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FUNCTIONS OF ULTRAEXPONENTIAL AND INFRALOGARITHM TYPES AND GENERAL SOLUTION OF THE ABEL FUNCTIONAL EQUATION

ФУНКЦІЇ УЛЬТРАЕКСПОНЕНЦІАЛЬНОГО ТА ІНФРАЛОГАРИФМІЧНОГО ТИПІВ І ЗАГАЛЬНИЙ РОЗВ'ЯЗОК ФУНКЦІОНАЛЬНОГО РІВНЯННЯ АБЕЛЯ

We propose generalized forms of ultraexponential and infralogarithm functions introduced and studied by the author earlier and present two classes of special functions, namely, ultraexponential and infralogarithm f-type functions. As a result of present investigation, we obtain general solution of the Abel equation $\alpha(f(x)) = \alpha(x) + 1$ under some conditions on a real function f and prove a new completely different uniqueness theorem for the Abel equation stating that the infralogarithm f-type function is its unique solution. We also show that the infralogarithm f-type function is an essentially unique solution of the Abel equation. Similar theorems are proved for the ultraexponential f-type functions and their functional equation $\beta(x) = f(\beta(x-1))$ which can be considered as dual to the Abel equation. We also solve certain problem being unsolved before, study some properties of two considered functional equations and some relations between them.

Запропоновано узагальнені форми ультраекспоненціальних та інфралогарифмічних функцій, що були введені і вивчені автором раніше, та наведено два класи спеціальних функцій — ультраекспоненціального та інфралогарифмічного f-типу. В результаті досліджень отримано загальний розв'язок рівняння Абеля $\alpha(f(x)) = \alpha(x) + 1$ за певних умов для реальної функції f і доведено нову цілком іншу теорему єдиності для рівняння Абеля з твердженням про те, що функція інфралогарифмічного f-типу є єдиним розв'язком цього рівняння. Також показано, що функція інфралогарифмічного f-типу є суттєво єдиним розв'язком рівняння Абеля. Подібні теореми доведено для функцій ультраекспоненціального f-типу та їх функціонального рівняння $\beta(x) = f(\beta(x-1))$, яке можна вважати дуальним для рівняння Абеля. Також розв'язано задачу, що не була розв'язана до теперішнього часу, вивчено властивості двох розглядуваних функціональних рівнянь та деякі співвідношення між ними.

1. Introduction and preliminaries. In [2] we solve the following functional equation completely and obtain its general solution

$$\alpha(a^x) = \alpha(x) + 1, \quad x \in \mathbb{R}\backslash[\delta_1, \delta_2].$$

In fact this equation is a special case of the Abel's equation (equation (7.14) in [3]), where $f = \exp_a$, $0 < a \neq 1$, $\delta_1 \leq \delta_2$ are the two zeros of $g(x) = a^{a^x} - x$ with this assumption that $\delta_2 = +\infty$ if $a > 1$, and

$$E = \mathbb{R}\backslash[\delta_1, \delta_2] = \begin{cases} \mathbb{R}, & a > e^{1/e}, \\ (-\infty, \delta_1), & 1 < a \leq e^{1/e}, \\ \mathbb{R} \setminus \{\delta_1 = \delta_2\}, & \left(\dfrac{1}{e}\right)^e \leq a < 1, \\ (-\infty, \delta_1) \cup (\delta_2, +\infty), & 0 < a < \left(\dfrac{1}{e}\right)^e. \end{cases}$$

It is very important to know that the general solution does not need any conditions on a and α, and it is new. The general solution is $\alpha = \varphi_1 \operatorname{Iog}_a + \overline{[\ ]}_a = (\phi)_1 \operatorname{Iog}_a + \overline{[\ ]}_a$, where Iog_a is the *infralogarithm function* as the dual of the *ultraexponential function*: $\operatorname{uxp}_a(x) = a^{\overset{x}{-}}$, $\overline{[\ ]}_a$ *ultra power part function*, $(\ )_1$ decimal part function, φ_1 every 1-periodic function and ϕ is any function defined on $[0, 1)$. For this reason we call the above equation *infralogarithm functional equation*. Moreover, we proved that the equation is equivalent to $f(a^{\overset{x}{-}}) = f((x)_1) + [x] + 1$, namely *co-infralogarithm functional equation*, if x is restricted to $(\mathbb{R}\backslash[\delta_1, \delta_2]) \cap D_{\operatorname{uxp}_a}$. Also, we prove a uniqueness theorem that states Iog_a is a unique solution of the Abel's $\exp_a$-functional equation (infralogarithm equation), under some conditions. Of course, similar theorems for the ultraexponential functions and their related functional equations ($\beta(x) = a^{\beta(x-1)}$ that is dual of the equation) are proved in [4].

Now, to generalize the above results for the Abel's equation in general (for a given real function $f : \mathbb{R} \to \mathbb{R}$), we first introduce ultraexponential and infralogarithm f-type functions and then obtain its general solution (by using the two classes of functions), under some conditions on f that is completely different to the previous assumptions in the main references such as [3, 5].

If f, φ are real functions and μ is an integer valued function ($\mu : \mathbb{R} \to \mathbb{Z}$), then we define the function f^μ_φ by

$$f^\mu_\varphi(x) = f^{\mu(x)}(\varphi(x)), \tag{1.1}$$

and call it *μ-composition of f at φ* or *μ-iteration of f at φ*. The domain of f^μ_φ is dependent on invertibility of f and the domains of each functions f, φ and μ. Therefore if f is not invertible, then $D_{f^\mu_\varphi} \subseteq \mu^{-1}([0, +\infty))$. Also $f^\mu_\varphi(x) = \varphi(x)$, for every $x \in \mu^{-1}(\{0\})$.

Notice that $f^\mu_I = f^\mu$ and $f^\mu_f = f^{\mu+1}$ (I is the identity function), also if $\mu = n$ is a constant function, then $f^\mu = f^n$ (n-composition of f).

If $\mu(x + k) = \mu(x) + k$ and φ is k-periodic, where k is an integer, then

$$f^\mu_\varphi(x + k) = f^{\mu(x+k)}(\varphi(x + k)) = f^{k+\mu(x)}(\varphi(x)) = f^k(f^\mu_\varphi(x)), \tag{1.2}$$

for every x such that $x, x + k \in D_{f^\mu_\varphi}$ (of course if f^k is defined). Especially if $k = 1$ and $g = f^\mu_\varphi$, then $g(x + 1) = f(g(x))$.

Denote by $[x]$ the largest integer not exceeding x and put $(x) = x - [x]$. Then, for any fixed real number $r \neq 0$, we set

$$(x)_r = r\left(\frac{x}{r}\right), \quad [x]_r = r\left[\frac{x}{r}\right] \quad \forall x \in \mathbb{R}$$

and call $(x)_r$ *r-decimal part of x* and $[x]_r$ *r-integer part of x*. Since $(x)_1 = (x)$ (to prevent any confusion between decimal and parentheses notation) sometimes we use the symbol $(x)_1$ instead of (x). Clearly $x = [x]_r + (x)_r$, and

$$[x]_r \in \langle r \rangle = r\mathbb{Z}, \qquad (x)_r \in r[0, 1) = [0, r) \quad \text{or} \quad (r, 0].$$

We call $(x)_r$, $[x]_r$ *r-parts of x*. It is easy to see that the r-decimal part function $(\ )_r$ is r-periodic (especially the decimal part function $(\ ) = (\ )_1$ is periodic of period 1).

Note. Every r-periodic function φ has the form $\varphi = \phi o(\ )_r$ where ϕ is a function defined on $r[0, 1)$. Since the composition of every function and a periodic function is periodic, the r-decimal part function is the basic r-period function. In fact $\varphi = \phi o(\ )_r$ is the general solution of the functional equation $\varphi(x + r) = \varphi(x)$.

2. Ultraexponential f-type functions. Here we generalize the ultraexponential function that is a unique extension of the Tetration.

Definition 2.1. *We call the function* $f_{(\,)}^{[\,]+1}$ "*semi-ultraexponential f-type function*" *and denote by* upt_f .

Recall that if $0 < a \neq 1$, then $\exp_a(x) = a^x$ and $\exp(x) = \exp_e(x) = e^x$ (for all x) and $\exp_a^{-1} = \log_a$, $\ln = \exp^{-1}$. Therefore $\mathrm{upt}_{\exp_a} = \mathrm{uxp}_a$ which is called ultraexponential function, and $\mathrm{uxp} = \mathrm{uxp}_e$ is the natural ultraexponential function. Hence $\mathrm{uxp}_a(x) = \exp_a^{[x]+1}((x)_1) = a^{\frac{x}{}}$ (see [4]).

Now let f be a function defined on $f^n([0,1))$, for every non-negative integer n. Then upt_f is defined on $[-1,+\infty)$ and we have

$$\mathrm{upt}_f(x) = f^{[x]+1}((x)_1) = \begin{cases} \vdots & \\ x+1, & -1 \leq x < 0, \\ f(x), & 0 \leq x < 1, \\ f^2(x-1), & 1 \leq x < 2, \\ f^3(x-2), & 2 \leq x < 3, \\ \vdots & \end{cases}$$

Note that if f is invertible, then the domain of upt_f may be larger than $[-1,+\infty)$. In fact with the mentioned hypothesis we have

$$D_{\mathrm{upt}_f} = [-1,+\infty) \cup ([-2,-1) \cap (D_{f^{-1}} - 2)) \cup \ldots = [-1,+\infty) \cup S_{f^{-1}}, \tag{2.1}$$

where $S_{f^{-1}} \subseteq (-\infty,-1)$ and $S_{f^{-1}} = \varnothing$ if f is not invertible.

The function upt_f satisfies the following well known functional equation

$$\beta(x) = f(\beta(x-1)), \tag{2.2}$$

for every $x \geq 0$. For this reason we call (2.2) "ultraexponential f-type functional equation".

***Example* 2.1.** If $f = \exp_a$, then $\mathrm{upt}_f = f_{(\,)}^{[\,]+1} = \mathrm{uxp}_a$. Now if $a > 1$, then (in (2.1) $S_{f^{-1}} = (-2,-1)$ so $D_{\mathrm{uxp}_a} = (-2,+\infty)$ and we have

$$\mathrm{uxp}_a(x) = \exp_a(\mathrm{uxp}_a(x-1)), \quad x > -1.$$

In this case the values of x in (2.2) are extended from $x \geq 0$ to $x > -1$.

But if $0 < a < 1$, then

$$S_{f^{-1}} = (-2,-1) \cup (-3,-2) \cup (-4+a,-3) \cup (-5+a,-5+a^a) \cup (-6+a^{a^a},-6+a^a) \cup \ldots,$$

and

$$\mathrm{uxp}_a(x) = \exp_a(\mathrm{uxp}_a(x-1)), \quad -1 \neq x > -2.$$

In fact the above equation holds for all x such that $x, x-1 \in D_{\mathrm{uxp}_a}$ (see [4]).

In [4] we proved the first uniqueness theorem about the *Tetration* that one of its corollaries states if $a = e$, then there is no any convex function on $[-1,+\infty)$ except that $f = \mathrm{uxp}_a$ such that satisfies the functional equation

$$\beta(x) = \exp_a(\beta(x-1)), \quad x > -1,$$

with the initial condition $f(0) = 1$. But for $a > e$ it was left as an unsolved problem as follows:

Question. Let $a > e$. Is there any convex function on $[-1, +\infty)$ except that $f = \mathrm{uxp}_a$ which satisfies the ultraexponential functional equation?

Now we claim that the answer is positive. Because, putting

$$\phi(x) = a^{\frac{2x+(-1+\ln a)x^2}{1+\ln a}}, \qquad f(x) = \exp_a^{[x]}(\phi(x-[x])) \quad \forall x > -2,$$

one can see that f satisfies the all conditions of the question (considering Corollary 2.3, 3.5 and Theorem 3.2 of [4]) and clearly $f \neq \mathrm{uxp}_a$. In fact there exist infinitely many solutions for it. For if $0 < \varepsilon \leq \dfrac{-1+\ln a}{1+\ln a}$, $\phi(x) = a^{(1-\varepsilon)x+\varepsilon x^2}$, then the function f defined by the above equation satisfies the all conditions.

The following lemma gives us general solution of the ultraexponential f-type equation for an arbitrary given function f.

Lemma 2.1. *Let c be a constant real number and f a given real function. Then, general solution of the functional equation*

$$\beta(x) = f(\beta(x-1)), \quad x \geq c, \tag{2.3}$$

is

$$\beta(x) = f^{[x-c]+1}(\varphi(x)), \quad x \geq c-1, \tag{2.4}$$

where φ is every 1-periodic real function.

Proof. Considering (1.2) if β has the form (2.4), then satisfies the equation. Conversely let β satisfies the equation and fix x. Then $\beta(x) = f^n(\beta(x-n)))$, for every non-negative integer n such that $x - n \geq c - 1$. Putting $n = [x-c]+1$ and $\varphi(x) = \beta((x-c)_1 + c - 1)$, we have $\beta(x) = f^{[x-c]+1}(\varphi(x))$ and φ is 1-periodic.

Note. If f is defined on $f^n([0,1))$ for every integer $n \geq 0$, then upt_f satisfies the equation (2.3), when $c = 0$. If f is invertible, then it may be satisfied the equation for some $c < 0$. For example if $f = \exp_a$, then upt_f satisfies the equation for $c = -1$ (see [4]).

But it is important to know that the function $\mathrm{upt}_f(x-c)$ satisfies (2.3), for every constant c (put $\varphi(x) = (x-c)_1$), and so always it is a solution of the equation.

An important problem about the ultra f-type equation is that when upt_f is its unique solution? (under which conditions?). To answer the question, in the following theorem we introduce a unique solution for the ultraexponential f-type equation.

Theorem 2.1 [A uniqueness conditions for the ultraexponential f-type functions]**.** *Let f be a function defined on $f^n([0,1))$ for every integer $n \geq 0$ and differentiable on $[0,1)$, $f(0) = 1$, $f'_+(0) \neq 0$.*

Then $\beta = \mathrm{upt}_f$ is the unique solution of the ultraexponential f-type equation on $[-1, +\infty)$ for which is increasing and differentiable on $[-1, 0)$ and β' is monotonic (non-decreasing or non-increasing) on it and

$$\beta(-1) = 0, \qquad \lim_{x \to 0^+} \beta'(x) = f'_+(0) \lim_{x \to 0^-} \beta'(x). \tag{2.5}$$

Proof. Clearly upt_f satisfies the conditions. Now if $\beta = g$ satisfies the conditions, then g is differentiable on $[-1,0]$, $(0,1)$ and continuous at zero $\big(\lim_{x\to 0} g(x) = g(0) = f(g(-1)) = f(0) = 1\big)$. Therefore $0 < g(x-1) < 1$ and $g'(x) = g'(x-1)f'(g(x-1))$, for every $0 < x < 1$, hence

$$f'_+(0)\lim_{x\to 0^-} g'(x) = \lim_{x\to 0^+} g'(x) = \lim_{x\to 0^+} g'(x-1)f'(g(x-1)) = f'_+(0)\lim_{t\to -1^+} g'(t).$$

So $\lim_{x\to 0^-} g'(x) = \lim_{x\to -1^+} g'(x)$ thus g' is constant on $(-1,0)$, because g' is monotonic on $(-1,0)$. Now considering $\lim_{x\to 0^-} g(x) = 1$, $\lim_{x\to -1^+} g(x) = 0$ we conclude that $g(x) = x+1$ on $[-1,0]$ and so the Lemma 2.1 completes this proof.

Now considering the above theorem and (2.5) we have the following corollary:

***Corollary* 2.1.** *Suppose f is a function defined on $f^n([0,1))$ for every integer $n \geq 0$ and differentiable on $[0,1)$ and $f'_+(0) = f(0) = 1$.*

Then $\beta = \mathrm{upt}_f$ is the only solution of the ultraexponential f-type equation on $[-1,+\infty)$ such that $\beta(-1) = 0$, β is increasing and differentiable on $[-1,0)$ and β' exists at zero and is monotonic on $(-1,0)$.

3. Infralogarithm *f*-type functions; a unique solution for the Abel functional equation. An interesting property of the ultraexponential f-type functions is that its inverse function (if exists) satisfies the Abel's equation

$$\alpha(f(x)) = \alpha(x) + 1.$$

Let f be a function defined on $f^n([0,1))$ for every integer $n \geq 0$ and $f(0) = 1$. Then f is defined on $f^n([0,1])$ for every integer $n \geq 0$ and

$$f^{n+1}(0) = f^n(1) = \mathrm{upt}_f(n), \quad n = 0,1,2,\ldots.$$

Now if f is continuous and increasing on $f^n([0,1])$ $(\forall n \geq 0)$, then the sequence $f^n(1)$ is increasing and $\lim_{n\to\infty} f^n(1) = f^{\infty}(1) = \delta$, where $1 < \delta \leq \infty$, and

$$\bigcup_{n=0}^{\infty} f^n([0,1]) = [0,\delta) \subseteq D_f.$$

Also $f\colon [0,\delta) \to [1,\delta)$ is increasing, continuous and invertible. Therefore $f^{-1}\colon [1,\delta) \to [0,\delta)$ is continuous, increasing and invertible too. In this case the function $\mathrm{upt}_f\colon [-1,+\infty) \to [0,\delta)$ is increasing, continuous and bijection too. So $\mathrm{upt}_f^{-1}\colon [0,\delta) \to [-1,+\infty)$ is continuous and increasing and we have

$$\mathrm{upt}_f^{-1}(x) = \begin{cases} \vdots & \\ x-1, & 0 \leq x \leq 1, \\ f^{-1}(x), & 1 \leq x \leq f(1), \\ 1 + f^{-2}(x), & f(1) \leq x \leq f^2(1), \\ 2 + f^{-3}(x), & f^2(1) \leq x \leq f^3(1), \\ \vdots & \end{cases}$$

Of course the domain of upt_f^{-1} may be larger than $[0,\delta)$ if the domain of f^{-1} is larger than $[1,\delta)$ (e.g. see Iog_a in [2]).

Definition 3.1. *Let f be a continuous and increasing real function on $f^n([0,1])$, for every integer $n \geq 0$, and $f(0) = 1$. Then we call* upt_f *"ultraexponential f-type function" and denote by* uxp_f *. Also, denote* uxp_f^{-1} *by* Iog_f *and call it "infralogarithm f-type function".*

Note that if $a > 1$ and $f = \exp_a$, then $\mathrm{uxp}_f = \mathrm{uxp}_a$ and $\mathrm{Iog}_f = \mathrm{uxp}_f^{-1} = \mathrm{uxp}_a^{-1} = = \mathrm{Iog}_a$.

Theorem 3.1. *The functions* Iog_f *and* $[\mathrm{Iog}_f]$ *satisfy the Abel's equation*

$$\alpha(f(x)) = \alpha(x) + 1, \quad 0 \leq x < \delta \tag{3.1}$$

(*where* $\delta = f^\infty(1)$ *and* $[\mathrm{Iog}_a] = [\,]o\,\mathrm{Iog}_a$). *Also, there exists an integer valued function μ such that* $\mathrm{Iog}_f = \mu + f^{-\mu-1}$ *(on* $[0,\delta)$*) and* $[\mathrm{Iog}_f] = \mu$, $(\mathrm{Iog}_f) = f^{-\mu-1}$.

Proof. Put $g = \mathrm{uxp}_f$ and $\mu = [g^{-1}]$. If $0 \leq x < \delta$, then $x = g(y)$ for some $y \geq -1$. So

$$\mu(f(x)) = [g^{-1}(f(x))] = [g^{-1}f(g(y))] = [g^{-1}(g(y+1))] = [y] + 1 = \mu(x) + 1.$$

Therefore $\mu = [\mathrm{Iog}_f]$ satisfies (3.1). Now putting $h = \mu + f^{-\mu-1}$, we have

$$hg(x) = \mu(g(x)) + f^{-\mu(g(x))-1}(g(x)) = [x] + f^{-[x]-1}(f^{[x]+1}((x))_1) = [x] + (x) = x,$$

for every $x \geq -1$. Now if $0 \leq x < \delta$, then $\mu(x) \leq g^{-1}(x) < \mu(x) + 1$ thus

$$g(\mu(x)) \leq x < g(\mu(x) + 1) \Rightarrow f^{\mu(x)+1}(0) \leq x < f^{\mu(x)+2}(0) \Rightarrow 0 \leq f^{-\mu(x)-1} < 1.$$

Hence $(h) = f^{-\mu-1}$, $[h] = \mu$ and if $0 \leq x < \delta$, then

$$gh(x) = f^{[h(x)]+1}((h(x))_1) = f^{\mu(x)+1}(f^{-\mu(x)-1}(x)) = x.$$

Now we have

$$\mathrm{Iog}_f(f(x)) = h(f(x)) = \mu(f(x)) + f^{-\mu(f(x))-1}(f(x)) = 1 + \mathrm{Iog}_f(x).$$

Theorem 3.1 is proved.

Note. The above theorem shows that we can consider the Abel's equation as an infralogarithm functional equation when $f(0) = 1$, f is continuous and increasing on $f^n([0,1])$ $(\forall n \geq 0)$. In this case we introduce general solution of the equation and proof a uniqueness theorem about it at the end of this section.

Now we proof a theorem that states an interesting relation between the ultraexponential f-type and Abel's equation.

Theorem 3.2. (i) *If α and β satisfy the Abel's equation and ultra f-type functional equation respectively, then $(\beta\alpha)f = f(\beta\alpha)$ and there exists 1-periodic function Φ such that $\alpha\beta(x) = x + \Phi(x)$* (*of course it holds for all x such that the compositions are possible*).

(ii) *If μ, h are two solutions of the* (3.1) (*Abel's equation on* $[0, f^\infty(1))$) *such that* $(h) = f^{-\mu}$, *then general solution of the equation is $\alpha = \mu + \varphi h$, where φ is every* 1*-periodic function.*

Proof. (i) Let α, β satisfy the equations, then

$$(\beta\alpha)f(x) = \beta(\alpha f(x)) = \beta(\alpha(x) + 1) = f(\beta(\alpha(x)) = f(\beta\alpha(x)).$$

Also, we have

$$\alpha\beta(x) = \alpha(f(\beta(x-1)) = \alpha(\beta(x-1)) + 1.$$

So $\alpha\beta$ satisfies the difference equation $\chi(x) = 1+\chi(x-1)$ and so there exists 1-periodic function Φ such that $\alpha\beta(x) = x + \Phi(x)$.

(ii) Let α has the mentioned form, then

$$\alpha(f(x)) = \mu(f(x)) + \varphi(h(f(x))) = 1 + \mu(x) + \varphi(h(x) + 1) = 1 + \alpha(x).$$

Conversely if α is an arbitrary solution of the equation, then $\alpha(f^n(x)) = \alpha(x) + n$, for every integer n such that f^n exists. Now considering the hypothesis we can put $n = -\mu(x_0)$, when x_0 is fix, and we have

$$\alpha(f^{-\mu(x_0)}(x_0)) = \alpha(x_0) - \mu(x_0) \Rightarrow \alpha((h(x_0)) = \alpha(x_0) - \mu(x_0).$$

So putting $\varphi = \alpha|_{[0,1)} o(\ )_1$ we have $\alpha(x) = \mu(x) + \varphi h(x)$ and φ is 1-periodic.

Theorem 3.2 is proved.

***Corollary* 3.1** [General solution of the Abel's equation]**.** *If f is continuous and increasing on $f^n([0,1])$, for every $n \geq 0$ and $f(0) = 1$, then the general solution of the* (3.1) *is*

$$\alpha = [\mathrm{Iog}_f] + \varphi\, \mathrm{Iog}_f, \tag{3.2}$$

where φ is every 1*-periodic function.*

Proof. Note that if h is a solution of the equation, then $h + c$ is so (c is constant). Now we get this result by Theorems 3.1, 3.2 (ii).

***Remark* 3.1.** Theorems 3.1, 3.2 and Corollary 3.1 state some important facts for the Abel's equation (for real functions) that one of them says the general solution can be gotten from the mentioned essential solutions (h and μ in Theorem 3.2 (ii)). Theorem 3.1 grantees that Iog_f and $[\mathrm{Iog}_f]$ (infralogarithm f-type function and its bracket) are the essential real solutions for the Abel's equation (under the conditions on f). On the other hand, replacing φ by α and putting $c = 1$ in the equation (1.48) of [3] we get (7.1) of [3] that is the Abel's equation $\alpha(f(x)) = \alpha(x) + 1$. Now comparing Theorem 1.9 of [3] about this equation and our results, we can see some similarities between the given form $\alpha = \varphi h + \mu$ of Theorem 3.2 (ii) and $\alpha = \varphi_0 a + d$ of [3] (the original form is $\alpha = \varphi_0[a(x)] + d(x)c$). Indeed, in these two similar forms both μ and d are integer valued functions and $\alpha = \varphi h+\mu$ gives general solution of the equation but $\alpha = \varphi_0 a+d$ gives a unique corresponded solution for the given function φ_0 with some conditions. Of course the conditions of the theorems are completely different (see [3] and [5]).

The above theorem states some interesting relations between the general solutions of the Abel's equation and the ultraexponential f-type equation specially when we consider the (3.1) ($0 \leq x < f^{\infty}(1)$). If $\phi = 0$ in (3.1), then it may $\alpha = \beta^{-1}$, e.g., $\alpha = \mathrm{Iog}_f$ and $\beta = \mathrm{uxp}_f$. Also, putting $\varphi = (\ )_1$ in the general solution (part (ii)) implies $\mu + f^{-\mu}$ is a solution of the equation.

Now we are ready to introduce a uniqueness conditions for the infralogarithm f-type function regarding to the (3.1) and similar to Theorem 2.1.

Theorem 3.3 [A uniqueness theorem for Abel's equation]**.** *Let f be continuous and increasing on $f^n([0,1])$, for every $n \geq 0$, and differentiable on $[0,1)$, $(1, f(1))$ and $f(0) = 1$.*

If h *is a solution of* (3.1) *such that* h' *is monotonic on* $(0,1)$ *and*

$$h(0) = -1, \quad \lim_{x\to 1^-} h'(x) = f'_+(0) \lim_{x\to 1^+} h'(x), \tag{3.3}$$

then $h = \mathrm{Iog}_f$ *(on* $[0, f^\infty(1))$*)*.

Proof. First note that Iog_f satisfies the conditions, clearly. Now if $0 < x < 1$, then $1 < f(x) < f(1)$ and $h'(x) = f'(x)h'(f(x))$. Since f is increasing and continuous on $[0,1)$ and $f(0) = 1$ we have

$$\lim_{x\to 0^+} h'(x) = \lim_{x\to 0^+} f'(x)h'(f(x)) = f'_+(0) \lim_{t\to 1^+} h'(t) = \lim_{x\to 1^-} h'(x).$$

Therefore $\lim_{x\to 0^+} h'(x) = \lim_{x\to 1^-} h'(x)$ so h' is constant on $(0,1)$, because it is monotonic on $(0,1)$. Now considering $-1 = h(0) = \lim_{x\to 0^+} h(x)$ and $0 = h(1) = \lim_{x\to 1^-} h(x)$ we conclude that $h(x) = x - 1$ on $[0,1]$.

Finally since $f\colon [0, f^\infty(1)) \to [1, f^\infty(1)) = \bigcup_{n=0}^\infty [f^n(1), f^{n+1}(1)]$ is continuous and increasing and $h(f^n(x)) = h(x) + n$ for $n \geq 0$, we conclude $h = \mathrm{Iog}_f$ on $[0, f^\infty(1))$.

Theorem 3.3 is proved.

Now considering the above theorem and (3.3), we have the following corollary:

***Corollary* 3.2.** *Let* f *be continuous and increasing on* $f^n([0,1])$, *for every* $n \geq 0$, *and differentiable on* $[0,1)$, $(1, f(1))$ *and* $f'_+(0) = f(0) = 1$. *Then* $\alpha = \mathrm{Iog}_f$ *is the only solution of the Abel's equation such that* $\alpha(0) = -1$ *and* α' *exists at* $x = 1$ *and monotonic on* $(0,1)$.

Moreover, in this case Iog_f *is differentiable on* $[0, f(1))$ *and if* f *is differentiable overall* $[0, \delta)$, *then* Iog_f *is so.*

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