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Xiao-Ming Zeng (Xiamen Univ., China)

APPROXIMATION FOR ABSOLUTELY CONTINUOUS FUNCTIONS BY STANCU BETA OPERATORS *

НАБЛИЖЕННЯ АБСОЛЮТНО НЕПЕРЕРВНИХ ФУНКЦІЙ БЕТА-ОПЕРАТОРАМИ СТАНКУ

In this paper, we obtain an exact estimate for the first-order absolute moment of Stancu Beta operators by means of the Stirling formula and integral operations. Then we use this estimate for establishing a theorem on approximation of absolutely continuous functions by Stancu Beta operators.

Отримано точну оцінку для абсолютного моменту бета-операторів Станку першого порядку із використанням формули Стірлінга та інтегральних операцій. Цю оцінку використано для встановлення теореми про наближення абсолютно неперервних функцій бета-операторами Станку.

1. Introduction and definitions. For Lebesgue integrable functions f on the interval $I = (0, \infty)$, Stancu Beta operators L_n are defined by

$$L_n(f;x) = \frac{1}{B(nx, n+1)} \int_0^\infty \frac{t^{nx-1}}{(1+t)^{nx+n+1}} f(t)dt, \tag{1}$$

where $B(p,q)$ is Beta function. The operators L_n were introduced first by Stancu [1] in 1995. Stancu [1] studied some approximation properties of these operators. Abel [2] derived the complete asymptotic expansion for the sequence of these operators. Abel, Gupta and others [3, 4] discussed the rates of convergence of the operators L_n for functions of bounded variation and functions with derivatives of bounded variation. For more related work, one can refer to [5 – 7]. In present paper we first give an exact estimate for the first order absolute moment of the operators L_n, then by means of this estimate we establish an approximation theorem of operators L_n for the absolutely continuous functions $f \in \Phi_{DB}$. The class of functions Φ_{DB} is defined as follows:

$$\Phi_{DB} = \left\{ f \mid f(x) - f(0) = \int_0^x \phi(t)dt; \right.$$

$$\phi \text{ is bounded on every finite subinterval of } \quad [0,\infty);$$

$$\left. \text{and} \quad f(t) = O(t^r) \quad \text{as} \quad t \to \infty \right\}.$$

For a bounded function f on every finite subinterval of $[0,\infty)$, we introduce the following metric form:

$$\Omega_x(f,\lambda) = \sup_{t \in [x-\lambda,\ x+\lambda] \cap [0,\infty]} |f(t) - f(x)|,$$

where $x \in [0,\infty)$ is fixed, $\lambda \geq 0$.

For the properties of $\Omega_x(f,\lambda)$, one can refer to reference [8].

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2. The first absolute moment of Stancu Beta operators. In this section we derive an exact estimate for the first order absolute moment of Stancu Beta operators. We need the following lemma.

Lemma 1 ([2], Proposition 2)**.** *For Stancu Beta operators L_n, we have*

$$L_n(1;x)=1, \qquad L_n(t;x)=x, \qquad L_n((t-x)^2;x)=\frac{x(1+x)}{n-1}. \tag{2}$$

Let $x\in(0,\infty)$ be fixed. For $r=0,1,2,\ldots$, there holds

$$L_n((t-x)^r;x)=O(n^{-\lfloor (r+1)/2\rfloor}). \tag{3}$$

Theorem 1. *For Stancu Beta operators L_n, we have*

$$L_n(|t-x|;x)=\frac{2x^{nx}}{n(1+x)^{nx+n}n!}\frac{\Gamma(nx+n+1)}{\Gamma(nx)}. \tag{4}$$

Proof. By direct computation and using Lemma 1, we have

$$L_n(|t-x|;x)=$$

$$=\frac{1}{B(nx,n+1)}\left[\int_0^x\frac{(x-t)t^{nx-1}}{(1+t)^{nx+n+1}}dt+\int_x^\infty\frac{(t-x)t^{nx-1}}{(1+t)^{nx+n+1}}dt\right]=$$

$$=\frac{2}{B(nx,n+1)}\int_0^x\frac{(x-t)t^{nx-1}}{(1+t)^{nx+n+1}}dt+L_n((t-x);x)=$$

$$=\frac{2}{B(nx,n+1)}\left(\int_0^x\frac{xt^{nx-1}}{(1+t)^{nx+n+1}}dt-\int_0^x\frac{t^{nx}}{(1+t)^{nx+n+1}}dt\right).$$

Change of variable and integration by parts derive

$$\int_0^x\frac{xt^{nx-1}}{(1+t)^{nx+n+1}}dt=x\int_0^{x/(1+x)}u^{nx-1}(1-u)^n du=$$

$$=\frac{\left(\frac{x}{1+x}\right)^{nx}\left(\frac{1}{(1+x)}\right)^{n}}{n}+\int_0^{x/(1+x)}u^{nx}(1-u)^{n-1}du, \tag{5}$$

and

$$\int_0^x\frac{t^{nx}}{(1+t)^{nx+n+1}}dt=\int_0^{x/(1+x)}u^{nx}(1-u)^{n-1}du. \tag{6}$$

From (5), (6) and simple computation, we obtain

$$L_n\left(|t-x|;x\right)=\frac{2x^{nx}}{n(1+x)^{nx+n}B(nx,\ n+1)}.$$

Note that

$$B(nx, n+1) = \frac{n!\Gamma(nx)}{\Gamma(nx+n+1)},$$

we obtain (4).

Theorem 1 is proved.

From Theorem 1 we derive the following inequalities, which are suitable for actual applications.

Proposition 1. *For Stancu Beta operators L_n, we have*

$$\left(\frac{2x(1+x)}{\pi n}\right)^{1/2}\left(1-\frac{1+x}{12nx}\right) < L_n(|t-x|;x) < \left(\frac{2x(1+x)}{\pi n}\right)^{1/2}. \qquad (7)$$

Proof. By Theorem 1 and using Stirling's formula (cf. [9, 10])

$$\Gamma(z+1) = \sqrt{2\pi z}(z/e)^z e^{\theta}, \qquad (12z+1)^{-1} < \theta < (12z)^{-1},$$

we have

$$L_n(|t-x|;x) = \frac{2x^{nx}}{n(1+x)^{nx+n}n!}\frac{\Gamma(nx+n+1)}{\Gamma(nx)} = \left(\frac{2x(1+x)}{\pi n}\right)^{1/2} e^{\theta_1-\theta_2-\theta_3},$$

where

$$\frac{1}{12(nx+n)+1} < \theta_1 < \frac{1}{12(nx+n)},$$

$$\frac{1}{12n+1} < \theta_2 < \frac{1}{12n}, \qquad \frac{1}{12nx+1} < \theta_3 < \frac{1}{12nx}.$$

Set $c(\theta) = \theta_1 - \theta_2 - \theta_3$, simple computation derives

$$-\frac{1+x}{12nx} < c(\theta) < 0.$$

Thus we obtain

$$\left(\frac{2x(1+x)}{\pi n}\right)^{1/2} e^{-(1+x)/(12nx)} < L_n(|t-x|;x) < \left(\frac{2x(1+x)}{\pi n}\right)^{1/2}.$$

It follows that

$$\left(\frac{2x(1+x)}{\pi n}\right)^{1/2}\left(1-\frac{1+x}{12nx}\right) < L_n(|t-x|;x) < \left(\frac{2x(1+x)}{\pi n}\right)^{1/2}.$$

Proposition 1 is proved.

***Corollary* 1.** *For Stancu Beta operators L_n and every $x \in (0,\infty)$, there holds*

$$L_n(|t-x|;x) = \left(\frac{2x(1+x)}{\pi n}\right)^{1/2} + O(n^{-3/2}).$$

3. Approximation for absolutely continuous functions. In this section we study the rate of convergence of Stancu Beta operators L_n for the functions $f \in \Phi_{DB}$. We need following two lemmas.

Lemma 2 ([3], Lemma 3)**.** *Let $x \in (0,\infty)$ be fixed, the functions $K_{n,x}$ and $R_{n,x}$ are defined by*

$$K_{n,x}(t) = \frac{1}{B(nx,n+1)} \int\limits_0^t \frac{u^{nx-1}}{(1+u)^{nx+n+1}} du, \tag{8}$$

$$R_{n,x}(t) = 1 - K_{n,x}(t) = \frac{1}{B(nx,n+1)} \int\limits_t^\infty \frac{u^{nx-1}}{(1+u)^{nx+n+1}} du. \tag{9}$$

Then for $n \geq 2$, we have

$$K_{n,x}(y) \leq \frac{x(1+x)}{(n-1)(x-y)^2}, \quad 0 \leq y < x, \tag{10}$$

$$R_{n,x}(z) = 1 - K_{n,x}(z) \leq \frac{x(1+x)}{(n-1)(z-x)^2}, \qquad x < z < \infty. \tag{11}$$

Lemma 3. *Let $r > 0$ and m be a positive integer satisfying $m > r/2$, we have*

$$\frac{1}{B(nx,n+1)} \int\limits_{2x}^\infty t^r \frac{t^{nx-1}}{(1+t)^{nx+n+1}} dt = \frac{(2x)^r}{x^{2m}} O\left(n^{-\lfloor m+1/2 \rfloor}\right). \tag{12}$$

Proof. For $m > r/2$ and $t \in [2x,\infty)$

$$\frac{d}{dt} \frac{t^r}{(t-x)^{2m}} = \frac{t^{r-1}(t-x)^{2m-1}(rt-2mt-rx)}{(t-x)^{4m}} < 0,$$

which implies that $\dfrac{t^r}{(t-x)^{2m}}$ is monotone decreasing for $t \in [2x,\infty)$. Thus

$$\frac{1}{B(nx,n+1)} \int\limits_{2x}^\infty t^r \frac{t^{nx-1}}{(1+t)^{nx+n+1}} dt \leq$$

$$\leq \frac{1}{B(nx,n+1)} \int\limits_{2x}^\infty \frac{(2x)^r}{x^{2m}} (t-x)^{2m} \frac{t^{nx-1}}{(1+t)^{nx+n+1}} dt \leq$$

$$\leq \frac{(2x)^r}{x^{2m}} L_n\left((t-x)^{2m}; x\right).$$

Equation (12) now follows from Lemma 1.

Now we state the main result of this section.

Theorem 2. *Let f be a function in Φ_{DB}. If $\phi(x+)$ and $\phi(x-)$ exist at a fixed point $x \in (0,\infty)$, write $\tau = \dfrac{\phi(x+)-\phi(x-)}{2}$, then for $n \geq 2$ we have*

$$\left| L_n(f;x) - f(x) - \tau \left(\frac{2x(1+x)}{n\pi}\right)^{1/2} \right| \leq$$

$$\leq \frac{3+7x}{n-1}\sum_{k=1}^{[\sqrt{n}\,]}\Omega_x(\phi_x, x/k) + \frac{(1+x)^{3/2}}{\sqrt{72x\pi}}n^{-3/2} + \frac{(2x)^r}{x^{2m}}O\left(n^{-\lfloor (m+1)/2\rfloor}\right), \qquad (13)$$

where m is a positive integer satisfying $m > r/2$, and

$$\phi_x(t) = \begin{cases} \phi(t) - \phi(x+), & x < t \leq 1, \\ 0, & t = x, \\ \phi(t) - \phi(x-), & 0 \leq t < x. \end{cases} \qquad (14)$$

Proof. By direct computation we find that

$$L_n(f;x) - f(x) = \frac{\phi(x+) - \phi(x-)}{2}L_n(|t-x|;x) - A_{n,x}(\phi_x) +$$

$$+ B_{n,x}(\phi_x) + D_{n,x}(\phi_x), \qquad (15)$$

where

$$A_{n,x}(\phi_x) = \int_0^x \left(\int_t^x \phi_x(u)du\right) d_tK_{n,x}(t),$$

$$B_{n,x}(\phi_x) = \int_x^{2x} \left(\int_x^t \phi_x(u)du\right) d_tK_{n,x}(t),$$

$$D_{n,x}(\phi_x) = \int_{2x}^{\infty} \left(\int_x^t \phi_x(u)du\right) d_tK_n(x,t),$$

and $K_{n,x}(t)$ is defined in (8).

Integration by parts derives

$$A_{n,x}(\phi_x) = \int_0^x \left(\int_t^x \phi_x(u)du\right) d_tK_{n,x}(t) =$$

$$= \int_t^x \phi_x(u)duK_{n,x}(t)\Bigg|_0^x + \int_0^x K_{n,x}(t)\phi_x(t)dt = \int_0^x K_{n,x}(t)\phi_x(t)dt =$$

$$= \left(\int_0^{x-x/\sqrt{n}} + \int_{x-x/\sqrt{n}}^x\right)K_{n,x}(t)\phi_x(t)dt.$$

Note that $K_{n,x}(t) \leq 1$ and $\phi_x(x) = 0$, by monotonicity of $\Omega_x(\phi_x, \lambda)$ it follows that

$$\left|\int_{x-x/\sqrt{n}}^x K_{n,x}(t)\phi_x(t)dt\right| \leq \frac{x}{\sqrt{n}}\Omega_x\left(\phi_x, \frac{x}{\sqrt{n}}\right) \leq \frac{2x}{n}\sum_{k=1}^{[\sqrt{n}\,]}\Omega_x(\phi_x, x/k).$$

On the other hand, by inequality (10) and using change of variable $t = x - x/u$, we have

$$\left|\int\limits_0^{x-x/\sqrt{n}} K_{n,x}(t)\phi_x(t)dt\right| \leq \frac{x(1+x)}{n-1}\int\limits_0^{x-x/\sqrt{n}} \frac{\Omega_x(\phi_x, x-t)}{(x-t)^2}dt =$$

$$= \frac{1+x}{n-1}\int\limits_1^{\sqrt{n}} \Omega_x(\phi_x, x/u)du \leq \frac{1+x}{n-1}\sum_{k=1}^{[\sqrt{n}]} \Omega_x(\phi_x, x/k).$$

Thus, it follows that

$$|A_{n,x}(\phi_x)| \leq \frac{1+3x}{n-1}\sum_{k=1}^{[\sqrt{n}]} \Omega_x(\phi_x, x/k). \tag{16}$$

Next we estimate $|B_{n,x}(\phi_x)|$

$$B_{n,x}(\phi_x) = \int\limits_x^{2x}\left(\int\limits_x^t \phi_x(u)du\right)d_tK_{n,x}(t) = -\int\limits_x^{2x}\left(\int\limits_x^t \phi_x(u)du\right)d_tR_{n,x}(t) =$$

$$= -\int\limits_x^t \phi_x(u)du \cdot R_{n,x}(t)\Bigg|_x^{2x} + \int\limits_x^{2x} \phi_x(t)R_{n,x}(t)dt =$$

$$= -\int\limits_x^{2x} \phi_x(u)du \cdot R_{n,x}(2x) + \int_x^{2x} \phi_x(t)R_{n,x}(t)dt. \tag{17}$$

By Lemma 2

$$\left|-\int\limits_x^{2x} \phi_x(u)du \cdot R_{n,x}(2x)\right| \leq x\Omega_x(\phi_x, x)\frac{x(1+x)}{(n-1)x^2} = \frac{(1+x)}{n-1}\Omega_x(\phi_x, x). \tag{18}$$

On the other hand, similar to the estimate of $|A_{n,x}(\phi_x)|$, we have

$$\left|\int\limits_x^{2x} \phi_x(t)R_n(x,t)dt\right| \leq \frac{1+3x}{n-1}\sum_{k=1}^{[\sqrt{n}]} \Omega_x(\phi_x, x/k).$$

For estimate of $|D_{n,x}(\phi_x)|$, note that $f(t) = O(t^r)$, thus there exists a constant M such that

$$|D_{n,x}(\phi_x)| \leq M\int\limits_{2x}^{\infty} t^r \frac{t^{nx-1}}{(1+t)^{nx+n+1}}dt.$$

Using Lemma 3 we obtain

$$|D_{n,x}(\phi_x)| = \frac{(2x)^r}{x^{2m}}O(n^{-\lfloor (m+1)/2\rfloor}). \tag{19}$$

Theorem 2 now follows from (15) – (19) combining with Proposition 1 and some simple computations.

Remark. If f is a function with derivative of bounded variation, then $f \in \Phi_{DB}$. Thus the approximation of functions with derivatives of bounded variation is a special case of Theorem 2. In this special case Theorem 2 is better than a result of Gupta, Abel and Ivan in [4].

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