

**S. Saracoglu** (Bartın Univ., Turkey),

**Y. Yayli** (Ankara Univ., Turkey)

## SPECIAL SPACE CURVES CHARACTERIZED BY $\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0$

### СПЕЦІАЛЬНІ ПРОСТОРОВІ КРИВІ,

### ЩО ХАРАКТЕРИЗУЮТЬСЯ УМОВОЮ $\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0$

In this study, by using the facts that  $\det(\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}) = 0$  characterizes a plane curve and  $\det(\alpha^{(2)}, \alpha^{(3)}, \alpha^{(4)}) = 0$  characterizes a curve of constant slope, we present the special space curves characterized by  $\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0$ , in different approaches. We show that the space curve is Salkowski if and only if  $\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0$ . The approach used in this paper is useful in understanding the role of the curves characterized by  $\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0$  in differential geometry.

За допомогою тих фактів, що умова  $\det(\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}) = 0$  характеризує плоску криву, а умова  $\det(\alpha^{(2)}, \alpha^{(3)}, \alpha^{(4)}) = 0$  — криву зі сталим нахилом, наведено спеціальні просторові криві, що характеризуються умовами  $\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0$ , в різних підходах. Показано, що просторова крива є кривою Салковського тоді і тільки тоді, коли  $\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0$ . Підхід, що використовується в роботі, є корисним для розуміння ролі кривих, що характеризуються умовою  $\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0$  в диференціальній геометрії.

**1. Introduction.** In the classical differential geometry general helices, Salkowski curves and spherical helices are well-known curves. These curves are defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the general helix) [2–5]. Among all the space curves with constant curvature, Salkowski curves are those for which the normal vector maintains a constant angle with a fixed direction in the space [3]. The study of these curves is given by Salkowski in [4], Monterde in [3], Kula, Ekmekçi, Yaylı, İlarıslan in [1] and Takenaka in [6]. Throughout this paper, by using some characterizations from [6], we are going to present the characterizations of the condition that  $\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0$  for special space curves in different approaches.

We first introduce the space curve  $\alpha$  as

$$\alpha(s) = (x(s), y(s), z(s)),$$

where  $s$  is the arclength parameter and denote two geometrical quantities, curvature and torsion, by  $\kappa$  and  $\tau$ . These structures play essential role in the theory of space curve. Such as, circles and circular helices are curves with constant curvature and torsion [3].

Furthermore, as it is seen from the known facts in [2, 5, 6],

(1) the condition that  $\det(\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}) = 0$  characterizes a great circle, where  $\alpha$  is a spherical curve,

(2) the condition that  $\det(\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}) = 0$  characterizes a plane curve,

(3) the condition that  $\det(\alpha^{(2)}, \alpha^{(3)}, \alpha^{(4)}) = 0$  characterizes a curve of constant slope.

In [6], Takenaka represents the difficulty in solving  $\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0$ . Therefore, he has given the following form to put the complicated terms away:

$$\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = \kappa^4 \det \begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ \kappa & \kappa' & \kappa'' \\ \tau & \tau' & \tau'' \end{pmatrix}, \quad (1)$$

where

$$\varphi_1 = -\left(\frac{1}{\kappa}\right)', \quad (2)$$

$$\varphi_2 = -\left(\frac{1}{\kappa}\right)'' - \frac{1}{\kappa}(\kappa^2 + \tau^2),$$

$$\varphi_3 = -\left(\frac{1}{\kappa}\right)''' - \left\{\frac{1}{\kappa}(\kappa^2 + \tau^2)\right\}' - \frac{1}{2\kappa}(\kappa^2 + \tau^2)'.$$

Moreover, by taking a curve with constant curvature, he shows that the following conditions are equivalent:

- (i)  $\det(\alpha^{(2)}, \alpha^{(3)}, \alpha^{(4)}) = 0$ ,  
 (ii)  $\tau(s) = \mp \frac{a^3(bs+c)}{\{1-a^4(bs+c)^2\}^{1/2}}$ , where  $b, c \in \mathbb{R}$  and

$$-\frac{1}{b}(1+c) \leq s \leq \frac{1}{b}(1-c).$$

Then for the constant curvature  $\kappa = a$ , he has found the torsion as:

$$\tau(s) = \pm \frac{a^3(bs+c)}{[1-a^4(bs+c)^2]^{1/2}}. \quad (3)$$

On the other hand, in [3], Monterde shows that normal vectors of the curve  $\alpha$  parametrized by arc-length with  $\kappa = 1$  make a constant angle with a fixed line in space if and only if

$$\tau(s) = \pm \frac{s}{[\tan^2 \phi - s^2]^{1/2}}. \quad (4)$$

As it is seen Salkowski curves are slant helices. In this study, we characterize Salkowski curves by the help of determinants. Accordingly, we present new special characterization with slant helices.

Analogously, in this paper, considering the relationship between the space curves, we have found important results. Moreover, we obtain that with constant curvature  $\kappa = 1$ , the space curve is Salkowski if and only if

$$\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0 \quad (5)$$

and also we have showed the following three conditions are equivalent:

- (1) the space curve  $\alpha$  is Salkowski curve,  
 (2)  $\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0$ ,  
 (3)  $\tau(s) = \pm \frac{s}{[\tan^2 \phi - s^2]^{1/2}}$ .

If we look previous studies in that field, we only meet the studies on the calculations of curvatures. In this study, we try to give some calculations on the family of determinants. and the special space curves that are characterized by  $\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0$ , in different approaches. We hope that this study will gain different interpretation to the other studies in this field.

**2. Spherical indicatrice curves.** In this section, we give the spherical indicatrice curves that are characterized by  $\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0$  and discuss the main properties.

**Proposition 1.** *Let*

$$\alpha: I \rightarrow E^3, \quad (6)$$

$$s \mapsto \alpha(s)$$

*be a space curve that is parametrized arc-length with  $\kappa \equiv 1$ . The tangent indicatrice of the space curve  $(T)$  is spherical helix if and only if*

$$\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0. \quad (7)$$

**Proof.** Let  $(T)$  be tangent curve as:

$$\alpha'(s) = (T) \quad (8)$$

and then

$$T' = \kappa N = N \quad (9)$$

for  $\kappa \equiv 1$ . Here,  $\|T'(s)\| = 1$  and the arc-parameter of the curve  $(T)$  is  $s$  and also the parameters of the curves  $\alpha$  and  $(T)$  are the same. According to all of these, if  $(T)$  is helix, then [6]

$$\det(T^{(2)}, T^{(3)}, T^{(4)}) = 0. \quad (10)$$

Thus

$$T = \alpha'(s) \quad (11)$$

and then

$$\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0. \quad (12)$$

On the contrary, it can be easily proved as above smiliarly.

**Theorem 1.** *Let*

$$\alpha: I \rightarrow E^3,$$

$$s \mapsto \alpha(s)$$

*be a space curve that is parametrized arc-length with  $\kappa \equiv 1$ . The tangent indicatrice  $(T)$  of  $\alpha$  is spherical helix if and only if*

$$2\tau''(1 + \tau^2) - 3\tau(1 + \tau^2)' = 0. \quad (13)$$

**Proof.** The tangent indicatrix of the space curve  $\alpha$  is spherical helix if and only if

$$\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0$$

and also from [6]

$$\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = \kappa^4 \det \begin{bmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ \kappa & \kappa' & \kappa'' \\ \tau & \tau' & \tau'' \end{bmatrix}. \quad (14)$$

Here, for  $\kappa \equiv 1$ ,

$$\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = \det \begin{bmatrix} 0 & -(1 + \tau^2) & -(1 + \tau^2)' - \frac{1}{2}(1 + \tau^2)' \\ 1 & 0 & 0 \\ \tau & \tau' & \tau'' \end{bmatrix} = 0$$

then we obtain

$$2\tau''(1 + \tau^2) - 3\tau(1 + \tau^2)' = 0. \quad (15)$$

Here,  $\tau$  is torsion of the curve  $\alpha(s)$ .

**Corollary 1.** *Let*

$$\alpha: I \rightarrow E^3,$$

$$s \mapsto \alpha(s)$$

be a space curve that is parametrized arc-length with  $\kappa \equiv 1$ . The tangent indicatrice ( $T$ ) of the space curve  $\alpha$  is spherical helix if and only if

$$\tau(s) = \pm \frac{bs + c}{[1 - (bs + c)^2]^{1/2}}, \quad (16)$$

where  $b \neq 0$  and  $b, c \in \mathbb{R}$ ,

$$-\frac{1}{b}(1 + c) \leq s \leq \frac{1}{b}(1 - c).$$

**Proof.** Under the condition  $\kappa \equiv 1$ , we have  $\varphi_1 = 0$ ,  $\varphi_2 = -(1 + \tau^2)$  and  $\varphi_3 = -(1 + \tau^2)' - \frac{1}{2}(1 + \tau^2)''$  from the proof of the theorem above. By giving similiar calculations from [6], the theorem can be easily proved.

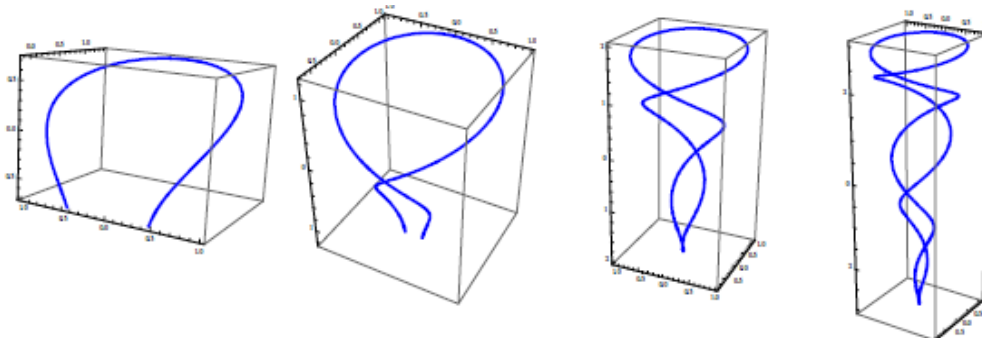
**Result.** By making similiar calculations we can get that

$$\tau = \pm \frac{bs + c}{[1 - (bs + c)^2]^{1/2}} \quad (17)$$

where  $b = \frac{1}{\tan \phi}$ , is the solution of differential equation as given

$$\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 2\tau''(1 + \tau^2) - 3\tau(1 + \tau^2)' = 0. \quad (18)$$

On the other hand, the following figure can be given as an example for showing the curve whose  $\kappa$  and  $\tau$  satisfy the condition above for  $\kappa = 1$ :



Salkowski curves with  $\kappa \equiv 1$  [3].

Now, we will introduce the condition that  $\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0$  for Salkowski curves in different approaches.

**Corollary 2.** *Let*

$$\begin{aligned}\alpha: I &\rightarrow E^3, \\ s &\mapsto \alpha(s)\end{aligned}$$

be a space curve that is parametrized arc-length with  $\kappa \equiv 1$ . The space curve  $\alpha$  is Salkowski curve if and only if

$$\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0.$$

**Proof.** If the space curve  $\alpha$  is Salkowski and slant helix, then

$$\langle N, d \rangle = \cos \theta = \text{constant}, \quad (19)$$

for a fixed line  $d$  in a space. Besides, it can be easily given that for the normal vector  $N$  of the space curve  $\alpha$ :

$$\beta(s) = T \quad \text{and} \quad \beta'(s) = N \quad \text{for} \quad \kappa \equiv 1. \quad (20)$$

Here,

$$\langle \beta'(s), d \rangle = \langle N, d \rangle = \text{constant}. \quad (21)$$

In this case, the curve  $\beta(s)$  is spherical helix. From the Theorem 1, for  $\beta = \alpha'$ ,

$$\det(\beta^{(2)}, \beta^{(3)}, \beta^{(4)}) = 0 \quad (22)$$

and then

$$\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0.$$

In contrary,

$$\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0$$

then for  $\beta = \alpha^{(1)}$ , we obtain

$$\det(\beta^{(2)}, \beta^{(3)}, \beta^{(4)}) = 0.$$

Thus,  $\beta(s)$  is a spherical helix. In that case, the curve  $\alpha(s)$  is slant helix. Then, we can easily get that  $\alpha(s)$  is Salkowski for  $\kappa \equiv 1$  from the calculations above.

**Corollary 3.** *Let*

$$\begin{aligned}\alpha: I &\rightarrow E^3, \\ s &\mapsto \alpha(s)\end{aligned} \quad (23)$$

be a space curve that is parametrized arc-length with  $\kappa \equiv 1$ . The following three conditions are equivalent:

- 1) the space curve  $\alpha$  is Salkowski curve,
- 2)  $\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0$ ,
- 3)  $\tau(s) = \pm \frac{s}{[\tan^2 \phi - s^2]^{1/2}}$ .

**Proof.** In the first step, it should be shown that the curve  $\alpha$  is Salkowski if and only if [3]

$$\tau(s) = \pm \frac{s}{[\tan^2 \phi - s^2]^{1/2}}.$$

In the second step, we should prove that  $\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0$  if and only if

$$\tau(s) = \pm \frac{s}{[\tan^2 \phi - s^2]^{1/2}}.$$

Here, by taking  $c = 0$  and  $b = \frac{1}{\tan \phi}$  in Corollary 1, we get

$$\tau(s) = \pm \frac{s}{[\tan^2 \phi - s^2]^{1/2}}.$$

Hence, the theorem is proved.

**3. Conclusions.** The starting point of this study is to give the special space curves that are characterized by  $\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0$ , in different approach. We have developed this approach with discussing main properties of spherical curves, slant helices, Salkowski curves and relationship between these curves. At this time, different approaches that we give here have showed us the space curve is Salkowski if and only if  $\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0$ . Additionally, it is obtained that the tangent indicatrice of the space curve ( $T$ ) is spherical helix if and only if  $\det(\alpha^{(3)}, \alpha^{(4)}, \alpha^{(5)}) = 0$ .

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