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**SINGULARLY PERTURBED PROBLEMS OF HYPERBOLIC-PARABOLIC TYPE WITH LIPSCHITZIAN NONLINEARITY**

We study the behavior of solutions of the Cauchy problem  $\varepsilon u''(t) + u'(t) + Au(t) + B(u(t)) = f(t), u(0) = u_0, u'(0) = u_1$  in the Hilbert space  $H$  as  $\varepsilon \rightarrow 0$ , where  $A$  is a linear, self-adjoint, strong positive operator and  $B$  is nonlinear Lipschitzian operator.

**1. Introduction.**

Let  $V$  and  $H$  be the real Hilbert spaces equipped with the norms  $\|\cdot\|$  and  $|\cdot|$ , respectively, such that  $V \subset H$ , where the embedding is defined densely and continuously. By  $(\cdot, \cdot)$  denote the scalar product in  $H$ . Let  $A : V \rightarrow H$  be a linear, self-adjoint operator and

$$(Au, u) \geq \omega \|u\|^2, \quad \forall u \in V, \quad \omega > 0. \tag{1}$$

Let  $B : H \rightarrow H$  be a nonlinear operator which satisfies the Lipschitz condition

$$|B(u) - B(v)| \leq L|u - v|, \quad \forall u, v \in H. \tag{2}$$

In this paper we shall study the behavior of the solutions of the problem

$$\begin{cases} \varepsilon u''(t) + u'(t) + Au(t) + B(u(t)) = f(t), \\ u(0) = u_0, u'(0) = u_1 \end{cases} \tag{P_\varepsilon}$$

as  $\varepsilon \rightarrow 0$ , where  $\varepsilon$  is a small positive parameter. Our aim is to show that  $u \rightarrow v$  as  $\varepsilon \rightarrow 0$ , where  $v$  is the solution of the problem

$$\begin{cases} v'(t) + Av(t) + B(v(t)) = f(t), \\ v(0) = u_0, \end{cases} \tag{P_0}$$

The main tool in our approach is the relation between the solutions of the problems  $(P_\varepsilon)$  and  $(P_0)$  in the linear case.

Let us remind some notations which will be used in the sequel.

For  $k \in \mathbb{N}, p \in [1, \infty)$  and  $(a, b) \subset (-\infty, +\infty)$  we denote by  $W^{k,p}(a, b; H)$  the usual Sobolev spaces of vectorial distributions:  $W^{k,p}(a, b; H) = \{f \in D'(a, b; H); u^{(l)} \in L^p(a, b; H), l = 0, 1, \dots, k\}$  equipped with the norm

$$\|f\|_{W^{k,p}(a,b;H)} = \left( \sum_{l=0}^k \|f^{(l)}\|_{L^p(a,b;H)}^p \right)^{1/p}.$$

For each  $k \in \mathbb{N}, W^{k,\infty}(a, b; H) = \{f \in D'(a, b; H); u^{(l)} \in L^\infty(a, b; H), l = 0, 1, \dots, k\}$  is the Banach space equipped with the norm

$$\|f\|_{W^{k,\infty}(a,b;H)} = \max_{0 \leq l \leq k} \|f^{(l)}\|_{L^\infty(a,b;H)}.$$

For  $s \in \mathbb{R}, k \in \mathbb{N}$  and  $p \in [1, \infty]$  we denote the following Banach spaces  $W_s^{k,p}(a, b; H) = \{f : (a, b) \rightarrow H; e^{-st} f^{(l)} \in L^p(a, b; H), l = 0, 1, \dots, k\}$  equipped with the norm

$$\|f\|_{W_s^{k,p}(a,b;H)} = \max_{0 \leq l \leq k} \|e^{-st} f^{(l)}(\cdot)\|_{L^p(a,b;H)}.$$

## 2. A priori estimates for solutions of the problem $(P_\varepsilon)$ .

In this section we shall prove an *a priori* estimates for the solutions of the problems  $(P_\varepsilon)$  which are uniform relative to the small values of parameter  $\varepsilon$ . First of all we shall remind the existence theorem for the solutions of the problems  $(P_\varepsilon)$  and  $(P_0)$ .

**THEOREM A.** [1]. *Let  $T > 0$ . Suppose that  $f \in W^{1,1}(0, T; H)$ ,  $u_0, u_1 \in V$  and the operators  $A$  and  $B$  satisfy the conditions (1) and (2) respectively. Then there exists a unique function  $u \in C(0, T; H) \cap L^\infty(0, T; V)$  satisfying the problem  $(P_\varepsilon)$  and the conditions:  $Au \in L^\infty(0, T; H)$ ,  $u' \in L^\infty(0, T; V)$ ,  $u'' \in L^\infty(0, T; H)$ .*

**THEOREM B.** [1]. *If  $f \in W^{1,1}(0, T; H)$ ,  $u_0 \in V$  and  $A$  and  $B$  satisfy the conditions (1), (2), then there exists a unique strong solution  $v \in W^{1,\infty}(0, T; H)$  of the problem  $(P_0)$  and the estimates*

$$|v(t)| \leq e^{(L-\omega)t} \left( |u_0| + \int_0^t e^{-(L-\omega)\tau} (|f(\tau) - B(0)|) d\tau \right),$$

$$|v'(t)| \leq e^{(L-\omega)t} \left( |Au_0 + B(u_0) - f(0)| + \int_0^t e^{-(L-\omega)\tau} |f'(\tau)| d\tau \right),$$

are true for  $0 \leq t \leq T$ .

We remind that a function  $v \in C([0, T]; H)$  is said to be a *strong solution* (in the following named *solution*) for Cauchy problem  $(P.v)$  if: a)  $v$  is absolutely continuous on any compact subinterval of  $(0, T)$ ; b)  $v(t) \in D(A)$  a.e.  $t \in (0, T)$ ; c)  $v(0) = u_0$  and  $v$  satisfies the equation from  $(P.v)$  a.e.  $t \in (0, T)$ .

Before to prove the estimates for solutions of problem  $(P_\varepsilon)$  we recall the following well-known lemma.

**LEMMA A.** [2]. *Let  $\psi \in L^1(a, b)$  ( $-\infty < a < b < \infty$ ) with  $\psi \geq 0$  a. e. on  $(a, b)$  and let  $c$  be a fixed real constant. If  $h \in C([a, b])$  verify*

$$\frac{1}{2}h^2(t) \leq \frac{1}{2}c^2 + \int_a^t \psi(s)h(s)ds, \quad \forall t \in [a, b],$$

then

$$h(t) \leq |c| + \int_a^t \psi(s)ds, \quad \forall t \in [a, b]$$

also holds.

Denote by

$$E_1(u, t) = \varepsilon|u'(t)| + |u(t)| + \left( \varepsilon \left( Au(t), u(t) \right) \right)^{1/2} + \left( \varepsilon \int_0^t |u'(\tau)|^2 d\tau \right)^{1/2} + \left( \int_0^t \left( Au(\tau), u(\tau) \right) d\tau \right)^{1/2}.$$

**LEMMA 1.** *Suppose that for any  $T > 0$   $f \in W^{1,1}(0, T; H)$ ,  $u_0, u_1 \in V$  and the operators  $A$  and  $B$  satisfy the conditions (1) and (2). Then there exists the positive constants  $\gamma$  and  $C$  depending on  $\omega$  and  $L$  such that for the solutions of the problem  $(P_\varepsilon)$  the following estimates*

$$E_1(u, t) \leq Ce^{\gamma t} \left( E_1(u, 0) + \int_0^t |f(\tau) - B(0)| e^{-\gamma\tau} d\tau \right), \quad 0 \leq t \leq T, \quad (3)$$

$$E_1(u', t) \leq C e^{\gamma t} \left( E_1(u', 0) + \int_0^t |f'(\tau)| e^{-\gamma \tau} d\tau \right), \quad 0 \leq t \leq T \quad (4)$$

are true. If  $B = 0$ , then in (3) and (4)  $\gamma = 0$ .

*Proof.* Denote by

$$E(u, t) = \varepsilon^2 |u'(t)|^2 + \frac{1}{2} |u(t)|^2 + \varepsilon \left( Au(t), u(t) \right) + \varepsilon \int_0^t |u'(\tau)|^2 d\tau + \\ + \varepsilon \left( u(t), u'(t) \right) + \int_0^t \left( Au(\tau), u(\tau) \right) d\tau.$$

The direct computations show that for every solution of the problem  $(P_\varepsilon)$  the following equality

$$\frac{d}{dt} E(u, t) = \left( f(t) - B(0), u(t) + 2\varepsilon u'(t) \right) - \left( (Bu(t)) - B(0), u(t) + \varepsilon u'(t) \right) \quad (5)$$

is true. Since  $|B(u) - B(0)| \leq L|u|$ ,  $E(u, t) \geq 0$  and  $|u|(|u| + 2\varepsilon|u'|) \leq 2\gamma E(u, t)$  with some  $\gamma > 0$ , then from (5) follows the inequality

$$\frac{d}{dt} E(u, t) \leq 2\gamma E(u, t) + \left( |f(t) - B(0)| (|u(t)| + 2\varepsilon|u'(t)|) \right). \quad (6)$$

As  $|u(t)| + 2\varepsilon|u'(t)| \leq 2C(E(u, t))^{1/2}$  with some  $C > 0$ , then from (6) we have

$$\frac{d}{dt} \left( e^{-2\gamma t} E(u, t) \right) \leq 2C |f(t) - B(0)| \left( E(u, t) \right)^{1/2} e^{-2\gamma t}.$$

Integrating the last inequality we obtain

$$\frac{1}{2} E(u, t) e^{-2\gamma t} \leq \frac{1}{2} E(u, 0) + C \int_0^t e^{-2\gamma \tau} \left( E(u, \tau) \right)^{1/2} |f(\tau) - B(0)| d\tau.$$

Using Lemma A from the last inequality we get the estimate

$$\left( E(u, t) \right)^{1/2} \leq e^{\gamma t} \left[ \left( E(u, 0) \right)^{1/2} + C \int_0^t |f(\tau) - B(0)| e^{-\gamma \tau} d\tau \right]. \quad (7)$$

It is easy to see that there exist positive constants  $C_0, C_1$  depending only on  $\omega$  such that

$$C_0 \left( E(u, t) \right)^{1/2} \leq E_1(u, t) \leq C_1 \left( E(u, t) \right)^{1/2}. \quad (8)$$

Using the inequalities (8) from (7) we obtain the estimate (3).

To prove the estimate (4) let us denote by  $u_h(t) = u(t+h) - u(t)$ ,  $h > 0$ ,  $t \geq 0$ . For any solution of the problem  $(P_\varepsilon)$  we have

$$\frac{d}{dt} E(u_h, t) = (2\varepsilon(u'_h(t) + u_h(t)), f_h - (B(u(t)))_h).$$

Since

$$|(B(u(t)))_h| = |B(u(t+h)) - B(u(t))| \leq L|u_h(t)|, \quad |2\varepsilon u'_h + u_h(t)| \leq 2C(E(u_h, t))^{1/2},$$

and

$$|2\varepsilon u'_h(t) + u_h(t)||u_h(t)| \leq 2\gamma E(u_h, t),$$

then we have

$$\frac{d}{dt}(e^{-2\gamma t} E(u_h, t)) \leq 2C(E(u_h, t))^{1/2} |f_h(t)| e^{-2\gamma t}.$$

Integrating the last inequality we get

$$E(u_h, t)e^{-2\gamma t} \leq E(u_h, 0) + \int_0^t e^{-2\gamma\tau} |f_h(\tau)|(E(u_h, \tau))^{1/2} d\tau.$$

Dividing the last inequality by  $h^2$  and then passing to the limit as  $h \rightarrow 0$  we get

$$E(u', t)e^{-2\gamma t} \leq E(u', 0) + \int_0^t e^{-2\gamma\tau} |f'(\tau)|(E(u', \tau))^{1/2} d\tau. \quad (9)$$

Since  $u'(0) = u_1, \varepsilon u''(0) = f(0) - Au_0 - u_1 - B(u_0)$ , then using Lemma A and (8) from (9) we obtain the estimate (4) in the same way as was obtained the the estimate (3). Lemma 1 is proved.

### 3. Relation between the solutions of the problems $(P_\varepsilon)$ and $(P_0)$ in the linear case.

In this section we shall give the relation between the solutions of the problem  $(P_\varepsilon)$  and  $(P_0)$  in the linear case, i. e. in the case when  $B = 0$ . This relation was inspired by the work [3]. At first we shall prove some properties of the kernel  $K(t, \tau, \varepsilon)$  of transformation which realizes this connection.

For  $\varepsilon > 0$  denote

$$K(t, \tau, \varepsilon) = \frac{1}{2\sqrt{\pi\varepsilon}} \left( K_1(t, \tau, \varepsilon) + 3K_2(t, \tau, \varepsilon) - 2K_3(t, \tau, \varepsilon) \right),$$

where

$$K_1(t, \tau, \varepsilon) = \exp \left\{ \frac{3t - 2\tau}{4\varepsilon} \right\} \lambda \left( \frac{2t - \tau}{2\sqrt{\varepsilon t}} \right), \quad K_2(t, \tau, \varepsilon) = \exp \left\{ \frac{3t + 6\tau}{4\varepsilon} \right\} \lambda \left( \frac{2t + \tau}{2\sqrt{\varepsilon t}} \right),$$

$$K_3(t, \tau, \varepsilon) = \exp \left\{ \frac{\tau}{\varepsilon} \right\} \lambda \left( \frac{t + \tau}{2\sqrt{\varepsilon t}} \right), \quad \lambda(s) = \int_s^\infty e^{-\eta^2} d\eta.$$

LEMMA 2. [4] *The function  $K(t, \tau, \varepsilon)$  possesses the following properties:*

- (i) For any fixed  $\varepsilon > 0$   $K \in C(\{t \geq 0\} \times \{\tau \geq 0\}) \cap C^\infty(\{t > 0\} \times \{\tau > 0\})$ ;
- (ii)  $K_t(t, \tau, \varepsilon) = \varepsilon K_{\tau\tau}(t, \tau, \varepsilon) - K_\tau(t, \tau, \varepsilon), \quad t > 0, \tau > 0$ ;
- (iii)  $\varepsilon K_\tau(t, 0, \varepsilon) - K(t, 0, \varepsilon) = 0, \quad t \geq 0$ ;
- (iv)  $K(0, \tau, \varepsilon) = \frac{1}{2\varepsilon} \exp \left\{ -\frac{\tau}{2\varepsilon} \right\}, \quad \tau \geq 0$ ;
- (v) For each fixed  $t > 0, s, q \in \mathbb{N}$  there exist constants  $C_1(s, q, t, \varepsilon) > 0$  and  $C_2(s, q, t) > 0$  such that

$$|\partial_t^s \partial_\tau^q K(t, \tau, \varepsilon)| \leq C_1(s, q, t, \varepsilon) \exp\{-C_2(s, q, t)\tau/\varepsilon\}, \quad \tau > 0;$$

(vi)  $K(t, \tau, \varepsilon) > 0, \quad t \geq 0, \quad \tau \geq 0;$

(vii) Let  $\varepsilon$  be fixed,  $0 < \varepsilon \ll 1$ . For any  $\varphi : [0, \infty) \rightarrow H$  continuous on  $[0, \infty)$  such that  $|\varphi(t)| \leq M \exp\{Ct\}, t \geq 0$ , the relation

$$\lim_{t \rightarrow 0} \int_0^\infty K(t, \tau, \varepsilon) \varphi(\tau) d\tau = \int_0^\infty e^{-\tau} \varphi(2\varepsilon\tau) d\tau,$$

is valid in  $H$ ;

(viii)  $\int_0^\infty K(t, \tau, \varepsilon) d\tau = 1, \quad t \geq 0;$

(ix) Suppose  $\rho : [0, \infty) \rightarrow \mathbb{R}$  possesses the following properties:  $\rho \in C^1[0, \infty)$ ,  $\rho$  and  $\rho'$  are increasing functions and  $|\rho(t)| \leq M \exp\{ct\}, |\rho'(t)| \leq M \exp\{ct\}$ , for  $t \in [0, \infty)$ . Then there exist positive constants  $C_1$  and  $C_2$  such that

$$\int_0^\infty K(t, \tau, \varepsilon) |\rho(t) - \rho(\tau)| d\tau \leq C_1 \sqrt{\varepsilon} \exp\{C_2 t\}, \quad t > 0;$$

(x) Let  $f(t) \in W_C^{1,\infty}(0, \infty; H)$  with some  $C \geq 0$ . Then there exist positive constants  $C_1, C_2$  such that

$$\left| f(t) - \int_0^\infty K(t, \tau, \varepsilon) f(\tau) d\tau \right|_H \leq C_1 \sqrt{\varepsilon} \exp\{C_2 t\} \|f'\|_{L^\infty(0, \infty; H)}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1;$$

(xi) There exists  $C > 0$  such that

$$\int_0^t \int_0^\infty K(\tau, \theta, \varepsilon) \exp\left\{-\frac{\theta}{\varepsilon}\right\} d\theta d\tau \leq C\varepsilon, \quad t \geq 0, \quad \varepsilon > 0.$$

Now we are ready to establish the relation between the solutions of the problem  $(P_\varepsilon)$  and the corresponding solutions of the problem  $(P_0)$  in the linear case, i. e. in the case when  $B = 0$ .

**THEOREM 1.** Let  $A : D(A) \subset H \rightarrow H$  be a linear and closed operator,  $f \in L^\infty(0, \infty; H)$  for some  $C \geq 0$ . If  $u$  is a solution of the problem  $(P_\varepsilon)$  such that  $u \in W_C^{2,\infty}(0, \infty; H)$  with some  $C \geq 0$ , then the function  $v_0$  which is defined by

$$v_0(t) = \int_0^\infty K(t, \tau, \varepsilon) u(\tau) d\tau$$

is a solution of the following problem:

$$\begin{cases} v_0'(t) + Av_0(t) = F_0(t, \varepsilon), & t > 0, \\ v_0(0) = \varphi_\varepsilon, \end{cases} \quad (P.v_0)$$

where

$$F_0(t, \varepsilon) = \frac{1}{\sqrt{\pi}} \left[ 2 \exp\left\{\frac{3t}{4\varepsilon}\right\} \lambda\left(\sqrt{\frac{t}{\varepsilon}}\right) - \lambda\left(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\right) \right] u_1 + \int_0^\infty K(t, \tau, \varepsilon) f(\tau) d\tau,$$

$$\varphi_\varepsilon = \int_0^\infty e^{-\tau} u(2\varepsilon\tau) d\tau.$$

*Proof.* Integrating by parts and using the properties (i) – (iii) and (v) of Lemma 2 we get

$$\begin{aligned} v'_0(t) &= \int_0^\infty K_t(t, \tau, \varepsilon) u(\tau) d\tau = \int_0^\infty \left( \varepsilon K_{\tau\tau}(t, \tau, \varepsilon) - K_\tau(t, \tau, \varepsilon) \right) u(\tau) d\tau = \\ &= \int_0^\infty K(t, \tau, \varepsilon) \left( \varepsilon u''(\tau) + u'(\tau) \right) d\tau + \varepsilon K(t, 0, \varepsilon) u_1 - Av_0(t) + \int_0^\infty K(t, \tau, \varepsilon) f(\tau) d\tau. \end{aligned}$$

Thus  $v_0(t)$  satisfies the equation from  $(P.v_0)$ . From property (viii) of Lemma 2 follows the validity of the initial condition of  $(P.v_0)$ . Theorem 1 is proved.

#### 4. Limits of the solutions of the problem $(P_\varepsilon)$ as $\varepsilon \rightarrow 0$ .

In this section we shall study the behavior of the solutions of the problem  $(P_\varepsilon)$  as  $\varepsilon \rightarrow 0$ .

**THEOREM 2.** *Suppose  $f \in W_C^{1,\infty}(0, \infty; H)$ , with some  $C \geq 0$ ,  $u_0, u_1 \in V$  and the operators  $A$  and  $B$  satisfy the condition (1) and (2) respectively. Then there exist positive constants  $C_1, C_2$  such that*

$$|u(t) - v(t)| \leq C_1 M e^{C_2 t} \sqrt{\varepsilon}, \quad t \geq 0, \quad 0 \leq \varepsilon \ll 1, \quad (10)$$

where  $u$  and  $v$  are the solutions of the problems  $(P_\varepsilon)$  and  $(P.v)$ , respectively,

$$M = |f(0)| + |u_0| + |Au_0| + |B(u_0)| + |u_1| + \|f'\|_{W_C^{1,\infty}(0, \infty; H)},$$

and  $C_1$  and  $C_2$  are independent on  $M$  and  $\varepsilon$ .

*Proof.* Under the conditions of the theorem from (4) follows the estimate

$$\|u'(t)\|_{L_{C_1}^\infty(0, \infty; H)} \leq CM, \quad t \geq 0. \quad (11)$$

According to Theorem 1 the function  $w$  which is defined by

$$w(t) = \int_0^\infty K(t, \tau, \varepsilon) u(\tau) d\tau$$

is a solution of the problem

$$\begin{cases} w'(t) + Aw(t) = F(t, \varepsilon), \\ w(0) = w_0, \end{cases} \quad (P.w)$$

where

$$\begin{aligned} F(t, \varepsilon) &= F_0(t, \varepsilon) + \int_0^\infty K(t, \tau, \varepsilon) f(\tau) d\tau - \int_0^\infty K(t, \tau, \varepsilon) B(u(\tau)) d\tau, \\ F_0(t, \varepsilon) &= \frac{1}{\sqrt{\pi}} \left[ 2 \exp \left\{ \frac{3t}{4\varepsilon} \right\} \lambda \left( \sqrt{\frac{t}{\varepsilon}} \right) - \lambda \left( \frac{1}{2} \sqrt{\frac{t}{\varepsilon}} \right) \right] u_1, \quad w_0 = \int_0^\infty e^{-\tau} u(2\varepsilon\tau) d\tau. \end{aligned}$$

Using the properties (vi), (viii) and (x) of Lemma 2 and the estimate (11) we get

$$|u(t) - w(t)| \leq C e^{C_2 t} \sqrt{\varepsilon} \|u'(t)\|_{L_{C_1}^\infty(0, \infty; H)} \leq CM e^{C_2 t} \sqrt{\varepsilon}, \quad t \geq 0. \quad (12)$$

Let us denote  $R(t) = v(t) - w(t)$ , where  $v$  is the solution of the problem  $(P.v)$  and  $w$  is the solution of the problem  $(P.w)$ . Then  $R(t)$  is the solution of the problem

$$\begin{cases} R'(t) + AR(t) = B(w(t)) - B(v(t)) + \mathcal{F}(t, \varepsilon), & t \geq 0, \\ R(0) = R_0, \end{cases}$$

where  $R_0 = u_0 - w_0$  and

$$\mathcal{F}(t, \varepsilon) = f(t) - \int_0^\infty K(t, \tau, \varepsilon) f(\tau) d\tau - F_0(t, \varepsilon) - B(w(t)) + \int_0^\infty K(t, \tau, \varepsilon) B(u(\tau)) d\tau.$$

As  $R(t) \in V$  and  $V$  is continuously embedded in  $H$  then

$$(AR(t), R(t)) \geq \omega \|R(t)\|^2 \geq \omega_0 |R(t)|^2, \quad \omega_0 > 0.$$

Therefore

$$\begin{aligned} \frac{d}{dt} |R(t)|^2 &= -2(AR(t), R(t)) + 2(R(t), B(w(t)) - B(v(t))) + 2(\mathcal{F}(t, \varepsilon), R(t)) \leq \\ &\leq 2\omega_1 |R(t)|^2 + 2|\mathcal{F}(t, \varepsilon)| |R(t)|, \quad t \geq 0, \quad \omega_1 = -\omega_0 + L, \end{aligned}$$

and hence

$$\frac{1}{2} |R(t)|^2 e^{-2\omega_1 t} \leq \frac{1}{2} |R_0|^2 + \int_0^t |\mathcal{F}(\tau, \varepsilon)| |R(\tau)| e^{-2\omega_1 \tau} d\tau, \quad t \geq 0,$$

then using Lemma A we obtain the estimate

$$|R(t)| \leq e^{\omega_1 t} \left( |R_0| + \int_0^t |\mathcal{F}(\tau, \varepsilon)| e^{-\omega_1 \tau} d\tau \right), \quad t \geq 0. \quad (13)$$

From (11) follows the estimate

$$\begin{aligned} |R_0| &\leq \int_0^\infty e^{-\tau} |u(2\varepsilon\tau) - u_0| d\tau \leq \int_0^\infty e^{-\tau} \int_0^{2\varepsilon\tau} |u'(s)| ds d\tau \leq \\ &\leq 2\varepsilon CM \int_0^\infty \tau e^{-\tau+2C_1\varepsilon\tau} d\tau \leq CM\varepsilon, \quad 0 < \varepsilon \leq (4C_1)^{-1}. \end{aligned} \quad (14)$$

Now let us estimate  $|\mathcal{F}(t, \varepsilon)|$ . Using the property **(x)** of Lemma 2 we have

$$\left| f(t) - \int_0^\infty K(t, \tau) f(\tau) d\tau \right| \leq C_1 M \sqrt{\varepsilon} e^{C_2 t}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1. \quad (15)$$

As  $e^\tau \lambda(\sqrt{\tau}) \leq C, \tau \geq 0$ , then for  $\varepsilon \in (0, (8|\omega_1|)^{-1}]$  we have

$$\begin{aligned} \int_0^t \exp \left\{ \frac{3\tau}{4\varepsilon} - \omega_1 \tau \right\} \lambda \left( \sqrt{\frac{\tau}{\varepsilon}} \right) d\tau &\leq \varepsilon \int_0^{\frac{t}{\varepsilon}} \exp \left\{ \frac{3\tau}{4} + |\omega_1| \tau \varepsilon \right\} \lambda(\sqrt{\tau}) d\tau \leq \\ &\leq C \int_0^\infty e^{7\tau/8} \lambda(\sqrt{\tau}) d\tau = C\varepsilon \int_0^\infty e^{-\tau/8} e^\tau \lambda(\sqrt{\tau}) d\tau \leq C\varepsilon, \quad t \geq 0, \quad 0 < \varepsilon \ll 1, \end{aligned}$$

and

$$\int_0^t e^{-\omega_1 \tau} \lambda \left( \frac{1}{2} \sqrt{\frac{\tau}{\varepsilon}} \right) d\tau \leq \varepsilon \int_0^\infty e^{|\omega_1| \varepsilon \tau} \lambda \left( \frac{1}{2} \sqrt{\tau} \right) d\tau \leq C\varepsilon, \quad t \geq 0, \quad 0 < \varepsilon \ll 1.$$

Therefore we get the estimate

$$\int_0^t e^{-\omega_1 \tau} |F_0(\tau, \varepsilon)| d\tau \leq C\varepsilon |u_1| \leq C\varepsilon M, \quad t \geq 0, \quad 0 < \varepsilon \ll 1. \quad (16)$$

Let us estimate the difference

$$I(t) = \int_0^\infty K(t, \tau, \varepsilon) B(u(\tau)) d\tau - B(w(t)) = I_1(t) + I_2(t), \quad (17)$$

where due to property **(viii)** of Lemma 2

$$I_1(t) = \int_0^\infty K(t, \tau, \varepsilon) (B(u(\tau)) - B(w(\tau))) d\tau,$$

$$I_2(t) = \int_0^\infty K(t, \tau, \varepsilon) (B(w(\tau)) - B(w(t))) d\tau.$$

Using condition (2), the estimate (12) and property **(ix)** of Lemma 2 we have

$$\begin{aligned} |I_1(t)| &\leq CLM\sqrt{\varepsilon} \int_0^\infty K(t, \tau, \varepsilon) e^{C_2 \tau} d\tau \leq \\ &\leq CM\sqrt{\varepsilon} \int_0^\infty K(t, \tau, \varepsilon) (|e^{C_2 \tau} - e^{C_2 t}| + e^{C_2 t}) d\tau \leq CM\sqrt{\varepsilon} e^{C_3 t}. \end{aligned} \quad (18)$$

To estimate  $I_2(t)$  we will evaluate the function  $w'(t)$ . Integrating by parts and using the properties **(ii)**, **(iii)** and **(iv)** of Lemma 2 we have

$$\begin{aligned} w'(t) &= \int_0^\infty K_t(t, \tau, \varepsilon) u(\tau) d\tau = \int_0^\infty (\varepsilon K_{\tau\tau}(t, \tau, \varepsilon) - K_\tau(t, \tau, \varepsilon)) u(\tau) d\tau = \\ &= - \int_0^\infty (\varepsilon K_\tau(t, \tau, \varepsilon) - K(t, \tau, \varepsilon)) u'(\tau) d\tau = \\ &= -\frac{3}{2} \int_0^\infty K(t, \tau, \varepsilon) u'(\tau) d\tau + \frac{3}{4\sqrt{\pi\varepsilon}} \int_0^\infty (K_2(t, \tau, \varepsilon) - K_3(t, \tau, \varepsilon)) u'(\tau) d\tau. \end{aligned} \quad (19)$$

Due the estimate (11) we get

$$\begin{aligned} \left| \int_0^\infty K(t, \tau, \varepsilon) u'(\tau) d\tau \right| &\leq CM \int_0^\infty K(t, \tau, \varepsilon) e^{C_2 \tau} d\tau \leq \\ &\leq CM \int_0^\infty K(t, \tau, \varepsilon) (|e^{C_2 \tau} - e^{C_2 t}| + e^{C_2 t}) d\tau \leq CM e^{C_2 t}, \quad t \geq 0. \end{aligned} \quad (20)$$

Also integrating by parts we obtain the estimates

$$\int_0^\infty K_i(t, \tau, \varepsilon) e^{C_2 \tau} d\tau \leq C\varepsilon e^{C_2 t}, \quad i = 2, 3, \quad t \geq 0, \quad 0 < \varepsilon \ll 1. \quad (21)$$

Using (19), (20) and (21) we get

$$|w'(t)| \leq CM e^{C_3 t}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1. \quad (22)$$



The estimate (22) and property (ix) of Lemma 2 permit to evaluate  $I_2(t)$ .

$$\begin{aligned} |I_2(t)| &\leq \int_0^\infty K(t, \tau, \varepsilon) |B(w(\tau)) - B(w(t))| d\tau \leq L \int_0^\infty K(t, \tau, \varepsilon) \left| \int_\tau^t w'(s) ds \right| d\tau \leq \\ &\leq LCM \int_0^\infty K(t, \tau, \varepsilon) |e^{C_2 t} - e^{C_2 \tau}| d\tau \leq CM e^{C_3 t} \sqrt{\varepsilon}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1. \end{aligned} \quad (23)$$

From (17), (18) and (23) we get

$$|I(t)| \leq CM e^{C_3 t} \sqrt{\varepsilon}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1. \quad (24)$$

From (15), (16) and (24) follows the estimate

$$\int_0^t e^{-\omega_1 \tau} |\mathcal{F}(\tau, \varepsilon)| d\tau \leq CM e^{C_3 t} \sqrt{\varepsilon}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1. \quad (25)$$

From (13), using the estimates (14) and (25) we get

$$|R(t)| \leq C_1 M e^{C_3 t} \sqrt{\varepsilon}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1. \quad (26)$$

Finally from the estimates (12) and (26) we have

$$|u(t) - v(t)| \leq |u(t) - w(t)| + |R(t)| \leq C_1 M e^{C_3 t} \sqrt{\varepsilon}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1.$$

The estimate (10) is proved.

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