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# USAGE OF THE APPARATUS OF ORDINARY DIFFERENTIAL EQUATIONS IN MODELLING OF ECONOMIC AND ENVIRONMENTAL SYSTEMS

***Abstract.*** *The ordinary differential equations techniques applying to investigate the economical and ecological systems has been considered in presented article. The interconnected economical complexes development for the countries with the different economical potential has been simulated. The population economical activity influence on the environment pollution and the state of region's flora has been investigated. The economical efficiency of the new technical diagnostics implementation has been studied. The methods of presented models realization has been presented and investigated, the results of tested calculations have been presented and one's analysis has been given. The directions of future investigations have been determined.*

***Keywords:*** *differential equation, mathematical modeling method, modeling, diagnostics.*

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## Introduction

The modeling of economic and environmental systems is performed with the use of linear and nonlinear systems of ordinary differential equations apparatus. As a basic, the development of which can be interpreted below, would be a model "predator – victim", created in 1925 by Alfred Lotka and Vito Volterra [2]. The basic equation system of this model is written as:

$$\begin{cases} \frac{dx}{dt} = k_1 x(t) - g_1 x(t) y(t) \\ \frac{dy}{dt} = -k_2 y(t) + g_2 x(t) y(t) \end{cases}. \tag{1}$$

There $x(t)$ – the count of victim's population, $y(t)$ – the count of predator's population, $k_i, g_i, i = 1,2$ – the model's coefficients, detailed content of which is given in [2]. Models of economic systems of different nature are considered in [1, 4–9, 11–13]. The offered research touches a construction and research of three models of economic and economically – ecological systems, and also them practical realization and researching.

## 1. Model problems formulation

In predicting the development of interconnected economies, the question arises whether economies with relatively low levels of development may not suffer significant economic losses at a time when the world's leading economies are suffering losses as a result of the economic crisis. The methods of mathematical design are used for research of the indicated question with the use of the systems as a "predator – victim", that allows to build mathematical models and define their descriptions that would allow to answer theset questions. The task is reduced to solution ' Liabilities system of differential equations of the form:

$$\begin{cases} \dfrac{dx_1}{dt} = A_1 x_1 (A_2 - x_1) - A_3 x_1 x_2 + A_4 x_1 x_3 \\ \dfrac{dx_2}{dt} = A_5 x_2 (A_6 - x_2) - A_7 x_1 x_2 + A_8 x_1 x_3 \\ \dfrac{dx_3}{dt} = A_9 x_3 (A_{10} - x_3) + A_{11} x_1 x_3 + A_4 x_1 x_2 \end{cases}, \quad (2)$$

there $x_1$ and $x_2$ – economically strong countries, $x_3$ a country with a low level of economy with appropriate initial conditions $x_1(0) = x_{10}$; $x_2(0) = x_{20}$; $x_3(0) = x_{30}$. The coefficients $A_i$ could be present as a time function $A_i = A_i(t)$.

Another model is related to a system described by three differential equations for functions: *x(t)* – population in the region; *y(t)* – the level of pollution and other non–harmful effects on the environment caused by the economic activity of the population; *z(t)* – the level of flora of the region (trees, agricultural products, forests, gardens, etc.), however, the equation system looks like:

$$\begin{cases} \dfrac{dx}{dt} = Ax - By + Cz \\ \dfrac{dy}{dt} = Dx - Ez \\ \dfrac{dz}{dt} = Hx - Gy + Fz \end{cases}. \quad (3)$$

Initial conditions must be specified for the correct formulation of the modeling task:

$$\begin{cases} X(0) = X_0 \\ Y(0) = Y_0 \\ Z(0) = Z_0 \end{cases}. \quad (4)$$

The third model describes a situation for which the functions *x(t)*, *y(t)*; *z(t)* are introduced with the following meaning: *x(t)* – costs for implementation of new technical diagnostics and control standards; *y(t)* – costs for elimination of emergencies consequences; *z(t)* is the efficiency of the studied industrial system's element. When a mathematical model is constructed, a differential equation system that describes how to modify the corresponding variables per unit of time in assuming the nature of the relationship between the quantities is recorded. As a result, the following system of ordinary differential equations that binds the variables *x(t)*; *y(t)*; *z(t)* is obtained:

$$\begin{cases} \frac{dx}{dt} = K_1 x(A-x) - K_2 y + K_3 z \\ \frac{dy}{dt} = K_4 x(A-x) + K_5 y(B-y) + K_6 z \\ \frac{dz}{dt} = K_7 x - K_8 y \end{cases}, \qquad (5)$$

with appropriate initial conditions.

Systems of type (2) – (5) are a certain extension of the known predator–prey model. The algorithms for finding the coefficients of systems (2) and (3) by the method of expert estimation are proposed, and in the modeling of system (2) additional conditions for its coefficients in terms of obtaining asymptotically stable solutions are established.. Introducing nonlinear components into systems (2), (3), and (5) allows us to obtain solutions that more accurately reflect the essence of the phenomena and processes being modeled. In the implementation of the mentioned models, the following approach was used: in the first step, all the mentioned models were selected as linear, suitable calculations and analysis of the obtained results were carried out.. If there were doubts about the correspondence of these results to the characteristics of real systems, then new nonlinear terms describing the level of interaction of the relevant factors were introduced into the respective systems. If the qualitative behavior of the solutions satisfied the researcher, then methods of practical evaluation of the coefficients of the systems based on the results of their statistical studies, real data on the characteristics of their functioning were created. When model solutions, that meet certain economic requirements and environmental standards, are obtained, recommendations to optimize the systems under consideration by the necessary criteria, which are responsible for the stable operation of the respective systems with the fulfillment of their functions. All models are brought to numerical realization in the form of software complexes by Runge-Kutta methods [3], it allows to carry out a wide class of calculations in order to estimate the dynamics of the process development depending on the suitable coefficients of the model.

Special kind of models is the simulation of an advertising campaign for goods and services is an important element in ensuring that they hit the market. Often, advertising is carried out haphazardly or using certain empirical methods, which can have the opposite effect when product advertising begins to act as a counter–advertisement. At the same time, mathematical methods, in particular, the apparatus of ordinary differential equations, to build a model of an advertising campaign are promising for studying the features of the advertising process.

The advertising model is based on the following assumptions. It is believed that $\frac{dN(t)}{dt}$ – the rate of change in the number of consumers who know about the product and are ready to buy it ($N(t)$– the number of informed customers), function $\alpha_1(t) \cdot (N_0 - N(t))$ – characterizes the intensity of the advertising company, $\alpha_1(t) > 0$ characterizes the cost of advertising, $N_0$ – the total number of potential buyers. It is also believed that people who know about the product, in one way or another, disseminate information about the product to customers who do not know about it (potential customers). This contribution is characterized by an addition $\alpha_2(t) \cdot N(t) \cdot (N_0 - N(t))$. The value $\alpha_2(t)$characterizes the degree of communication of clients with potential clients.

Based on the made assumptions made, the equation [1] is obtained:

$$\frac{dN(t)}{dt} = [\alpha_1(t) + \alpha_2(t)N(t)](N_0 - N(t)) \tag{6}$$

with initial condition $N(0) = N_1, N_1 < N_0$. If $\alpha_1(t) >> \alpha_2(t)$, the next equation can be received (matches classic advertising campaigns):

$$\frac{dN(t)}{dt} = \alpha(t)N(t). \tag{7}$$

Otherwise, it is possible to receive the equation that describes the so-called "network marketing":

$$\begin{aligned} &\frac{dN(t)}{d\tau} = N(t)(N_0 - N(t)), \\ &d\tau = \alpha_2(t)dt. \end{aligned} \tag{8}$$

Obviously, equations (7) and (8) are squared:

$$N(t) = \frac{N_0 e^{N0t}}{1 + e^{N0t}}, \tag{9}$$

In this case, the number of informed consumers if $t \to \infty$ remains constant:

$$\lim_{t \to \infty} N = \lim_{t \to \infty} \frac{N_0 e^{N0t}}{1 + e^{N0t}} = N_0 \tag{10}$$

and is equal to the number of potential customers.

When the value $[\alpha_1(t) + \alpha_2(t)N(t)]$ becomes negative ($\alpha_2(t) < 0$ – negative evaluation of consumers of the quality of the goods) – manufacturers should further analyze and evaluate the possibilities of direct advertising in the promotion of products in the market.

Depending on the values $\alpha_1(t)$, $\alpha_2(t)$ and $N(t)$, product promotion activities can be aimed at improving the results of both direct advertising ($\alpha_1(t)$) and promoting indirect advertising ($\alpha_2(t)$).

If $N << N_0$ (a little-known product), and as a result, $\alpha_2(t) \cdot N << \alpha_1(t)$, then the equation (6) comes to view $\frac{dN(t)}{dt} = \alpha_1(t) N_0$ and has the solution

$$N = N_0 \int_0^t \alpha_1(t) dt . \tag{11}$$

As for (8), the solution (6) with conditions $\alpha_1(t) = \alpha_1$ i $\alpha_2(t) = \alpha_2$

$$N(t) = \frac{N_0 e^{(N0\alpha 2 + \alpha 1)} - N_0 \cdot \dfrac{\alpha_1}{\alpha_2}}{1 + e^{(N0\alpha 2 + \alpha 1)}} \tag{12}$$

And, based on (10):

$$\lim_{t \to \infty} N = N_0 . \tag{13}$$

Thus, the solution (1) is stable, which substantiates the correctness of the proposed models.

In the general case, integration (6) is performed using numerical methods [3] – for example, fourth-order Runge-Kutta methods. The order of precision of a method is made on the basis of solving model equations.

Based on the proposed model, the following tasks are solved:

– the task of estimating the profit of an advertising company,

– the dependence of the number of potential clients on the methods and intensity of the advertising campaign is investigated.

Methods for determining or selecting $\alpha_1(t)$, $\alpha_2(t)$ based on data on a planned (or ongoing) advertising campaign, have been developed to evaluate its effectiveness. The areas of further research may be related to the processing of statistics on various advertising campaigns, their effectiveness, duration over time in order to restore the analytical structure of the features introduced, and the possible correction of the model (1).

## 2. Analysis of the obtained results

In the implementation of model (2) the values of coefficients at which periodic crisis manifestations of countries with higher levels of economic development (series 1, 2 in Figure 1) do not affect the level of economy of a country with relatively weaker economic indicators (row 3) were established.

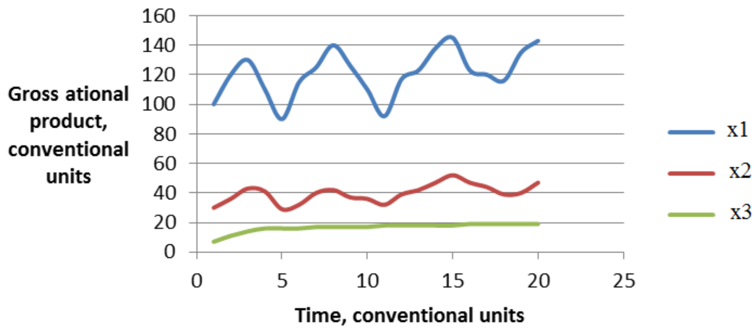


Figure 1 – Countries economics indicators dynamics

Implementation of the model (3), (4) allows you to set these coefficients, in which the stability of solutions with the desired asymptotic values (Figure 2) is provided.

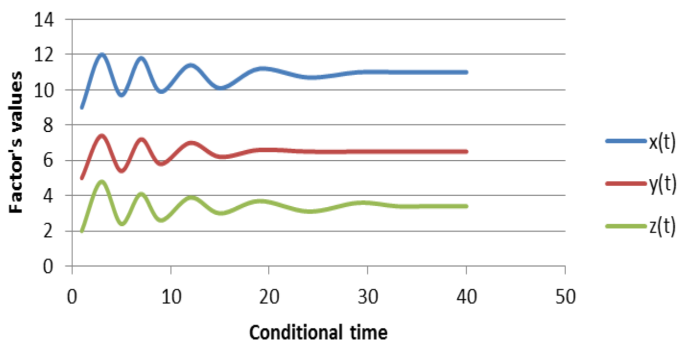


Figure 2 – The change of economic and ecological system indicators dynamics

The implementation of model (5) allows us to set the values of coefficients in which the stability of the solutions with the desired asymptotic values of the indicators and the dynamics of their change over time is ensured (figure 3):

$t0 := 0 \qquad t1 := 13$

Given

$$\frac{d}{dt}x(t) = 0.2x(t)\cdot(10 - x(t)) - 0.3y(t) + 0.4z(t) \qquad x(t0) = 2$$

$$\frac{d}{dt}y(t) = 0.2(10 - x(t))\cdot x(t) + 0.3y(t)\cdot(5 - y(t)) + 0.3z(t) \qquad y(t0) = 4$$

$$z(t0) = 3$$

$$\frac{d}{dt}z(t) = 0.4x(t) - 0.3y(t)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} := \text{Odesolve}\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix}, t, t1\right]$$

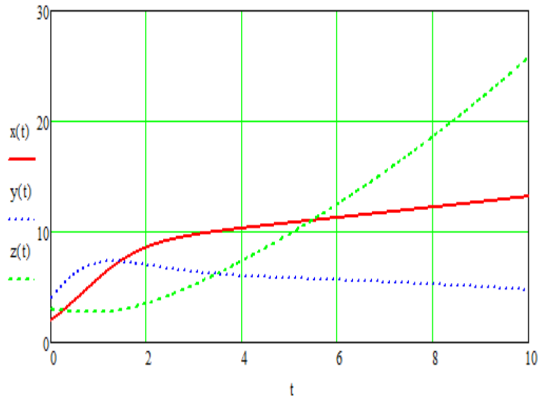


Figure 3 – The change of cost indicators for standards of technical diagnostics implementation, for elimination of emergencies consequences and efficiency of gas transmission system's work

The results of advertising campaign intensity model calculations are presented in Figure 4. The results of the calculations can determine the time at which the advertising campaign can be rolled – further investment has no proper effect.

Advertising campaign intesity

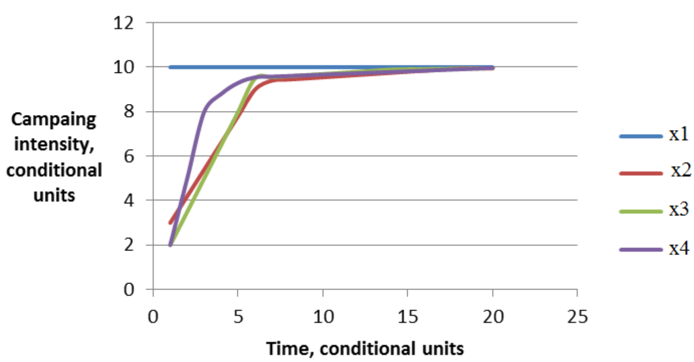


Figure 4 – The results of advertising campaign intensity model calculations

The conducted researches testify high efficiency of economic systems mathematical modeling method for their study, description and optimization. In fact, the modeling problem is divided into two stages – at the first, the relationships between its elements are studied in order to obtain qualitative indicators that adequately reflect its behavior, and the second, in fact, the inverse problem of selecting of the model coefficients that allow to determine the quantitative indices of the simulated systems is solved. The areas of further research are related to the adaptation of the developed models to real economic systems.

In particular, for the model of interconnected economies it is necessary to establish coefficients that quantitatively characterize the level of interconnection between each pair of economic complexes, and in the first stage of modeling these coefficients are normalized, their numerical values are chosen at the interval (0; 1) and are determined by the method of expert estimates, however in the analysis of real economic systems, it is necessary to switch to the dimensional values of these coefficients, which requires the interconnected efforts of specialists in the fields of economics and applied mathematics.

For the second model, which describes the functions *x(t)* – population in the region; *y(t)* – the level of pollution and other non–harmful effects on the environment caused by the economic activity of the population; *z(t)* – the level of flora of the region (trees, agricultural products, forests, gardens, etc.) to determine the model coefficients it is necessary to take into account the peculiarities of each of the studied regions – it should be noted that the values of the coefficients will depend on the characteristics of the studied region – the coefficients for the industrial regions will differ from the coefficients determined for the Carpathian or Bukovina region. It is also advisable to use the method of expert estimation for model calculations, having in mind again the scaling of coefficients by the interval (0; 1).

To practical realization the third model with investigated functions *x(t)* – costs for implementation of new technical diagnostics and control standards; *y(t)* – costs for elimination of emergencies consequences; *z(t)* is the efficiency of the studied industrial system's element in determining the model values of the system coefficients by the method of expert assessments, an important point is the selection of experts who should be specialists in the operation and evaluation of the technical condition of the systems under study and on economic issues to find the best ways to distribute investments. At the same time, it is important to ensure the objectivity of expert assessments – the opinion of experts on only one issue should not prevail.

For the practical implementation of the proposed model of the advertising company it is necessary to take into account the fact that only in this model its coefficients are functions of time. This necessitates the need for a wide-ranging survey of advertising campaigns for different types of advertising items – although widely used consumer electronics campaigns, the number of such campaigns provides a great deal of material for determining the appropriate features – model equation coefficients, based on analysis of relevant statistics.

The peculiarity of each of the proposed models is the fact that they are most effective in predicting the behavior of the simulated systems, since the construction of sufficiently accurate forecasts allows you to solve the following problems:

– forecasting the development of interdependent economies based on the study of economic trends characteristic of previous periods;

– forecasting of economic and ecological characteristics of regions taking into account their peculiarities:

– predicting the technical and economic efficiency of implementing new diagnostic systems and the effectiveness of advertising campaigns.

**Conclusions**

1. The technique of ordinary differential equations can be successfully applied to the simulation of the interconnected economies development, to estimation the change of economic and ecological system indicators dynamics and the change of cost indicators for standards of technical diagnostics implementation, for elimination of emergencies consequences and efficiency of gas transmission system's work and to predictions the advertising campaign efficiency.

2. All this models can be realized using the simple linear and quadratic type relationships, which, however, allow to receive the numerical results that are sufficiently accurate in terms of practical needs.

3. Construction and implementation of models 1 – 3 is carried out in two stages – the first of them is a simulation of the desired behavior of the system (prediction) b and in the second – the model is corrected by developing a methodology for determining its coefficients (correction).

4. Model 4 uses only one differential equation, which in most cases can be solved analytically. The obtained solutions are stable, which is a confirmation of the correctness of the proposed models.

5. The above four models do not exhaust the entire class of problems of modeling environmental, economic and other types of systems – for the scope of this work, the results of the authors' work on modeling systems in medicine, as well as models of dimensions above three are presented. However, these models have been successfully used in the approaches presented in the presented work.

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