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**ON THE BEHAVIOUR OF SOLUTIONS OF DEGENERATE NONLINEAR ELLIPTIC EQUATIONS**

The behaviour of singular solution  $u(x)$  of quasilinear elliptic equation

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} a_i \left( x, \frac{\partial u}{\partial x} \right) - |u|^{q-2} u = 0, \quad x \in B_1(0) \setminus \{0\} \tag{1}$$

is studied. We assume Caratheodory's conditions for coefficients  $a_i(x, \xi)$  and inequalities

$$\sum_{i=1}^n a_i(x, \xi) \xi_i \geq \nu_1 |x|^\sigma |\xi|^p, \quad |a_i(x, \xi)| \leq \nu_2 |x|^\sigma |\xi|^{p-1}$$

with positive constants  $\nu_1, \nu_2, \sigma \in (p - n, n(p - 1)), 1 < p < n$ .

Following results are established:

1) the boundedness of the solution  $u(x)$  of the equation (1) under conditions  $q > 1$ ,

$$|u(x)| \leq M_1 |x|^{-P+\delta} \quad \text{for } 0 < |x| \leq \frac{1}{2}, \quad P = \frac{n-p+\sigma}{p-1}, \quad \delta > 0;$$

2) the estimate

$$|u(x)| \leq M_3 |x|^{-Q} \quad \text{for } 0 < |x| \leq \frac{1}{2}, \quad Q = \frac{p-\sigma}{q-p}, \quad q > p;$$

3) the removability of each isolated singularity of the solution of the equation (1) if  $Q < P$ .

These assertions are generalized well-known results of V.A.Kondratiev and E.M.Landis established for  $\sigma = 0$  for the equation (1) with  $p = 2$  and linear principal part.

1. We study the behaviour of a singular solution  $u(x)$  of the following quasilinear elliptic equation:

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} a_i \left( x, \frac{\partial u}{\partial x} \right) - |u|^{q-2} u = 0, \quad x \in \Omega = B_1(0) \setminus \{0\} \tag{1}$$

where  $B = B_1(0)$  is the ball of radius 1 with a center 0, and  $q$  is a real positive number precised later.

We assume that the coefficients  $a_i(x, \xi)$  are Caratheodory's functions and that they satisfy the following ellipticity condition:

$$\sum_{i=1}^n a_i(x, \xi) \xi_i \geq \nu_1 |x|^\sigma |\xi|^p \tag{2}$$

and the growth condition:

$$|a_i(x, \xi)| \leq \nu_2 |x|^\sigma |\xi|^{p-1} \tag{3}$$

for  $(x, \xi) \in \Omega \times \mathbb{R}^n$  and with positive constants  $\nu_1, \nu_2$  and numbers  $\sigma, p$  such that

$$p - n < \sigma < n(p - 1), 1 < p < n. \tag{4}$$

A function  $u(x) \in W_{loc}^{1,p}(\Omega) \cap L_{loc}^q(\Omega)$  is called a solution of the equation (1) in  $B \setminus \{0\}$  if for an arbitrary function  $\varphi \in W^{1,p}(B) \cap L^q(B)$  that is equal to zero near  $\partial B \cup \{0\}$  we have

$$\int_{\Omega} \sum_{i=1}^n a_i(x, \frac{\partial u}{\partial x}) \frac{\partial \varphi}{\partial x_i} dx + \int_{\Omega} |u|^{q-2} u \varphi dx = 0. \tag{5}$$

We will say that the solution  $u(x)$  of the equation (1) in  $B \setminus \{0\}$  has at  $\{0\}$  removable singularity if the integral identity (5) is true for an arbitrary function  $\varphi \in W^{1,p}(B) \cap L^q(B)$  that is equal to zero near  $\partial B$ .

Main Results of this paper are following Theorems.

**THEOREM 1.** *Let  $u(x)$  be a solution of the equation (1) in  $B \setminus \{0\}$ . Assume that  $q > 1$ , the inequalities (2), (3) and the estimate*

$$|u(x)| \leq M_1 |x|^{-P+\delta}, \quad P = \frac{n-p+\sigma}{p-1} \tag{6}$$

are satisfied for  $0 < |x| \leq \frac{1}{2}$  with some positive numbers  $M_1, \delta$ . Then there exists a positive constant  $M_2$  such that the estimate

$$|u(x)| \leq M_2 \quad 0 < |x| \leq \frac{1}{2} \tag{7}$$

holds.

**THEOREM 2.** *Assume that conditions (2), (3) are satisfied and  $q > p$ . Let  $u(x)$  be a solution of the equation (1) in  $B \setminus \{0\}$ . Then the estimate*

$$|u(x)| \leq M_3 |x|^{-Q}, \quad 0 < |x| \leq \frac{1}{2} \tag{8}$$

holds with  $Q = \frac{p-\sigma}{q-p}$  and some positive constant  $M_3$ .

**THEOREM 3.** *Assume that conditions (2), (3) are satisfied and  $q > \frac{np-p+\sigma}{n-p+\sigma}$ . Then for an arbitrary solution  $u(x)$  in  $B \setminus \{0\}$  the singularity at  $\{0\}$  is removable.*

Note that V.A. Kondratiev and E.M. Landis in [1] established analogous result for linear equation of type (1), that corresponds to  $p = 2, \sigma = 0$ .

**2. Proof of the Theorem 1.** Let us substitute in (5) a test function

$$\varphi = (1 + |u|)^{-\alpha} u \psi^p(x) \eta_r^p(x) \quad , 0 < \alpha < 1,$$

where  $\psi(x) = 1$  in  $B_{\frac{1}{2}}(0), 0 \leq \psi(x) \leq 1, \psi(x) = 0$  outside  $B_{\frac{3}{4}}(0), |\frac{\partial \psi}{\partial x}| \leq c_0, \eta_r(x) = 1$  outside  $B_{2r}(0), 0 \leq \eta_r(x) \leq 1, \eta_r(x) = 0$  inside  $B_r(0), |\frac{\partial \eta_r}{\partial x}| \leq \frac{c_0}{r}$ , where  $r$  is enough small.

Using inequalities (2), (3), Young inequality and simple calculations we obtain the estimate

$$\begin{aligned} & \int_B (1 + |u|)^{-\alpha} \left| \frac{\partial u}{\partial x} \right|^p \psi^p(x) \eta_r^p(x) |x|^\sigma dx \leq \\ & \leq C_1 \int_{B \setminus B_r(0)} (1 + |u|)^{-\alpha} |x|^\sigma |u|^p \left( \left| \frac{\partial \psi}{\partial x} \right|^p + \left| \frac{\partial \eta_r}{\partial x} \right|^p \right) dx. \end{aligned} \tag{9}$$



The constant  $C_1$  here and other constants  $C_i$  in the proof of the Theorem 1 depend only on  $\nu_1, \nu_2, n, p, \sigma, M_1$ .

Now we want to pass to the limit when  $r \rightarrow 0$ . The term of sum in right hand side of (9) with a derivative of  $\psi$  is estimated as following

$$\int_{B \setminus B_r(0)} (1 + |u|)^{-\alpha} |u|^p |x|^\sigma \left| \frac{\partial \psi}{\partial x} \right|^p dx \leq C_2 \left\{ 1 + r^{n+\sigma - \left( \frac{n-p+\sigma}{p-1} - \delta \right) (p-\alpha)} \right\}. \quad (10)$$

The right-hand side of (10) is bounded for  $r \rightarrow 0$  if

$$\alpha > \frac{n + \sigma - [p + \delta(p - 1)]p}{n + \sigma - [p + \delta(p - 1)]}. \quad (11)$$

The second term of the right-hand side of (9) has the estimate

$$\int_{B \setminus B_r(0)} (1 + |u|)^{-\alpha} |u|^p \left| \frac{\partial \eta_r}{\partial x} \right|^p |x|^\sigma dx \leq C_3 r^{n+\sigma - \left( \frac{n-p+\sigma}{p-1} - \delta \right) (p-\alpha) - p}. \quad (12)$$

The last expression will go to zero when  $r \rightarrow 0$  if

$$n + \sigma - \left( \frac{n - p + \sigma}{p - 1} - \delta \right) (p - \alpha) - p > 0,$$

this implies the condition on  $\alpha$ :

$$\alpha > \frac{n + \sigma - p - \delta p(p - 1)}{\sigma + n - p - \delta(p - 1)}. \quad (13)$$

We choose  $\alpha$  as in (13) with  $\alpha < 1$ . By monotone convergence theorem applied to the inequality (9) we obtain:

$$\int_B (1 + |u|)^{-\alpha} \left| \frac{\partial u}{\partial x} \right|^p \psi^p(x) |x|^\sigma dx \leq C_4 \int_B (1 + |u|)^{-\alpha} |u|^p \left| \frac{\partial \psi}{\partial x} \right|^p |x|^\sigma dx. \quad (14)$$

Now we want to prove boundedness of  $u(x)$  in  $B$ . We shall use Moser iteration process. We substitute in (5)

$$\varphi = (1 + |u|_k)^l (1 + |u|)^{-\alpha} u \psi^s(x) \eta_r^s(x) \quad , s > 0, l \geq 0$$

where  $|u|_k = \min\{|u(x)|, k\}$ ,  $k > 0$ ,  $\psi(x), \eta_r(x)$  are the same functions as before.

After standard calculations we have the inequality

$$\begin{aligned} & \int_B (1 + |u|_k)^l (1 + |u|)^{-\alpha} \left| \frac{\partial u}{\partial x} \right|^p \psi^s(x) \eta_r^s(x) |x|^\sigma dx \leq \\ & \leq C_5 s^p \int_B (1 + |u|)^{-\alpha} |u|^p (1 + |u|_k)^l \psi^{s-p}(x) \eta_r^{s-p}(x) \cdot \\ & \cdot \left( \left| \frac{\partial \psi}{\partial x} \right|^p + \left| \frac{\partial \eta_r}{\partial x} \right|^p \right) |x|^\sigma dx. \end{aligned} \quad (15)$$

We pass to the limit in (15) if  $r \rightarrow 0$  and obtain

$$\begin{aligned} & \int_B (1 + |u|_k)^l (1 + |u|)^{-\alpha} \left| \frac{\partial u}{\partial x} \right|^p \psi^s(x) |x|^\sigma dx \leq \\ & \leq C_5 s^p \int_B (1 + |u|_k)^l (1 + |u|)^{-\alpha} (1 + |u|)^p \psi^{s-p}(x) |x|^\sigma dx. \end{aligned} \quad (16)$$

We will apply Moser method. We shall use embedding theorem

$$W_p^1(\Omega, |x|^\sigma) \subset L_{p\kappa}(\Omega, |x|^\sigma) \quad \text{with} \quad 1 < \kappa < \frac{\sigma + n}{\sigma + n - p}.$$

We apply the embedding theorem to the right hand side of (16) and we obtain

$$\begin{aligned} & \int_B (1 + |u|_k)^l (1 + |u|)^{p-\alpha} \psi^{s-p}(x) |x|^\sigma dx \leq \\ & \leq C_6 l^\kappa \left\{ \int_B (1 + |u|_k)^{\frac{l}{\kappa} + (\alpha-p)(1-\frac{1}{\kappa})} \right. \\ & \left. \cdot (1 + |u|)^{p-\alpha} \psi^{\frac{s-p}{\kappa} - p}(x) dx \right\}^\kappa. \end{aligned} \quad (17)$$

Now we can apply Moser method, which gives the inequality

$$\max_{x \in B_{\frac{1}{2}}} (1 + |u(x)|_k)^{l_0} \leq C_7 \int_B (1 + |u|_k)^{l_0} (1 + |u|)^{p-\alpha} |x|^\sigma \psi^{s_0-p}(x) dx \quad (18)$$

with numbers  $l_0, s_0$  such that  $l_0 > 0, s_0 \geq p$ .

We need to choose  $l_0$  such that

$$\int_B (1 + |u|)^{p-\alpha+l_0} |x|^\sigma dx \leq C_8 \quad (19)$$

or, equivalently we must choose  $l_0$  from the condition

$$n + \sigma - \left( \frac{n - p + \sigma}{p - 1} - \delta \right) (p - \alpha + l_0) > 0.$$

So we can choose positive  $l_0$  if  $\alpha$  satisfies the inequality (13). Now inequalities (18), (19) imply a boundedness of  $u(x)$  in  $B_{\frac{1}{2}}$ . This is the end of the proof of the theorem.

**3. Proof of the Theorem 2.** We introduce a function  $\xi_\rho(x)$  such that  $\xi_\rho(x) = 1$  for  $\frac{\rho}{2} < |x| < \rho$ ,  $\xi_\rho(x) = 0$  outside of  $\{\frac{\rho}{4} \leq |x| \leq \frac{5\rho}{4}\}$  and  $\left| \frac{\partial \xi_\rho(x)}{\partial x} \right| \leq \frac{c_9}{\rho}$ .

We substitute in (5) a test function

$$\varphi = |u(x)|^{r-1} u(x) \xi_\rho^s(x) \quad r \geq 1, s \geq p.$$

Standard calculations lead to following estimate

$$\begin{aligned} \frac{r\nu_1}{2} \int_{\Omega} |x|^{\sigma} |\nabla u|^p |u(x)|^{r-1} \xi_{\rho}^s(x) dx + \int_{\Omega} |u(x)|^{q+r-1} \xi_{\rho}^s(x) dx &\leq \\ &\leq C_{10} \int_{\Omega} \left(\frac{s}{r}\right)^p |x|^{\sigma} |u(x)|^{r+p-1} \frac{1}{\rho^p} \xi_{\rho}^{s-p}(x) dx. \end{aligned} \quad (20)$$

We recall now the weighted Sobolev embedding theorem (see [4])

$$\left( \frac{1}{\rho^{\sigma+n}} \int_{B_{\rho}(x_0)} |\phi|^{\kappa p} |x|^{\sigma} dx \right)^{\frac{1}{\kappa p}} \leq C_{11} \rho \left( \frac{1}{\rho^{\sigma+n}} \int_{B_{\rho}(x_0)} |\nabla \phi|^p |x|^{\sigma} dx \right)^{\frac{1}{p}}. \quad (21)$$

We apply this last inequality to the function  $\phi = |u(x)|^{\frac{r+p-1}{\kappa p}} \xi_{\rho}^{\frac{s-p}{\kappa p}}(x)$

$$\begin{aligned} \int_{\Omega} \left[ u^{\frac{r+p-1}{\kappa p}}(x) \xi_{\rho}^{\frac{s-p}{\kappa p}}(x) \right]^{\kappa p} |x|^{\sigma} dx &\leq \frac{C_{12}(r+s+p)^{\kappa p}}{\rho^{(\sigma+n)(\kappa-1)}} \rho^{\kappa p}. \\ \left[ \int_{\Omega} |u(x)|^{\frac{r+p-1}{\kappa} - p} |\nabla u|^p \xi_{\rho}^{\frac{s-p}{\kappa}}(x) |x|^{\sigma} + |u(x)|^{\frac{r+p-1}{\kappa} - p} \xi_{\rho}^{\frac{s-p}{\kappa} - p}(x) \left(\frac{c}{\rho}\right)^p |x|^{\sigma} dx \right]^{\kappa} & \end{aligned} \quad (22)$$

Using the inequality (20) we get the estimate

$$\begin{aligned} \frac{1}{\rho^{\sigma+n}} \int_{\Omega} |x|^{\sigma} \left( |u(x)|^{\frac{r+p-1}{\kappa p}} \xi_{\rho}^{\frac{s-p}{\kappa p}}(x) \right)^{\kappa p} dx &\leq \\ &\leq \frac{C_{13}(r+s+p)^{2\kappa p}}{r^{\kappa p}} \left( \frac{1}{\rho^{\sigma+n+p}} \int_{\Omega} |u(x)|^{\frac{r+p-1}{\kappa}} \xi_{\rho}^{\frac{s-p}{\kappa} - p}(x) |x|^{\sigma} dx \right)^{\kappa}. \end{aligned} \quad (23)$$

Denote  $r_i = \kappa^i p + 1 - p$ ,  $s_i = \left(\frac{pq}{q-p} - \frac{p}{1-\kappa}\right) \kappa^i + \frac{p}{1-\kappa}$ ,

$$I_i = \frac{1}{\rho^{\sigma+n}} \int_{\Omega} |x|^{\sigma} |u(x)|^{r_i+p-1} \xi_{\rho}^{s_i-p}(x) dx.$$

Using these notations we can write the inequality (23) with  $r = r_i$ ,  $s = s_i$  in following way

$$I_i \leq C_{14} \kappa^{2\kappa p i} I_{i-1}^{\kappa}. \quad (24)$$

Iterating this inequality we obtain

$$\max_{\frac{\rho}{2} < |x| < \rho} |u(x)|^p \leq \frac{C_{15}}{\rho^{\sigma+n}} \int_{\Omega} |x|^{\sigma} |u(x)|^p \xi_{\rho}^{s_0-p}(x) dx. \quad (25)$$

Now we use (20) (if  $r = r_0 = 1$  and  $s = s_0$ ) and we find

$$\int_{\Omega} u^q(x) \xi_{\rho}^{s_0}(x) dx \leq \frac{C_{16}}{\rho^p} \int_{\Omega} |x|^{\sigma} |u(x)|^p \xi_{\rho}^{s_0-p}(x) dx. \quad (26)$$



So, by using Young's inequality and (26), we have

$$\begin{aligned} \int_{\Omega} |x|^{\sigma} |u(x)|^p \xi_{\rho}^{s_0-p}(x) dx &= \int_{\Omega} \rho^{\frac{p^2}{q}} |u(x)|^p \rho^{-\frac{p^2}{q}} [\xi_{\rho}(x)]^{-p} |x|^{\sigma} \xi_{\rho}^{s_0}(x) dx \leq \\ &\leq C_{17} \varepsilon^{\frac{q}{p}} \int_{\Omega} |x|^{\sigma} |u(x)|^p \xi_{\rho}^{s_0-p}(x) dx + C_{18} \varepsilon^{-\frac{q}{q-p}} \int_{\Omega} \rho^{-\frac{p^2}{q-p}} |x|^{\frac{\sigma q}{q-p}} \xi_{\rho}^{s_0-\frac{pq}{q-p}}(x) dx, \end{aligned}$$

so

$$\int_{\Omega} |x|^{\sigma} |u(x)|^p \xi_{\rho}^{s_0-p}(x) dx \leq C_{19} \rho^{-\frac{p^2}{q-p} + n + \frac{\sigma q}{q-p}}$$

and then

$$\max_{\frac{\rho}{2} < |x| < \rho} |u(x)|^p \leq \frac{C_{20}}{\rho^{\sigma + \frac{p^2}{q-p} - \frac{\sigma q}{q-p}}}.$$

In the end we have

$$\max_{\frac{\rho}{2} < |x| < \rho} |u(x)| \leq \frac{C_{21}}{\rho^{\frac{p-\sigma}{q-p}}}$$

and the proof of the Theorem 2 is completed.

**4. Proof of the Theorem 3.** Theorems 1, 2 imply that an arbitrary solution of the equation (1) in  $B \setminus \{0\}$  is bounded in  $B_{\frac{1}{2}}$  if  $Q < P$ . It means the boundedness of the solution  $u(x)$  in  $B_{\frac{1}{2}}$  for  $q > \frac{np-p+\sigma}{n-p+\sigma}$ . For the proof of removability of a singularity of the solution  $u(x)$  it is sufficient to substitute in (5)  $\varphi(x) = \psi(x) \eta_r(x)$  where  $\psi(x) \in W^{1,p}(B) \cap L^q(B)$  and is equal to zero near  $\partial B$ ,  $\eta_r(x)$  is the same function that was used in the proof of the Theorem 1. Then passing to the limit if  $r \rightarrow 0$  we obtain that  $u(x)$  is the solution of (1) in  $B$ . This is the end of the proof of the Theorem 3.

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