)$ **Dini-continuous** *at zero. Suppose, in addition,*

$$g(x) \in \overset{\circ}{W}{}^{\frac{1}{2}}_{4-n}(\partial G), \quad \text{as well} \quad a(x) \in \overset{\circ}{W}{}^{0}_{4-n}(G), \gamma(x) \in \overset{\circ}{W}{}^{\frac{1}{2}}_{2-n}(\partial G), \ \text{if}\, u(0) \ne 0$$

and there are the numbers

$$\sup_{\varrho>0} \varrho^{-s}\left(\|f\|_{\overset{\circ}{W}{}^{0}_{4-n}(G_0^\varrho)} + \|g\|_{\overset{\circ}{W}{}^{1/2}_{4-n}(\Gamma_0^\varrho)} + |u(0)|\left(\|a\|_{\overset{\circ}{W}{}^{0}_{4-n}(G_0^\varrho)} + \|\gamma\|_{\overset{\circ}{W}{}^{1/2}_{2-n}(\Gamma_0^\varrho)}\right)\right) =: k_s,$$

$$\sup_{\varrho>0} \varrho^{1-s}\left(\|f\|_{n,G^\varrho_{\varrho/2}} + |u(0)|\|a\|_{n,G^\varrho_{\varrho/2}}\right) =: \varkappa_s. \tag{1.3}$$

Then there are $d \in (0,1)$ and a constant $C > 0$ depends only on $\nu, \mu, d, s, n, \lambda, \gamma_0, g_1$, $\|\gamma\|_{C^1(\partial G)}$, meas G and on the quantity $\int_0^d \frac{\mathcal{A}(r)}{r} dr$ such that $\forall x \in G_0^d$

$$|u(x) - u(0)| \le C\Big(|u|_{0,G} + k_s + \varkappa_s + \|f\|_{\overset{\circ}{W}{}^0_{4-n}(G)} + \|g\|_{\overset{\circ}{W}{}^{1/2}_{4-n}(\partial G)} + g_1 + \\ + |u(0)|\big(1 + \|a\|_{\overset{\circ}{W}{}^0_{4-n}(G)} + \|\gamma\|_{\overset{\circ}{W}{}^{1/2}_{2-n}(\partial G)}\big)\Big) \begin{cases} |x|^\lambda, & \text{if } s > \lambda, \\ |x|^\lambda \ln^{3/2}\Big(\frac{1}{|x|}\Big), & \text{if } s = \lambda, \\ |x|^s, & \text{if } s < \lambda. \end{cases} \quad (1.4)$$

If, in addition, there is a number

$$\sup_{\varrho > 0} \varrho^{-s} \left(\|f + u(0)a\|_{V^0_{p,2p-n}(G^\varrho_{\varrho/2})} + \|g\|_{V^{1-\frac{1}{p}}_{p,2p-n}(\Gamma^\varrho_{\varrho/2})} + |u(0)| \|\gamma\|_{V^{1-\frac{1}{p}}_{p,p-n}(\Gamma^\varrho_{\varrho/2})} \right) =: \tau_s, \quad (1.5)$$

then

$$|\nabla u(x)| \le C\Big(|u|_{0,G} + k_s + \varkappa_s + \tau_s + \|f\|_{\overset{\circ}{W}{}^0_{4-n}(G)} + \|g\|_{\overset{\circ}{W}{}^{1/2}_{4-n}(\partial G)} + g_1 + \\ + |u(0)|\big(1 + \|a\|_{\overset{\circ}{W}{}^0_{4-n}(G)} + \|\gamma\|_{\overset{\circ}{W}{}^{1/2}_{2-n}(\partial G)}\big)\Big) \begin{cases} |x|^{\lambda-1}, & \text{if } s > \lambda, \\ |x|^{\lambda-1} \ln^{3/2}\Big(\frac{1}{|x|}\Big), & \text{if } s = \lambda, \\ |x|^{s-1}, & \text{if } s < \lambda. \end{cases} \quad (1.6)$$

THEOREM 1.2.*Let u be a strong solution of problem (L) and assumptions of Theorem 1.1 are satisfied with $\mathcal{A}(r)$, which is a continuous at zero function, but* **not** *Dini-continuous at zero. Then there are $d \in (0,1)$ and for each $\varepsilon > 0$ a constant $C_\varepsilon > 0$ depends only on $\nu, \mu, d, s, n, \lambda, \gamma_0, g_1, \|\gamma\|_{C^1(\partial G)}$, meas G and on $\mathcal{A}(diamG)$ such that $\forall x \in G_0^d$*

$$|u(x) - u(0)| \le C_\varepsilon\Big(|u|_{0,G} + \|f\|_{\overset{\circ}{W}{}^0_{4-n}(G)} + \|g\|_{\overset{\circ}{W}{}^{1/2}_{4-n}(\partial G)} + g_1 + k_s + \varkappa_s + \\ + |u(0)|\big(1 + \|a\|_{\overset{\circ}{W}{}^0_{4-n}(G)} + \|\gamma\|_{\overset{\circ}{W}{}^{1/2}_{2-n}(\partial G)}\big)\Big) \begin{cases} |x|^{\lambda-\varepsilon}, & \text{if } s > \lambda, \\ |x|^{s-\varepsilon}, & \text{if } s \le \lambda \end{cases} \quad (1.7)$$

and

$$|\nabla u(x)| \le C_\varepsilon\Big(|u|_{0,G} + \|f\|_{\overset{\circ}{W}{}^0_{4-n}(G)} + \|g\|_{\overset{\circ}{W}{}^{1/2}_{4-n}(\partial G)} + g_1 + k_s + \varkappa_s + \tau_s + \\ + |u(0)|\big(1 + \|a\|_{\overset{\circ}{W}{}^0_{4-n}(G)} + \|\gamma\|_{\overset{\circ}{W}{}^{1/2}_{2-n}(\partial G)}\big)\Big) \begin{cases} |x|^{\lambda-1-\varepsilon}, & \text{if } s > \lambda, \\ |x|^{s-1-\varepsilon}, & \text{if } s \le \lambda. \end{cases} \quad (1.8)$$

THEOREM 1.3. *Let u be a strong solution of problem (L) and assumptions of Theorem 1.1 are satisfied with $s \ge \lambda$, $\mathcal{A}(r) \ln \frac{1}{r} \le const$, $r > 0$ and $\mathcal{A}(0) = 0$. Then there are $d \in (0,1)$*

and the constants $C > 0, c > 0$ depends only on $\nu, \mu, d, n, \lambda, \gamma_0, g_1, \|\gamma\|_{C^1(\partial G)}$, meas G and on $\mathcal{A}(diamG)$ such that $\forall x \in G_0^d$

$$|u(x) - u(0)| \le C\Big(|u|_{0,G} + \|f\|_{\overset{\circ}{W}{}^{0}_{4-n}(G)} + \|g\|_{\overset{\circ}{W}{}^{1/2}_{4-n}(\partial G)} + g_1 + k_s + \varkappa_s + \\ + |u(0)|\big(1 + \|a\|_{\overset{\circ}{W}{}^{0}_{4-n}(G)} + \|\gamma\|_{\overset{\circ}{W}{}^{1/2}_{2-n}(\partial G)}\big)\Big)|x|^{\lambda} \ln^{c+1}\frac{1}{|x|} \tag{1.9}$$

and

$$|\nabla u(x)| \le C\Big(|u|_{0,G} + \|f\|_{\overset{\circ}{W}{}^{0}_{4-n}(G)} + \|g\|_{\overset{\circ}{W}{}^{1/2}_{4-n}(\partial G)} + g_1 + k_s + \varkappa_s + \tau_s + \\ + |u(0)|\big(1 + \|a\|_{\overset{\circ}{W}{}^{0}_{4-n}(G)} + \|\gamma\|_{\overset{\circ}{W}{}^{1/2}_{2-n}(\partial G)}\big)\Big)|x|^{\lambda-1} \ln^{c+1}\frac{1}{|x|}. \tag{1.10}$$

These Theorems are proved in the same way as in [2] for Dirichlet problem with corresponding modifications, considering the other boundary condition. There the idea of the proof is based on the using so-called the Hardy -Friedrichs - Wirtinger inequality. Therefore we restrict oneself here only to the deduction of the similar inequality.

2. The Hardy -Friedrichs - Wirtinger inequality. The comparison principle.

DEFINITION 2.*Function u is called a* **weak** *solution of problem (EVP) provided that $u \in W^1(\Omega)$ and satisfies the integral identity*

$$\imath\Omega\left\{\frac{1}{q_i}\frac{\partial u}{\partial \omega_i}\frac{\partial \eta}{\partial \omega_i} - \vartheta u\eta\right\} d\Omega + \imath\partial\Omega\gamma(x)u\eta d\sigma = 0 \tag{II}$$

for all $\eta(x) \in W^1(\Omega)$.

Now, let introduce us the functionals on $W^1(\Omega)$:

$$F[u] = \int\limits_{\Omega} |\nabla_\omega u|^2 d\Omega + \int\limits_{\partial\Omega} \gamma(x)u^2 d\sigma, \quad G[u] = \int\limits_{\Omega} u^2 d\Omega,$$

$$H[u] = \int\limits_{\Omega} \Big\langle |\nabla_\omega u|^2 - \vartheta u^2 \Big\rangle d\Omega + \int\limits_{\partial\Omega} \gamma(x)u^2 d\sigma$$

and the corresponding to them bilinear forms

$$F(u,\eta) = \imath\Omega\frac{1}{q_i}\frac{\partial u}{\partial \omega_i}\frac{\partial \eta}{\partial \omega_i}d\Omega + \int\limits_{\partial\Omega} \gamma(x)u\eta d\sigma, \ G(u,\eta) = \imath\Omega u\eta d\Omega.$$

We define yet the set

$$K = \big\{u \in W^1(\Omega)\Big|\ G[u] = 1\big\}.$$

Since $K \subset W^1(\Omega)$, $F[u]$ is bounded from below for $u \in K$. The greatest lover bound of $F[u]$ for this family we denote by ϑ :

$$\inf_{u \in K} F[u] = \vartheta$$

We formulate the following statement:

THEOREM 2.1.(see Theorem of Subsection 4 §2.5, p. 123 [17]). *Let $\Omega \subset S^{n-1}$ be bounded domain with smooth boundary $\partial\Omega$. Let $\gamma(x) \in C^0(\partial\Omega)$, $\gamma(x) \geq \gamma_0 > 0$. There exist $\vartheta > 0$ and a function $u \in K$ such that*

$$F(u,\eta) - \vartheta G(u,\eta) = 0 \text{ for arbitrary } \eta \in W^1(\Omega).$$

In particular $F[u] = \vartheta$. In addition, on Ω, u has continuous derivatives of arbitrary order and satisfies the equation $\triangle_\omega u + \vartheta u = 0$, $\omega \in \Omega$ and the boundary condition of (EVP) in the weak sense (see Remark on p. 121 - 122 [17]).

Next from the variational principle we obtain

THEOREM 2.2. *Let ϑ be the smallest positive eigenvalue of problem (EVP) (it there exists according to Theorem 2.1). Let $\Omega \subset S^{n-1}$ be bounded domain . Let $u \in W^1(\Omega)$ and $\gamma(x) \in C^0(\partial\Omega)$, $\gamma(x) \geq \gamma_0 > 0$. Then*

$$\vartheta \int\limits_\Omega u^2(\omega)d\Omega \leq \int\limits_\Omega |\nabla_\omega u(\omega)|^2 d\Omega + \int\limits_{\partial\Omega} \gamma(\omega)u^2(\omega)d\sigma. \tag{2.1}$$

Proof. Consider described above functionals $F[u], G[u], H[u]$ on $W^1(\Omega)$. We will find the pair (ϑ, u) that gives the minimum of the functional $F[u]$ in the set K. For this we investigate the minimization of the quadratic functional $H[u]$ on all functions $u(\omega)$, for which the integrals exist and which satisfy the boundary condition from (EVP). The necessary condition of existence of the functional minimum is $\delta H[u] = 0$. By the calculation of the first variation δH we have

$$\delta H[u] = -2\int\limits_\Omega (\triangle_\omega u + \vartheta u)\delta u d\Omega + 2\int\limits_{\partial\Omega} \frac{\partial u}{\partial \overrightarrow{\nu}}\delta u d\sigma + 2\int\limits_{\partial\Omega} \gamma(x) u \delta u d\sigma.$$

Hence we obtain the Euler equation and boundary condition that are our (EVP).

Backwards, let $u(\omega)$ be the solution of (EVP). By Theorem 2.1 $u \in C^2(\Omega)$. Therefore we can multiply both sides of equation (EVP) by u and integrate over Ω, using the Gauss-Ostrogradskiy formula:

$$\begin{gathered} 0 = \int\limits_\Omega (u\triangle_\omega u + \vartheta u^2)d\Omega = \vartheta \int\limits_\Omega u^2 d\Omega - \int\limits_\Omega |\nabla_\omega u|^2 d\Omega + \\ + \int\limits_\Omega \frac{\partial}{\partial \omega_i}\left(u \frac{J}{q_i}\frac{\partial u}{\partial \omega_i}\right) d\omega = \vartheta \int\limits_\Omega u^2 d\Omega - \int\limits_\Omega |\nabla_\omega u|^2 d\Omega + \int\limits_{\partial\Omega} u \frac{\partial u}{\partial \overrightarrow{\nu}} d\sigma = \\ = \vartheta \int\limits_\Omega u^2 d\Omega - \int\limits_\Omega |\nabla_\omega u|^2 d\Omega - \int\limits_{\partial\Omega} \gamma(x) u^2 d\sigma \underset{(\text{by } K)}{=} \vartheta - F[u] \Rightarrow \\ \vartheta = F[u], \end{gathered}$$

consequently required minimum is the least eigenvalue of (EVP).

The existence of a function $u \in K$ such that

$$F[u] \leq F[v] \text{ for all } v \in K \tag{2.2}$$

was proved by Theorem 2.1. □

Throughout what follows we work with the value λ defined by (1.2). Therefore the Friedrichs - Wirtinger inequality will be written in the next form

$$\lambda(\lambda+n-2)\int_\Omega \psi^2 d\Omega \le \int_\Omega |\nabla_\omega \psi|^2 d\Omega + \int_{\partial\Omega} \gamma(x)\psi^2 d\sigma, \qquad (2.3)$$
$$\forall \psi \in W^1(\Omega),\ \gamma(x)\in C^0(\partial\Omega),\ \gamma(x)\ge \gamma_0 > 0.$$

The Hardy -Friedrichs - Wirtinger inequality: *Let $u \in C^0(\overline{G})\cap W^1(G)$ and $\gamma(x)\in C^0(\partial\Omega)$, $\gamma(x)\ge\gamma_0>0$. Then*

$$\int_{G_0^d} r^{\alpha-4}u^2dx \le H(\lambda,n,\alpha)\left\{\int_{G_0^d} r^{\alpha-2}|\nabla u|^2dx + \int_{\Gamma_0^d} r^{\alpha-3}\gamma(x)u^2(x)ds\right\}, \qquad (2.4)$$

$$H(\lambda,n,\alpha) = \frac{1}{\lambda(\lambda+n-2)+\frac{1}{4}(4-n-\alpha)^2},\ \alpha\le 4-n \qquad (2.5)$$

provided that integrals on the right are finite.

Proof. By Theorem 2.2 the inequality (2.4) holds. Multiplying it by $r^{n-5+\alpha}$ and integrating over $r\in(0,d)$ we obtain

$$\int_{G_0^d} r^{\alpha-4}u^2dx \le \frac{1}{\lambda(\lambda+n-2)}\left\{\int_{G_0^d} r^{\alpha-2}\frac{1}{r^2}|\nabla_\omega u|^2dx + \int_{\Gamma_0^d} r^{\alpha-3}\gamma(x)u^2(x)ds\right\},$$
$$\forall\alpha\le 4-n. \qquad (2.6)$$

Hence it follows (2.4) for $\alpha = 4-n$. Now, let $\alpha < 4-n$. We prove that $u(0)=0$. In fact, from the representation $u(0)=u(x)-\big(u(x)-u(0)\big)$ by the Cauchy inequality we have $\frac{1}{2}|u(0)|^2 \le |u(x)|^2 + |u(x)-u(0)|^2$. Putting $v(x)=u(x)-u(0)$ we obtain

$$\frac{1}{2}|u(0)|^2\int_{G_0^d} r^{\alpha-4}dx \le \int_{G_0^d} r^{\alpha-4}u^2(x)dx + \int_{G_0^d} r^{\alpha-4}|v|^2dx < \infty \qquad (2.7)$$

(the first integral from the right is finite by (2.6) and the second is finite also in virtue of the Hardy inequalities (see Theorem 330 [7])). Since

$$\int_{G_0^d} r^{\alpha-4}dx = \text{mes}\Omega\int_0^d r^{\alpha+n-5}dr = \infty,$$

by $\alpha+n-4<0$, the assumption $u(0)\ne 0$ contradicts (2.7). Thus $u(0)=0$.

Therefore we can use the Hardy inequalities and we obtain

$$\int_{G_0^d} r^{\alpha-4}u^2dx \le \frac{4}{|4-n-\alpha|^2}\int_{G_0^d} r^{\alpha-2}u_r^2dx. \qquad (2.8)$$

Adding the inequalities (2.6), (2.8) and using the formula $|\nabla u|^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}|\nabla_\omega u|^2$, we get the desired (2.4). □

LEMMA 2.3.*Let G_0^d be the conical domain and*

$$V(\rho) = \int\limits_{G_0^\varrho} r^{2-n}|\nabla v|^2 dx + \int\limits_{\Gamma_0^\varrho} r^{1-n}\gamma(x)v^2(x)ds < \infty, \ \varrho \in (0, d).$$

Then

$$\int\limits_{\Omega}\left(\varrho v\frac{\partial v}{\partial r} + \frac{n-2}{2}v^2\right)\Bigg|_{r=\varrho} d\Omega \le \frac{\varrho}{2\lambda}V'(\varrho).$$

Proof. Writing $V(\varrho)$ in spherical coordinates

$$V(\varrho) = \int\limits_0^\varrho r^{2-n}\left(\int\limits_\Omega |\nabla v|^2 d\Omega\right) r^{n-1}dr + \int\limits_0^\varrho r^{1-n}\left(\int\limits_{\partial\Omega}\gamma(x)|v|^2 d\sigma\right) r^{n-2}dr =$$
$$= \int\limits_0^\varrho r\left(\int\limits_\Omega |\nabla v|^2 d\Omega\right) dr + \int\limits_0^\varrho \frac{1}{r}\left(\int\limits_{\partial\Omega}\gamma(x)|v|^2 d\sigma\right) dr$$

and differentiating by ϱ we obtain

$$V'(\varrho) = \int\limits_\Omega \left(\varrho\left(\frac{\partial v}{\partial r}\right)^2 + \frac{1}{\varrho}|\nabla_\omega v|^2\right)\Bigg|_{r=\varrho} d\Omega + \frac{1}{\varrho}\int\limits_{\partial\Omega}\gamma(x)v^2(x)d\sigma.$$

Moreover, by Cauchy's inequality, we have for all $\varepsilon > 0$

$$\rho u\frac{\partial u}{\partial r} \le \frac{\varepsilon}{2}u^2 + \frac{1}{2\varepsilon}\rho^2\left(\frac{\partial u}{\partial r}\right)^2.$$

Thus choosing $\varepsilon = \lambda$ we obtain, by the Friedrichs - Wirtinger inequality (2.3),

$$\int\limits_\Omega\left(\varrho v\frac{\partial v}{\partial r} + \frac{n-2}{2}v^2\right)\Bigg|_{r=\varrho} d\Omega \le \frac{\varepsilon+n-2}{2}\int\limits_\Omega v^2 d\Omega + \frac{\varrho^2}{2\varepsilon}\int\limits_\Omega\left(\frac{\partial v}{\partial r}\right)^2 d\Omega \le$$
$$\le \frac{\varepsilon+n-2}{2\lambda(\lambda+n-2)}\int\limits_\Omega |\nabla_\omega v|^2 d\Omega + \frac{\varrho^2}{2\varepsilon}\int\limits_\Omega\left(\frac{\partial v}{\partial r}\right)^2 d\Omega +$$
$$+ \frac{\varepsilon+n-2}{2\lambda(\lambda+n-2)}\int\limits_{\partial\Omega}\gamma(x)v^2(x)d\sigma = \frac{\varrho}{2\lambda}V'(\varrho).$$

□

PROPOSITION 2.4. The comparison principle. *Let G_0^d be a convex rotational cone with vertex at $\mathcal{O}$ and the aperture $\omega_0 \in (\frac{\pi}{2}, \pi)$. Let $\mathcal{L}$ be uniformly elliptic in G_0^d; $a^i(x), a(x) \in$*

$L^\infty_{loc}(G_0^d)$, $a(x) \le 0$ in G_0^d. Let $\gamma(x) \in C^0(\Gamma_0^d)$, $\gamma(x) \ge \gamma_0 > 0$ on Γ_0^d. Suppose that v and w are functions in $W^{2,n}_{loc}(G_0^d) \cap C^0(\overline{G_0^d})$ satisfying

$$\begin{cases} \mathcal{L}[w(x)] \le \mathcal{L}[v(x)], & x \in G_0^d; \\ \mathcal{B}[w(x)] \ge \mathcal{B}[v(x)], & x \in \Gamma_0^d; \\ \quad w(x) \ge v(x), & x \in \Omega_d \cup \mathcal{O}. \end{cases} \tag{2.9}$$

Then $v(x) \le w(x)$ in G_0^d.

This Proposition is the direct consequence of the Lieberman global and strong maximum principles for Lipschitz domains (Lemma 1.1 [11], Proposition 2.1 [14], Corollary 3.2 [14]). Here it should be noted that our condition $\omega_0 \in (\frac{\pi}{2}, \pi)$ guarantees the obliqueness of the operator $\mathcal{B}$ at $\forall x_0 \in \Gamma_0^d$ with the modulus of obliqueness lesser than 1.

3. The barrier function. The preliminary estimate of the solution modulus.

Let G_0^d be convex rotational cone with a solid angle $\omega_0 \in (0, \pi)$ and the lateral surface Γ_0^d, such that $G_0^d \subset \{x_1 \ge 0\}$. Let define the linear elliptic operator:

$$\mathcal{L}_0 \equiv a^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j};\ a^{ij}(x) = a^{ji}(x),\ x \in G_0^d;$$

$$\nu\xi^2 \le a^{ij}(x)\xi_i\xi_j \le \mu\xi^2, \forall x \in G_0^d, \forall \xi \in \mathbb{R}^n; \nu, \mu = const > 0$$

and the boundary operator:

$$\mathcal{B} \equiv \frac{\partial}{\partial \vec{n}} + \frac{1}{|x|}\gamma(x), \qquad \gamma(x) \ge \gamma_0 > 0,\ x \in \Gamma_0^d \setminus \mathcal{O}.$$

LEMMA 3.1. (About the existence of the barrier function). *Let be given the numbers $\gamma_0 > \tan\frac{\omega_0}{2}, \delta > 0, g_1 \ge 0, d \in (0,1)$. There exist $h > 0$, which are defined only by G_0, the numbers $B, \varkappa_0 > 0$ and the function $w(x) \in C^1(\overline{G_0}) \cap C^2(G_0)$, which depend only on the domain G_0,the ellipticity constants ν, μ of the operator $\mathcal{L}_0$ and the quantities γ_0, δ, g_1, such that for any $\varkappa \in (0; \min(\delta, \varkappa_0, \gamma_0 h - 1))$are satisfied the following:*

$$\mathcal{L}_0[w(x)] \le -\nu h^2|x|^{\varkappa-1};\ \ x \in G_0^d; \tag{3.1}$$

$$\mathcal{B}[w(x)] \ge g_1|x|^{\delta};\ \ x \in \Gamma_0^d \setminus \mathcal{O}; \tag{3.2}$$

$$0 \le w(x) \le C_0(\varkappa, B, \omega_0)|x|^{\varkappa+1};\ \ x \in \overline{G_0^d}; \tag{3.3}$$

$$|\nabla w(x)| \le C_1(\varkappa, B, \omega_0)|x|^{\varkappa};\ \ x \in \overline{G_0^d}. \tag{3.4}$$

Proof. Let $(x, y, x') \in \mathbb{R}^N$, where $x = x_1, y = x_2, x' = (x_3, ..., x_N)$. In $\{x_1 \ge 0\}$ we consider the cone K with the vertex in $\mathcal{O}$, such that $K \supset G_0^d$ (we recall that $G_0^d \subset \{x_1 \ge 0\}$). Let ∂K- the lateral surface of K and let $\partial K \cap y\mathcal{O}x = \Gamma_\pm$ is $x = \pm hy$, where $h = \cot\frac{\omega_0}{2}$, $0 < \omega_0 < \pi$, such that in the interior of K holds the inequality $x > h|y|$. We shall consider the function:

$$w(x; y, x') \equiv x^{\varkappa-1}(x^2 - h^2y^2) + Bx^{\varkappa+1}, \text{ with some } \varkappa \in (0; 1), B > 0. \tag{3.5}$$

Let coefficients of the operator $\mathcal{L}_0$ are: $a^{2,2} = a$, $a^{1,2} = b$, $a^{1,1} = c$. Then we have:

$$\mathcal{L}_0 w = a w_{yy} + 2b w_{xy} + c w_{xx};\ \nu\eta^2 \le a\eta_1^2 + 2b\eta_1\eta_2 + c\eta_2^2 \le \mu\eta^2;$$
$$\eta^2 = \eta_1^2 + \eta_2^2;\ \forall \eta_1, \eta_2 \in \mathbb{R}. \quad (3.6)$$

Let calculate the operator $\mathcal{L}_0$ on the function (3.5). For $t = \frac{y}{x}, |t| < \frac{1}{h}$ we obtain: $\mathcal{L}_0 w = -h^2 x^{\varkappa-1}\phi(\varkappa)$, where

$$\begin{aligned}\phi(\varkappa) &= 2a - 4bt + 4bt\varkappa - ch^{-1}(1+B)(\varkappa^2+\varkappa) + ct^2\varkappa^2 - 3ct^2\varkappa + 2ct^2 = \\ &= c(t^2 - h^{-2}(1+B))\varkappa^2 + (4bt - ch^{-2}(1+B) - 3ct^2)\varkappa + 2(ct^2 - 2bt + a);\end{aligned}$$
$$c(t^2 - h^{-2}(1+B)) = c\left(\frac{y^2}{x^2} - \frac{1+B}{h^2}\right) \le -c\frac{B}{h^2} < 0.$$

Because of (3.6), we have $\phi(0) = 2(ct^2 - 2bt + a) \ge 2\nu$, and since $\phi(\varkappa)$ is the square function, there exists the number $\varkappa_0 > 0$ depending only on ν, μ, h, such that $\phi(\varkappa) \ge \nu$ for $\varkappa \in [0; \varkappa_0]$. Therefore we obtain (3.1).

Now, let notice $\Gamma_\pm : x = \pm hy,\ h = \cot\frac{\omega_0}{2},\ 0 < \omega_0 < \pi$, then we have

$$\text{on } \Gamma_+ : \begin{cases} x &= r\cos\frac{\omega_0}{2}, \\ y &= r\sin\frac{\omega_0}{2} \end{cases} \begin{cases} \angle(\vec{n}, x) = \frac{\pi}{2} + \frac{\omega_0}{2}, \\ \angle(\vec{n}, y) = \frac{\omega_0}{2} \end{cases}$$
$$\text{on } \Gamma_- : \begin{cases} x &= r\cos\frac{\omega_0}{2}, \\ y &= -r\sin\frac{\omega_0}{2} \end{cases} \begin{cases} \angle(\vec{n}, x) = \frac{\pi}{2} + \frac{\omega_0}{2}, \\ \angle(\vec{n}, y) = \pi + \frac{\omega_0}{2} \end{cases}$$
$$\sin\frac{\omega_0}{2} = \frac{1}{\sqrt{1+h^2}}, \quad \cos\frac{\omega_0}{2} = \frac{h}{\sqrt{1+h^2}}.$$

Therefore we obtain:

$$\begin{aligned} w_x = (1+\varkappa)x^\varkappa(1+B) - (\varkappa-1)h^2y^2x^{\varkappa-2} &\Rightarrow w_x\big|_{\Gamma_\pm} = [2 + B(1+\varkappa)]x^\varkappa, \\ w_y = -2h^2yx^{\varkappa-1} &\Rightarrow w_y\big|_{\Gamma_\pm} = \mp 2hx^\varkappa. \end{aligned} \quad (3.7)$$

Because of $\frac{\partial w}{\partial \vec{n}}\Big|_{\Gamma_\pm} = w_x \cos\angle(\vec{n}, x)\Big|_{\Gamma_\pm} + w_y \cos\angle(\vec{n}, y)\Big|_{\Gamma_\pm}$ and (3.7), we get:

$$\frac{\partial w}{\partial \vec{n}}\bigg|_{\Gamma_\pm} = -r^\varkappa \frac{h^\varkappa}{(1+h^2)^{\frac{\varkappa+1}{2}}}[2(1+h^2) + B(1+\varkappa)].$$

Hence it follows:

$$\mathcal{B}[w]\Big|_{\Gamma_\pm} \ge \frac{h^\varkappa}{(1+h^2)^{\frac{\varkappa+1}{2}}} r^\varkappa \big[Bh\gamma_0 - B(1+\varkappa) - 2(1+h^2)\big].$$

Since $h > \frac{1}{\gamma_0}$, we obtain:

$$\mathcal{B}[w]\Big|_{\Gamma_\pm^d} \ge \frac{h^\varkappa r^\varkappa}{(1+h^2)^{\frac{\varkappa+1}{2}}} \left[B(h\gamma_0 - 1 - \varkappa) - 2(1+h^2)\right] \ge g_1 r^\delta,\ 0 < r < d < 1,$$

if we choose: $\varkappa \le \delta \Rightarrow r^{\varkappa} \ge r^{\delta}$ and

$$B \ge \Big\{ \frac{g_1(1+h^2)^{\frac{\varkappa+1}{2}}}{h^{\varkappa}} + 2(1+h^2) \Big\} \cdot \frac{1}{h\gamma_0 - 1 - \varkappa}. \tag{3.8}$$

Now we'll proof (3.3). Let rewrite the function (3.5) in spherical coordinates. Recalling that $h = \cot \frac{\omega_0}{2}$, we obtain:

$$w(x; y, x') = (1+B)(r\cos\omega)^{1+\varkappa} - h^2 r^2 \sin^2\omega (r\cos\omega)^{\varkappa-1} =$$
$$= r^{1+\varkappa} \cos^{\varkappa-1}\omega \left(B\cos^2\omega + \frac{\chi(\omega)}{\sin^2\frac{\omega_0}{2}} \right), \quad \forall \omega \in \big[-\frac{\omega_0}{2}; \frac{\omega_0}{2}\big],$$

where $\chi(\omega) = \sin\left(\frac{\omega_0}{2} - \omega\right) \cdot \sin\left(\frac{\omega_0}{2} + \omega\right)$. We find $\chi'(\omega) = -\sin 2\omega$ and $\chi'(\omega) = 0$ for $\omega = 0$. Now we see that $\chi''(0) = -2\cos 0 = -2 < 0$. This way we have $\max\limits_{\omega \in [-\omega_0/2, \omega_0/2]} \chi(\omega) = \chi(0) = \sin^2 \frac{\omega_0}{2}$ and therefore:

$$w(x; y, x') \le r^{1+\varkappa} \cos^{\varkappa-1}\omega (B\cos^2\omega + 1) \le r^{1+\varkappa} \cos^{\varkappa+1} \left(B + \frac{1}{\cos^2\omega} \right) \le$$
$$\le r^{1+\varkappa} \left(B + \frac{1}{\cos^2\omega} \right).$$

Hence it follows (3.3). Finally, from (3.7) it follows (3.4). □

With help of Lemma 3.1 and Proposition 2.4 we estimate $|u(x)|$ for (L) in the neighborhood of conical point.

THEOREM 3.2.*Let $u(x)$ be a strong solution of problem (L) and satisfy assumptions (a)-(d). Then there exist the numbers $d \in (0,1)$ and $\varkappa > 0$ depending only on $\nu, \mu, n, \varkappa_0$, $\omega_0, f_1, \beta, \gamma_0, s, g_1, M_0$ and the domain $G_0^{d_0}$ such that*

$$|u(x) - u(0)| \le C_0 |x|^{\varkappa+1}, \quad x \in G_0^d, \tag{3.9}$$

where the positive constant C_0 not depends on $u(x)$ but only on $\nu, \mu, n, f_1, g_1, \beta, s, \gamma_0, M_0$ and the domain G.

Now the proof of Theorems 1.1–1.3 is based on the deduction of the global and local integral weighted estimates of the (L) solutions with the sharp estimating constants. For this we use the Hardy -Friedrichs - Wirtinger inequality, Lemma 2.3 and the Theorem about the solutions of the Cauchy problem for a differential inequality (see [2, 1]).

4. Examples.

We present examples which show that conditions of Theorems 1.1– 1.3 (in particular the Dini condition on the function $\mathcal{A}(r)$ in condition (b) at the point $\mathcal{O}$ in Theorem 1.1) are essential for their validity. Suppose $n = 2$, the domain G lies inside the corner

$$G_0 = \{(r,\omega)\,|\,r > 0;\ -\frac{\omega_0}{2} < \omega < \frac{\omega_0}{2}\}, \quad \omega_0 \in]0, \pi[;$$

$\mathcal{O} \in \partial G$ and in some neighborhood of $\mathcal{O}$ the boundary ∂G coincides with the sides of the corner $\omega = -\frac{\omega_0}{2}$ and $\omega = \frac{\omega_0}{2}$. We denote

$$\Gamma_\pm = \{(r,\omega)\ |\ r > 0;\ \omega = \pm\frac{\omega_0}{2}\}$$

and we put
$$\gamma(x)\Big|_{\omega=\pm\frac{\omega_0}{2}} = \gamma_\pm = const > 0.$$

I. We consider the following problem:
$$\begin{cases} \triangle u = 0, & x \in G_0; \\ \left(\frac{\partial u}{\partial n} + \frac{1}{r}\gamma_\pm u\right)\Big|_{\Gamma_\pm} = 0. \end{cases}$$

We verify that the function $u(r,\omega) = r^\lambda \psi(\omega)$ is a solution of our problem, if λ is the least positive eigenvalue and $\psi(\omega)$ is regular eigenfunction associated to this λ of the problem
$$\begin{cases} \psi'' + \lambda^2\psi = 0, & \omega \in \left(-\frac{\omega_0}{2}, \frac{\omega_0}{2}\right) \\ (\pm\psi' + \gamma_\pm\psi)\Big|_{\omega=\pm\frac{\omega_0}{2}} = 0. \end{cases}$$

Precisely λ is defined from transcedence equation
$$\tan(\lambda\omega_0) = \frac{\lambda(\gamma_+ + \gamma_-)}{\lambda^2 - \gamma_+\gamma_-}. \tag{4.1}$$

And then we find eigenfunction
$$\psi(\omega) = \lambda\cos\left[\lambda\left(\omega - \frac{\omega_0}{2}\right)\right] - \gamma_+\sin\left[\lambda\left(\omega - \frac{\omega_0}{2}\right)\right]. \tag{4.2}$$

The existence of positive solution of (4.1) may be verify by the graphic method. This example shows that the exponent λ in (1.3) cannot be increased.

REMARK 2.*In order that $\lambda > 1$ we show that the condition $\gamma(x) \geq \gamma_0 > \tan\frac{\omega_0}{2}$ from the assumption (c) of our Theorems is justified. In fact, we rewrite the equation (4.1) in the equivalent form*
$$\lambda = \frac{1}{\omega_0}\left(\arctan\frac{\gamma_+}{\lambda} + \arctan\frac{\gamma_-}{\lambda}\right). \tag{4.3}$$

Hence it follows that must be fulfilled
$$\begin{gathered} 1 < \lambda < \tfrac{1}{\omega_0}\left(\arctan\gamma_+ + \arctan\gamma_-\right) \quad \Rightarrow \\ \omega_0 < \arctan\tfrac{\gamma_+ + \gamma_-}{1-\gamma_+\gamma_-}, \ \textit{provided}\ \gamma_+\gamma_- < 1. \end{gathered} \tag{4.4}$$

But our condition from the assumption (c) means that $\gamma_\pm \geq \gamma_0 > \tan\frac{\omega_0}{2}$. Hence we obtain
$$\frac{\gamma_+ + \gamma_-}{1-\gamma_+\gamma_-} \geq \frac{2\gamma_0}{1-\gamma_0^2} > \frac{2\tan\frac{\omega_0}{2}}{1-\tan^2\frac{\omega_0}{2}} = \tan\omega_0,\ \omega_0 < \frac{\pi}{2}$$
thus we get (4.4). In case $\gamma_\pm \geq \gamma_0 > \tan\frac{\omega_0}{2} \geq 1$ for $\omega_0 \in [\frac{\pi}{2}, \pi)$ inequality $\lambda > 1$ is fulfilled a fortiori, because of the property of monotonic increase of the eigenvalues together with the increase of $\gamma(x)$ (see for example Theorem 6 §2, chapter VI [3]).

II. The function

$$u(r,\omega) = r^{\lambda}\left(\ln\frac{1}{r}\right)^{\frac{\lambda-1}{\lambda+1}}\psi(\omega)$$

with defined by (4.1) - (4.2) $\lambda, \psi(\omega)$ is a solution in the corner G_0 of the problem

$$\begin{cases} \sum\limits_{i,j=1}^{n} a^{ij}(x)u_{x_ix_j} = 0, \ x \in G_0, \\ \left(\frac{\partial u}{\partial n} + \frac{1}{r}\gamma_{\pm}u\right)\Big|_{\Gamma_{\pm}} = 0, \ \gamma_{\pm} > 0, \end{cases}$$

where

$$a^{11}(x) = 1 - \frac{2}{\lambda+1}\cdot\frac{x_2^2}{r^2\ln 1/r}, \quad r > 0;$$
$$a^{12}(x) = a^{21}(x) = \frac{2}{\lambda+1}\cdot\frac{x_1x_2}{r^2\ln 1/r}, \quad r > 0;$$
$$a^{22}(x) = 1 - \frac{2}{\lambda+1}\cdot\frac{x_1^2}{r^2\ln 1/r}, \quad r > 0;$$
$$a^{ij}(0) = \delta_i^j, \ (i,j = 1,2).$$

In the domain G_0^d, $d < e^{-2}$ the equation is uniformly elliptic with ellipticity constants $\mu = 1$ and $\nu = 1 - \frac{2}{\ln(1/d)}$. Further,

$$\mathcal{A}(r) = \frac{2}{\lambda+1}\ln^{-1}\left(\frac{1}{r}\right),$$

i.e., the function $\mathcal{A}(r)$ does not satisfy a Dini condition at zero. Moreover, $a^{ij}(x)$ are continuous at the point $\mathcal{O}$. This example shows that the condition of Theorem 1.1 about Dini-continuity of leading coefficients of (L) are essential, as well illustrates the precision of assumptions of Theorem 1.3.

III. The function

$$u(r,\omega) = r^{\lambda}\ln\frac{1}{r}\psi(\omega)$$

with defined by (4.1) - (4.2) $\lambda, \psi(\omega)$ is a solution in the corner G_0 of the problem

$$\begin{cases} \triangle u + \frac{2\lambda}{r^2\ln\frac{1}{r}}u = 0, \ x \in G_0, \\ \left(\frac{\partial u}{\partial n} + \frac{1}{r}\gamma_{\pm}u\right)\Big|_{\Gamma_{\pm}} = 0, \ \gamma_{\pm} > 0. \end{cases}$$

This example shows that assumptions of Theorems 1.1 and 1.3 on lowest coefficients of (L) are precise and essential.

IV. The function

$$u(r,\omega) = r^{\lambda}\ln\frac{1}{r}\psi(\omega)$$

with defined by (4.1) - (4.2) $\lambda, \psi(\omega)$ is a solution in the corner G_0 of the problem

$$\begin{cases} \triangle u = -2\lambda r^{\lambda-2}\psi(\omega), \ x \in G_0, \\ \left(\frac{\partial u}{\partial n} + \frac{1}{r}\gamma_{\pm} u\right)\Big|_{\Gamma_{\pm}} = 0, \ \gamma_{\pm} > 0. \end{cases}$$

All assumptions of Theorem 1.1 are fulfilled with $s = \lambda$. This example shows the precision of assumptions for right sides of (L) in Theorem 1.1.

1. *Bochniak M., Borsuk M.* Dirichlet problem for linear elliptic equations degene- rating at a conical boundary point // Analysis. Munchen.- Germany.- 2003.- 23(3).- p. 225– 248.
2. *Borsuk M.V.* Best-possible estimates of solutions of the Dirichlet problem for linear elliptic nondivergence equations of second order in a neighborhood of a conical point on the boundary, // Math. USSR Sbornik.- v. 74.- 1993.- 185–201.
3. *Courant R., Hilbert D.* Methoden der mathematischen Physik, Bd. 1 // Springer-Verlag.- Berlin.- 1931.
4. *Faierman M.* Regularity of solutions of an elliptic boundary value problem in a rectangle // Comm. in PDE.- v. 12.- 1987.- p. 285–305.
5. *Garroni M.G., Solonikov V.A. and Vivaldi M.A.* On the oblique derivative problem in an infinite angle // Topological methods in nonlinear analysis.- v. 7.- 1996.- p. 299–325.
6. *Gilbarg D. and Trudinger N.S.* Elliptic Partial Differential Equations of Second Order // Springer-Verlag.- Berlin/Heidelberg/New York.- 1977. - Revised Third Printing.- 1998.
7. *Hardy G.H., Littlewood J.E. and Pólya G.* Inequalities.- University Press, Cambridge.- 1952.
8. *Kondrat'ev V.A.* Boundary value problem for elliptic equations in domains with conical or angular points // Trudy Moscov. Mat. Obshch.- v. 16.- 1967.- p. 209–292.
9. *Kozlov V.A., Maz'ya V.G. and Rossman J.* Elliptic boundary value problem in domains with point singularities // AMS. Mathmatical surveys and monographs.- v. 52.- 1997.
10. *Kozlov V.A., Maz'ya V.G. and Rossman J.* Spectral problems associated with corner singularities of solutions to elliptic equations // AMS. Mathmatical surveys and monographs.- v. 85.- 2001.
11. *Lieberman G.M.* Local estimates for subsolutions and supersolutions of oblique derivative problems for general second order elliptic equations // Trans. of AMS..- v. 304.- 1987.- p. 343–353.
12. *Lieberman G.M.* Oblique derivative problems in Lipschitz domains. Continous boundary data // Bull. Un. Mat. Ital. B.- v. 7(1).- 1987.- p. 1185–1210.
13. *Lieberman G.M.* The Perron process applied to oblique derivative problems // Advances in Math.- v. 55.- 1985. p. 161–172.
14. *Lieberman G.M.* Pointwise estimates for oblique derivative problems in nonsmooth domains// J. Diff. Equat.- v. 173.- 2001.- no.1.- p. 178 – 211.
15. *Lieberman G.M.* The conormal derivative problem for equations of variational type in nonsmooth domains // Trans. of AMS. v. 330.- 1992.- p. 41-67.
16. *Reisman H.* Second order elliptic boundary value problem in a domain with edges // Comm. in PDE.- v. 6.- 1995.- p. 1023-1042.
17. *Sobolev S.L.* Some applications of functional analysis in mathematical physics // Trans. of Math. Monographs.- v. 90.- 1991.- AMS, Providence, Rhode Island.

Department of Mathematics and Informatics,
University of Warmia and Mazury in Olsztyn,
10-957 Olsztyn-Kortowo, Poland
`borsuk@uwm.edu.pl`

Received 30.08.2003