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VALUE-SHARING AND UNIQUENESS OF ENTIRE FUNCTIONS*

ПОДІЛ ДАНИХ ТА ЄДИНІСТЬ ЦІЛИХ ФУНКЦІЙ

We study the uniqueness of entire functions sharing a nonzero value and obtain some results improving the results due to Fang, J. F. Chen, X. Y. Zhang and W. C. Lin et al.

Вивчається єдиність цілих функцій, що поділяють ненульове значення. Отримано деякі результати, що поліпшують результати Фанга, Дж. Ф. Чена, С. Й. Жанга, В. Ц. Ліна та інших.

1. Introduction and main results. In this paper, a meromorphic function will mean meromorphic in the whole complex plane. We will use the standard notations of Nevanlinna's value distribution theory such as $T(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, $m(r,f)$ and so on, as explained in Hayman [1], Yang [2] and Yi and Yang [3]. We denote by $S(r,f)$ any function satisfying $S(r,f)=o(T(r,f))$, as $r\to\infty$ possibly outside a set r of finite linear measure.

Let a be a finite complex number, and k be a positive integer. We denote by $E_{k)}(a,f)$ the set of zeros of $f-a$ with multiplicities at most k, where each zero is counted according to its multiplicity. We denote by $\overline{E}_{k)}(a,f)$ the set of zeros of $f-a$ with multiplicities are not greater than k, where each zero is counted only once. In addition, we denote by $N_{(k}\left(r,\dfrac{1}{f-a}\right)$ $\left(\text{or } \overline{N}_{(k}\left(r,\dfrac{1}{f-a}\right)\right)$ the counting function with respect to the set $E_{k)}(a,f)$ $\left(\text{or } \overline{E}_{k)}(a,f)\right)$.

Set

$$N_k\left(r,\frac{1}{f-a}\right)=\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}_{(2}\left(r,\frac{1}{f-a}\right)+\ldots+\overline{N}_{(k}\left(r,\frac{1}{f-a}\right).$$

We define

$$\Theta(a,f)=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}\left(r,\dfrac{1}{f-a}\right)}{T(r,f)}$$

and

$$\delta_k(a,f)=1-\overline{\lim_{r\to\infty}}\frac{N_k\left(r,\dfrac{1}{f-a}\right)}{T(r,f)}.$$

Let f and g be two nonconstant meromorphic functions, a be a finite complex number. We say that f,g share the value a CM (counting multiplicities) if f,g have the same a-points with the same multiplicities and we say that f,g share the value a IM (ignoring multiplicities) if we do not consider the multiplicities. We denote by $\overline{N}_L\left(r,\dfrac{1}{f-a}\right)$ the counting function for a-points of both f and g about which f has larger multiplicity than g , with multiplicity not be counted. Similarly, we have the notation $\overline{N}_L\left(r,\dfrac{1}{g-a}\right)$. Next we denote by $N_0(r,1/F')$ the counting function of those zeros of F' that are not the zeros of $F(F-1)$.

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Definition 1.1. *Let k be a positive integer. Let f and g be two nonconstant meromorphic functions such that f and g share the value* 1 IM. *Let z_0 be a 1-point of f with multiplicity p and a 1-point of g with multiplicity q. We denote by $\overline{N}_{f>k}\left(r, \dfrac{1}{g-1}\right)$ the reduced counting function of those 1-points of f and g such that $p > q = k$. $\overline{N}_{g>k}\left(r, \dfrac{1}{f-1}\right)$ is defined analogously.*

Definition 1.2. *Let f and g satisfy $E_{p)}(1, f) = E_{p)}(1, g)$, where $p \geq 2$ is an integer. We denote by $\overline{N}_{(p+1}(r, 1; f|g \neq 1)$ the reduced counting function of those 1-points of f with multiplicities at least $p+1$, which are not the 1-points of g. Also $N_{(p+1}(r, 1; g|f \neq 1)$ is defined analogously.*

In [4], Fang got the following results.

Theorem A. *Let f and g be two nonconstant entire functions and n, k be two positive integers with $n > 2k+4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share* 1 CM, *then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ or $f = tg$ for a constant t such that $t^n = 1$.*

Theorem B. *Let f and g be two nonconstant entire functions and n, k be two positive integers with $n > 2k+8$. If $\big(f^n(f-1)\big)^{(k)}$ and $\big(g^n(g-1)\big)^{(k)}$ share* 1 CM, *then $f = g$.*

In 2008, J. F. Chen, X. Y. Zhang [5] improved the above result and obtained the following result.

Theorem C. *Let f and g be two nonconstant entire functions and n, k be two positive integers with $n > 5k+7$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share* 1 IM, *then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ or $f = tg$ for a constant t such that $t^n = 1$.*

Theorem D. *Let f and g be two nonconstant entire functions and n, k be two positive integers with $n > 5k+13$. If $\big(f^n(f-1)\big)^{(k)}$ and $\big(g^n(g-1)\big)^{(k)}$ share* 1 IM, *then $f = g$.*

In 2008, X. Y. Zhang, J. F. Chen and W. C. Lin [6] extended the above result by proving the following result.

Theorem E. *Let $f(z)$ and $g(z)$ be two entire functions, n, m and k be three positive integers with $n > 3m + 2k + 5$, and $P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0$ or $P(z) = C$, where $a_0 \neq 0, a_1, \ldots, a_{m-1}, a_m \neq 0, C \neq 0$ are complex constants. If $\big[f^n P(f)\big]^{(k)}$ and $\big[g^n P(g)\big]^{(k)}$ share* 1 CM, *then either $f(z) = c_1 e^{\lambda z}$, $g(z) = e^{-\lambda z}$, where λ_1, λ_2, λ are three constants satisfying $(-1)^k (\lambda_1 \lambda_2)^n (n\lambda)^{2k} C^2 = 1$, or $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f, g) = 0$, where $R(\omega_1, \omega_2) = \omega_1^n p(\omega_1) - \omega_2^n p(\omega_2)$.*

In this paper we always use $L(z)$ denoting a arbitrary polynomial of degree n, i.e.,

$$L(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0 = a_n (z - c_1)^{l_1} (z - c_2)^{l_2} \ldots (z - c_s)^{l_s}, \tag{1.1}$$

where a_i, $i = 0, 1, \ldots, n$, $a_n \neq 0$ and c_j, $j = 1, 2, \ldots, s$, are finite complex number constants, and $c_1, c_2, \ldots, c_s$ are all the distinct zeros of $L(z)$, $l_1, l_2, \ldots, l_s, s, n$ are all positive integers satisfying

$$l_1 + l_2 + \ldots + l_s = n \qquad \text{and let} \quad l = \max\{l_1, l_2, \ldots, l_s\}. \tag{1.2}$$

Corresponding to the above results, some authors considered the uniqueness problems of entire functions that have fixed points, see M. L. Fang and H. Qiu [7], W. C. Lin and H. X. Yi [8], J. Dou, X. G. Qi and L. Z. Yang [9]. In this paper, we consider the existence of solutions of $[L(f)]^{(k)} - P$ and the corresponding uniqueness theorems, and we obtain the following results which generalize the above theorems.

Theorem 1.1. *Let f be a transcendental entire function. If $n > k + s$, then $[L(f)]^{(k)} = P$ has infinitely many solutions, where $P \not\equiv 0$ is a polynomial.*

***Remark* 1.1.** It is easy to see that a polynomial $Q(z) - P(z)$ has exactly $\max\{m, n\}$ solutions (counting multiplicities), where deg$Q = m$, deg$P = n$, but a transcendental entire function may have no solution. For example, let $f(z) = e^{\alpha(z)} + P(z)$, then function $f(z) - P(z)$ has no any solution, where $\alpha(z)$ is an entire function.

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions and n, k, l and $p(\geq 2)$ be four positive integers satisfying $5l > 4n + 5k + 7$. If $\overline{E}_{p)}\Big(1, (L(f))^{(k)}\Big) = \overline{E}_{p)}\Big(1, (L(g))^{(k)}\Big)$, then either $f = b_1e^{bz} + c$, $g = b_2e^{-bz} + c$ or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where b_1, b_2, b are three constants satisfying $(-1)^k(b_1b_2)^n(nb)^{2k} = 1$ and $R(\omega_1, \omega_2) = L(\omega_1) - L(\omega_2)$.*

***Corollary* 1.1.** *Let $f(z)$ and $g(z)$ be two nonconstant entire functions and n, k and l be three positive integers satisfying $5l > 4n + 5k + 7$. If $\big[L(f)\big]^{(k)}$ and $\big[L(g)\big]^{(k)}$ share* 1 IM, *then either $f = b_1e^{bz} + c, g = b_2e^{-bz} + c$ or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where b_1, b_2, b are three constants satisfying $(-1)^k(b_1b_2)^n(nb)^{2k} = 1$ and $R(\omega_1, \omega_2) = L(\omega_1) - L(\omega_2)$.*

***Remark* 1.2.** When $l = n, l = n-1, c = 0$, from Corollary 1.1 we can easily get Theorems C, D.

***Corollary* 1.2.** *Let $f(z)$ and $g(z)$ be two nonconstant entire functions and n, k and l be three positive integers satisfying $2l > n + 2k + 4$. If $\big[L(f)\big]^{(k)}$ and $\big[L(g)\big]^{(k)}$ share* 1 CM, *then either $f = b_1e^{bz} + c$, $g = b_2e^{-bz} + c$ or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where b_1, b_2, b are three constants satisfying $(-1)^k(b_1b_2)^n(nb)^{2k} = 1$ and $R(\omega_1, \omega_2) = L(\omega_1) - L(\omega_2)$.*

***Remark* 1.3.** When $l = n$, $c = 0$, from Corollary 1.2 we can easily get Theorem A. When $l = n - 1$, $l = n - m$, $c = 0$, Corollary 1.2 promotes Theorems B, E.

***Remark* 1.4.** If $L(f) \equiv L(g)$, we obtain

$$a_nf^n + a_{n-1}f^{n-1} + \ldots + a_1f \equiv a_ng^n + a_{n-1}g^{n-1} + \ldots + a_1g.$$

Let $h = f/g$. If h is a constant, then substituting $f = gh$ into above equation we deduce

$$a_ng^n(h^n - 1) + a_{n-1}g^{n-1}(h^{n-1} - 1) + \ldots + a_1g(h - 1) \equiv 0,$$

which implies $h^d = 1$, $d = (n, \ldots, n-i, \ldots, 1), a_{n-i} \neq 0$ for some $i = 0, 1, \ldots, n-1$. Thus $f \equiv tg$ for a constant t such that $t^d = 1$. If h is not a constant, then we know by above equation that f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = L(\omega_1) - L(\omega_2)$.

***Remark* 1.5.** Moreover, let $L(z)$ is a generic polynomial of degree at least 5. Then from the equation $L(f) \equiv L(g)$, one can conclude that $f \equiv g$ and no other nonconstant meromorphic solutions f and g. In [15] Yang and Hua exhibits some classes of such polynomials. And some related definitions and results, we refer the reader to [16, 17].

2. Some lemmas.

Lemma 2.1 (see [1]). *Let $f(z)$ be a nonconstant meromorphic function and $a_1(z)$, $a_2(z)$ be two meromorphic functions such that $T(r, a_i) = S(r, f)$, $i = 1, 2$. Then*

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}\left(\frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - a_2}\right) + S(r, f).$$

Lemma 2.2 (see [1]). *Let f be a nonconstant meromorphic function, k be a positive integer, and c be a nonzero finite complex number. Then*

$$T(r,f) \le \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{(k)}-c}\right) - N\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f) \le$$

$$\le \overline{N}(r,f) + N_{k+1}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-c}\right) - N_0\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f),$$

where $N_0\left(r,\frac{1}{f^{(k+1)}}\right)$ denotes the counting function which only counts those points such that $f^{(k+1)}=0$ but $f(f^{(k)}-c)\neq 0$.

Lemma 2.3 (see [11]). *Let $a_n(\neq 0), a_{n-1}, \ldots, a_0$ be constants and f be a nonconstant meromorphic function, then*

$$T\big(r, a_n f^n + a_{n-1}f^{n-1} + \ldots + a_0\big) = nT(r,f).$$

Lemma 2.4 (see [12]). *Let $f(z)$ be a nonconstant meromorphic function, s, k be two positive integers. Then*

$$N_s\left(r,\frac{1}{f^{(k)}}\right) \le k\overline{N}(r,f) + N_{s+k}\left(r,\frac{1}{f}\right) + S(r,f).$$

Clearly, $\overline{N}\left(r,\frac{1}{f^{(k)}}\right) = N_1\left(r,\frac{1}{f^{(k)}}\right)$.

Lemma 2.5 (see [13]). *Let f be a nonconstant meromorphic function, $k(\ge 1)$ be a positive integer and let $\varphi(\not\equiv 0,\infty)$ be a small function of f. Then*

$$T(r,f) \le \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{(k)}-\varphi}\right) - N\left(r,\frac{1}{\big(f^{(k)}/\varphi\big)'}\right) + S(r,f). \tag{2.1}$$

Lemma 2.6 (see [14]). *Let $f(z)$ be a nonconstant meromorphic function and k be a positive integer. Suppose that $f^{(k)}\not\equiv 0$, then*

$$N\left(r,\frac{1}{f^{(k)}}\right) \le N\left(r,\frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f).$$

Lemma 2.7 (see [18]). *Let f, g share* (1, 0). *Then*

(i) $\overline{N}_{f>1}\left(r,\frac{1}{g-1}\right) \le \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}(r,f) - N_0\left(r,\frac{1}{f'}\right) + S(r,f)$,

(ii) $\overline{N}_{g>1}\left(r,\frac{1}{f-1}\right) \le \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}(r,g) - N_0\left(r,\frac{1}{g'}\right) + S(r,g)$.

Lemma 2.8. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, $k, p(\ge 2)$ be two positive integers. If $\overline{E}_{p)}(1,f^{(k)}) = \overline{E}_{p)}(1,g^{(k)})$ and*

$$\Delta = 2\delta_{k+1}(0,g) + \delta_{k+1}(0,f) + \delta_{k+2}(0,g) + \delta_{k+2}(0,f) > 4,$$

then $\frac{1}{f^{(k)}-1} = \frac{bg^{(k)}+a-b}{g^{(k)}-1}$, *where a, b are two constants.*

Proof. Let

$$\Phi(z) = \frac{f^{(k+2)}}{f^{(k+1)}} - 2\frac{f^{(k+1)}}{f^{(k)}-1} - \frac{g^{(k+2)}}{g^{(k+1)}} + 2\frac{g^{(k+1)}}{g^{(k)}-1}. \tag{2.2}$$

Clearly $m(r,\Phi) = S(r,f) + S(r,g)$. We consider the cases $\Phi(z) \not\equiv 0$ and $\Phi(z) \equiv 0$.

Let $\Phi(z) \not\equiv 0$, then if z_0 is a common simple 1-point of $f^{(k)}$ and $g^{(k)}$, substituting their Taylor series at z_0 into (2.2), we see that z_0 is a zero of $\Phi(z)$. Thus, we have

$$N_{11}\left(r, \frac{1}{f^{(k)}-1}\right) =$$

$$= N_{11}\left(r, \frac{1}{g^{(k)}-1}\right) \le \overline{N}\left(r, \frac{1}{\Phi}\right) \le T(r,\Phi) + O(1) \le N(r,\Phi) + S(r,f) + S(r,g). \tag{2.3}$$

Noting that $\overline{E}_{p)}(1, f^{(k)}) = \overline{E}_{p)}(1, g^{(k)})$, we deduce from (2.2) that

$$N(r,\Phi) \le \overline{N}_{(p+1}\left(r, 1; f^{(k)} | g^{(k)} \ne 1\right) + \overline{N}_{(p+1}\left(r, 1; g^{(k)} | f^{(k)} \ne 1\right) +$$

$$+ \overline{N}_{(2}\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}_{(2}\left(r, \frac{1}{g^{(k)}}\right) + N_0\left(r, \frac{1}{f^{(k+1)}}\right) + N_0\left(r, \frac{1}{g^{(k+1)}}\right) +$$

$$+ \overline{N}_L\left(r, \frac{1}{f^{(k)}-1}\right) + \overline{N}_L\left(r, \frac{1}{g^{(k)}-1}\right). \tag{2.4}$$

Here $N_0\left(r, \dfrac{1}{f^{(k+1)}}\right)$ has the same meaning as in Lemma 2.2. From Lemma 2.2, we obtain

$$T(r,g) \le N_{k+1}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g^{(k)}-1}\right) - N_0\left(r, \frac{1}{g^{(k+1)}}\right) + S(r,g). \tag{2.5}$$

Since

$$\overline{N}\left(r, \frac{1}{g^{(k)}-1}\right) = N_{11}\left(r, \frac{1}{g^{(k)}-1}\right) + \overline{N}_{(2}\left(r, \frac{1}{f^{(k)}-1}\right) +$$

$$+ \overline{N}_{g^{(k)}>1}\left(r, \frac{1}{f^{(k)}-1}\right) + \overline{N}_{(p+1}(r, 1; g^{(k)} | f^{(k)} \ne 1). \tag{2.6}$$

Thus we deduce from (2.3)–(2.6) that

$$T(r,g) \le \overline{N}_{(p+1}\left(r, 1; f^{(k)} \mid g^{(k)} \ne 1\right) + 2\overline{N}_{(p+1}\left(r, 1; g^{(k)} \mid f^{(k)} \ne 1\right) + N_{k+1}\left(r, \frac{1}{g}\right) +$$

$$+ \overline{N}_{(2}\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}_{(2}\left(r, \frac{1}{g^{(k)}}\right) + N_0\left(r, \frac{1}{f^{(k+1)}}\right) + \overline{N}_{(2}\left(r, \frac{1}{f^{(k)}-1}\right) +$$

$$+ \overline{N}_L\left(r, \frac{1}{f^{(k)}-1}\right) + \overline{N}_L\left(r, \frac{1}{g^{(k)}-1}\right) + \overline{N}_{g^{(k)}>1}\left(r, \frac{1}{f^{(k)}-1}\right) + S(r,f) + S(r,g). \tag{2.7}$$

From the definition of $N_0\left(r,\dfrac{1}{f^{(k+1)}}\right)$, we see that

$$N_0\left(r,\frac{1}{f^{(k+1)}}\right)+\overline{N}_{(2}\left(r,\frac{1}{f^{(k)}-1}\right)+N_{(2}\left(r,\frac{1}{f^{(k)}}\right)-\overline{N}_{(2}\left(r,\frac{1}{f^{(k)}}\right)\leq N\left(r,\frac{1}{f^{(k+1)}}\right).$$

The above inequality and Lemma 2.6 give

$$N_0\left(r,\frac{1}{f^{(k+1)}}\right)+\overline{N}_{(2}\left(r,\frac{1}{f^{(k)}-1}\right)\leq N\left(r,\frac{1}{f^{(k+1)}}\right)-N_{(2}\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}_{(2}\left(r,\frac{1}{f^{(k)}}\right)\leq$$

$$\leq N\left(r,\frac{1}{f^{(k)}}\right)-N_{(2}\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}_{(2}\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)\leq \overline{N}\left(r,\frac{1}{f^{(k)}}\right)+S(r,f). \tag{2.8}$$

Substituting (2.8) in (2.7), we get

$$T(r,g)\leq \overline{N}_{(p+1}\left(r,1;f^{(k)}\mid g^{(k)}\neq 1\right)+2\overline{N}_{(p+1}\left(r,1;g^{(k)}\mid f^{(k)}\neq 1\right)+N_{k+1}\left(r,\frac{1}{g}\right)+$$

$$+\overline{N}_{(2}\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}_{(2}\left(r,\frac{1}{g^{(k)}}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}_L\left(r,\frac{1}{f^{(k)}-1}\right)+$$

$$+\overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right)+\overline{N}_{g^{(k)}>1}\left(r,\frac{1}{f^{(k)}-1}\right)+S(r,f)+S(r,g). \tag{2.9}$$

Since $E_{p)}(1,f^{(k)})=E_{p)}(1,g^{(k)})$, we have

$$p\overline{N}_{(p+1}\left(r,1;g^{(k)}\mid f^{(k)}\neq 1\right)+\overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right)\leq N\left(r,\frac{1}{g^{(k)}-1}\right)-\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right).$$

From Lemma 2.6, we obtain

$$N\left(r,\frac{1}{g^{(k)}-1}\right)-\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)+N\left(r,\frac{1}{g^{(k)}}\right)-\overline{N}\left(r,\frac{1}{g^{(k)}}\right)+$$

$$+\overline{N}_0\left(\frac{1}{g^{(k+1)}}\right)\leq N\left(r,\frac{1}{g^{(k+1)}}\right)\leq N\left(r,\frac{1}{g^{(k)}}\right)+S(r,g).$$

This shows

$$N\left(r,\frac{1}{g^{(k)}-1}\right)-\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)\leq \overline{N}\left(r,\frac{1}{g^{(k)}}\right)-\overline{N}_0\left(\frac{1}{g^{(k+1)}}\right).$$

Therefore,

$$p\overline{N}_{(p+1}\left(r,1;g^{(k)}|f^{(k)}\neq 1\right)+\overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right)\leq \overline{N}\left(r,\frac{1}{g^{(k)}}\right)-\overline{N}_0\left(\frac{1}{g^{(k+1)}}\right).$$

From this, we have

$$2\overline{N}_{(p+1}\left(r,1;g^{(k)}|f^{(k)}\neq 1\right)+\overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right)\leq$$

$$\leq \overline{N}\left(r, \frac{1}{g^{(k)}}\right) \leq N_{k+1}\left(r, \frac{1}{g}\right) + S(r, g). \tag{2.10}$$

Because

$$\overline{N}_{(p+1}\left(r, 1; f^{(k)}|g^{(k)} \neq 1\right) + \overline{N}_L\left(r, \frac{1}{f^{(k)} - 1}\right) \leq \overline{N}_{(2}\left(r, \frac{1}{f^{(k)} - 1}\right) \leq$$

$$\leq \overline{N}\left(r, \frac{f^{(k)}}{f^{(k+1)}}\right) + S(r, f) \leq T\left(r, \frac{f^{(k+1)}}{f^{(k)}}\right) + S(r, f) \leq$$

$$\leq N\left(r, \frac{f^{(k+1)}}{f^{(k)}}\right) + S(r, f) \leq \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + S(r, f). \tag{2.11}$$

Combining (2.9) – (2.11), Lemmas 2.4 and 2.7, we get

$$T(r, g) \leq 2N_{k+1}\left(r, \frac{1}{g}\right) + N_{k+1}\left(r, \frac{1}{f}\right) + N_{k+2}\left(r, \frac{1}{g}\right) + N_{k+2}\left(r, \frac{1}{f}\right) + S(r, f) + S(r, g).$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r, f) \leq T(r, g)$ for $r \in I$. Hence

$$T(r, g) \leq \Big\{2\big[1 - \delta_{k+1}(0, g)\big] + \big[1 - \delta_{k+1}(0, f)\big] +$$

$$+\big[1 - \delta_{k+2}(0, g)\big] + \big[1 - \delta_{k+2}(0, f)\big] + \varepsilon\Big\}T(r, g) + S(r, g)$$

for $r \in I$ and $0 < \varepsilon < \Delta - 4$, that is $\{\Delta - 4 - \varepsilon\}T(r, g) \leq S(r, g)$, i.e., $\Delta - 4 \leq 0$ or $\Delta \leq 4$, which is a contradiction to our hypotheses $\Delta > 4$.

Hence, we get $\Phi(z) \equiv 0$. Then by (2.2), we have

$$\frac{f^{(k+2)}}{f^{(k+1)}} - \frac{2f^{(k+1)}}{f^{(k)} - 1} \equiv \frac{g^{(k+2)}}{g^{(k+1)}} - \frac{2g^{(k+1)}}{g^{(k)} - 1}.$$

By integrating two sides of the above equality, we obtain

$$\frac{1}{f^{(k)} - 1} = \frac{bg^{(k)} + a - b}{g^{(k)} - 1}, \tag{2.12}$$

where $a (\neq 0)$ and b are constants.

Lemma 2.8 is proved.

Lemma 2.9. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share a nonzero polynomial* 1 IM *and*

$$\Delta = \delta_{k+2}(0, g) + \delta_{k+2}(0, f) + \delta_{k+1}(0, f) + 2\delta_{k+1}(0, g) > 4,$$

then $\dfrac{1}{f^{(k)} - 1} = \dfrac{bg^{(k)} + a - b}{g^{(k)} - 1}$, *where a, b are two constants.*

Proof. Since $f^{(k)}$ and $g^{(k)}$ share the value 1 IM, we have $\overline{N}_{(p+1}(r, 1; g^{(k)} \mid f^{(k)} \neq 1) = 0$. Proceeding as in the proof of Lemma 2.8, we obtain conclusion of Lemma 2.9.

Lemma 2.10. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share a nonzero polynomial 1 CM and*

$$\Delta = \delta_{k+2}(0,g) + \delta_{k+2}(0,f) > 1,$$

then $\dfrac{1}{f^{(k)}-1} = \dfrac{bg^{(k)}+a-b}{g^{(k)}-1}$, *where a, b are two constants.*

Proof. Since $f^{(k)}$ and $g^{(k)}$ share the value 1 CM, we have $\overline{N}_L\left(r,\dfrac{1}{F-1}\right) = \overline{N}_L\left(r,\dfrac{1}{G-1}\right) = 0$ and $\overline{N}_{(p+1}\left(r,1;g^{(k)}|f^{(k)} \neq 1\right) = 0$. Proceeding as in the proof of Lemma 2.8, we obtain conclusion of Lemma 2.10.

Lemma 2.11 (see [19]). *Let $f(z)$ be a nonconstant entire function and $k(\geq 2)$ be a positive integer. If $ff^{(k)} \neq 0$, then $f = e^{az+b}$, where $a \neq 0, b$ are constants.*

3. Proof of theorems. ***3.1. Proof of Theorem* 1.1.** Because f is a transcendental entire, we get $T(r,P) = o(T(r,f))$. Suppose that $z_0 \notin \{z\colon P(z)=0\}$ is a zero of $L(f)$ with its multiplicity $l \geq k+2$, then z_0 is a zero of $(L(f)^{(k)}/P)'$ with its multiplicity $l-k-1 \geq 1$. From Lemmas 2.3 and 2.5, we have

$$nT(r,f) = T(r,L(f)) + S(r,f) \leq$$

$$\leq N\left(r,\frac{1}{L(f)}\right) + N\left(r,\frac{1}{L(f)^{(k)}-P}\right) - N\left(r,\frac{1}{(L(f)^{(k)}/P)'}\right) + S(r,f) \leq$$

$$\leq N_{k+1}\left(r,\frac{1}{L(f)}\right) + \overline{N}\left(r,\frac{1}{L(f)^{(k)}-P}\right) - N_0\left(r,\frac{1}{(L(f)^{(k)}/P)'}\right) + S(r,f) \leq$$

$$\leq N_{k+1}\left(r,\frac{1}{(f-c_1)^{l_1}}\right) + \ldots + N_{k+1}\left(r,\frac{1}{(f-c_s)^{l_s}}\right) + \overline{N}\left(r,\frac{1}{L(f)^{(k)}-P}\right) + S(r,f) \leq$$

$$\leq (k+s)T(r,f) + \overline{N}\left(r,\frac{1}{L(f)^{(k)}-P}\right) + S(r,f).$$

Thus, we get

$$(n-k-s)T(r,f) \leq \overline{N}\left(r,\frac{1}{L(f)^{(k)}-P}\right) + S(r,f).$$

Noting that $n > k+s$, we get $\big[L(f)]^{(k)}\big] = P$ has infinitely many solutions.

***3.2. Proof of Theorem* 1.2.** Let $L(z)$ and l be given by (1.1), (1.2), respectively. Without loss of generality, we suppose that $a_n = 1$, $l = l_1$, and $c = c_1$, we get

$$\Theta(0,L(f)) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r,\dfrac{1}{L(f)}\right)}{T\big(r,L(f)\big)} \geq 1 - \overline{\lim_{r\to\infty}} \frac{\sum_{j=1}^{s}\overline{N}\left(r,\dfrac{1}{f-c_j}\right)}{nT(r,f)} \geq 1 - \frac{s}{n} \geq \frac{l-1}{n}.$$

Similarly, we have $\Theta\big(0,L(g)\big) \geq \dfrac{l-1}{n}$.

Moreover, we have

$$\delta_{k+1}\big(0, L(f)\big) = 1 - \varlimsup_{r\to\infty} \frac{N_{k+1}\left(r, \dfrac{1}{L(f)}\right)}{T(r, L(f))} \geq$$

$$\geq 1 - \varlimsup_{r\to\infty} \frac{\sum_{j=2}^{s} N_{k+1}\left(r, \dfrac{1}{(f-c_j)^{l_j}}\right) + N_{k+1}\left(r, \dfrac{1}{(f-c)^l}\right)}{nT(r,f)} \geq$$

$$\geq 1 - \varlimsup_{r\to\infty} \frac{(s-1)T(r,f) + (k+1)T(r,f) + S(r,f)}{nT(r,f)} \geq$$

$$\geq 1 - \frac{s+k}{n} \geq \frac{l-k-1}{n}.$$

Similarly, we obtain

$$\delta_{k+1}\big(0, L(g)\big) \geq 1 - \frac{s+k}{n} \geq \frac{l-k-1}{n},$$

$$\delta_{k+2}\big(0, L(g)\big) \geq 1 - \frac{s+k+1}{n} \geq \frac{l-k-2}{n},$$

$$\delta_{k+2}\big(0, L(f)\big) \geq 1 - \frac{s+k+1}{n} \geq \frac{l-k-2}{n}.$$

Because $5l > 4n + 5k + 7$, we get

$$\Delta = 2\delta_{k+1}\big(0, L(g)\big) + \delta_{k+1}\big(0, L(f)\big) + \delta_{k+2}\big(0, L(g)\big) + \delta_{k+2}\big(0, L(f)\big) > 4.$$

By Lemma 2.8, we can have

$$\frac{1}{L(f)^{(k)} - 1} = \frac{bL(g)^{(k)} + a - b}{L(g)^{(k)} - 1}. \tag{3.1}$$

Next, we consider the following three cases:

Case 1. $b \neq 0$ and $a = b$. Then from (3.1) we obtain

$$\frac{1}{L(f)^{(k)} - 1} = \frac{bL(g)^{(k)}}{L(g)^{(k)} - 1}. \tag{3.2}$$

1.1. If $b = -1$, then it follows from (3.2) that $\big[L(f)\big]^{(k)}\big[L(g)\big]^{(k)} \equiv 1$.
That is

$$\big[(f-c)^l(f-c_2)^{l_2}\ldots(f-c_s)^{l_s}\big]^{(k)}\big[(g-c)^l(g-c_2)^{l_2}\ldots(g-c_s)^{l_s}\big]^{(k)} = 1. \tag{3.3}$$

1.1.1. When $s = 1$, we can rewrite (3.3)

$$\big[(f-c)^n\big]^{(k)}\big[(g-c)^n\big]^{(k)} = 1.$$

Since $5l > 4n + 5k + 7$, $l = n$, then $n > 5k + 7$. So $f - c \neq 0$, $g - c \neq 0$, according Lemma 2.11, we have

$$f = b_1 e^{bz} + c, \qquad g = b_2 e^{-bz} + c,$$

where b_1, b_2, b are three constants satisfying $(-1)^k (b_1 b_2)^n (nb)^{2k} = 1$.

1.1.2. When $s \geq 2$, we notice that $5l > 4n + 5k + 7$. Hence $l > 5k + 7$. Suppose that z_0 is a l-fold zero of $f - c$, we know that z_0 must be a $(l-k)$-fold zero of $\left[(f-c)^l (f-c_2)^{l_2} \ldots (f-c_s)^{l_s}\right]^{(k)}$. Noting that g is an entire function, it follows from (3.3), which is a contradiction. Hence $f - c \neq 0$, $g - c \neq 0$. So we get $f = e^{\alpha(z)} + c$, where $\alpha(z)$ is a nonconstant entire function. Thus we have

$$[f^i]^{(k)} = \left[(e^\alpha + c)^i\right]^{(k)} = p_i\left(\alpha', \alpha'', \ldots, \alpha^{(k)}\right) e^{i\alpha}, \quad i = 1, 2, \ldots, n, \tag{3.4}$$

where p_i, $i = 1, 2, \ldots, n$, is differential polynomials about α', $\alpha'', \ldots, \alpha^{(k)}$. Obviously, $p_i \not\equiv 0$, $T(r, p_i) = S(r, f)$, $i = 1, 2, \ldots, n$, we get from (3.3) and (3.4) that

$$N\left(r, \frac{1}{p_n e^{(n-1)\alpha} + \ldots + p_1}\right) = S(r, f).$$

According to Lemmas 2.1, 2.3 and $f = e^\alpha + c$, we get

$$(n-1)T(r, f - c) = T(r, p_n e^{(n-1)\alpha} + \ldots + p_1) + S(r, f) \leq$$

$$\leq \overline{N}\left(r, \frac{1}{p_n e^{(n-1)\alpha} + \ldots + p_1}\right) + \overline{N}\left(r, \frac{1}{p_n e^{(n-1)\alpha} + \ldots + p_2 e^\alpha}\right) + S(r, f) \leq$$

$$\leq \overline{N}\left(r, \frac{1}{p_n e^{(n-2)\alpha} + \ldots + p_2}\right) + S(r, f) \leq$$

$$\leq (n-2)T(r, f - c) + S(r, f),$$

which is a contradiction.

1.2. If $a = b \neq -1$, then (3.2) that can be written as

$$L(g)^{(k)} = \frac{-1}{b} \cdot \frac{1}{L(f)^{(k)} - (1+b)/b}. \tag{3.5}$$

From (3.5) we get

$$\overline{N}\left(r, \frac{1}{L(f)^{(k)} - (1+b)/b}\right) = \overline{N}(r, g) = S(r, f). \tag{3.6}$$

By (3.6) and Lemma 2.2, we have

$$nT(r, f) = T(r, L(f)) + O(1) \leq$$

$$\leq N_{k+1}\left(r, \frac{1}{L(f)}\right) + N\left(r, \frac{1}{L(f)^{(k)} - (1+b)/b}\right) + S(r, f) \leq$$

$$\leq N_{k+1}\left(r, \frac{1}{(f-c)^l}\right) + N_{k+1}\left(r, \frac{1}{(f-c_2)^{l_2} \ldots (f-c_s)^{l_s}}\right) + S(r, f) \leq$$

$$\leq (k+s)T(r, f) \leq (k + n - l + 1)T(r, f) + S(r, f),$$

which is a contradiction, because $5l > 4n + 5k + 7$.

Case 2. $b \neq 0$ and $a \neq b$. We discuss the following we subcases:

2.1. Suppose that $b = -1$, then $a \neq 0$ and (3.1) can be rewritten as

$$L(f)^{(k)} = \frac{a}{a + 1 - L(g)^{(k)}}. \tag{3.7}$$

From (3.7) we obtain

$$\overline{N}\left(r, \frac{1}{a + 1 - L(g)^{(k)}}\right) = \overline{N}(r, f) = S(r, g). \tag{3.8}$$

From (3.8), Lemmas 2.2 and 2.4, we get

$$nT(r, g) = T(r, L(g)) + O(1) \leq N_{k+1}\left(r, \frac{1}{L(g)}\right) + S(r, g).$$

Next, by using the argument as in Case 1.2, we have a contradiction.

2.2. Suppose that $b \neq -1$, then (3.1) be rewritten as

$$L(f)^{(k)} - \frac{b+1}{b} = \frac{-a}{b^2} \frac{1}{L(g)^{(k)} + (a - b)/b}. \tag{3.9}$$

From (3.9), we have

$$\overline{N}\left(r, \frac{1}{L(f)^{(k)} - (b+1)/b}\right) = \overline{N}(r, g). \tag{3.10}$$

From (3.10), Lemmas 2.2, 2.4 and in the same manner as in Case 1.2, we can get a contradiction.

Case 3. $b = 0$ and $a \neq 0$. Then (3.1) can be rewritten as

$$L(g)^{(k)} = aL(f)^{(k)} + (1 - a), \tag{3.11}$$

$$L(g) = aL(f) + (1 - a)p_1(z), \tag{3.12}$$

where p_1 is a polynomial with its deg$p_1 \leq k$. If $a \neq 1$, then $(1 - a)p_1 \not\equiv 0$. This together with (3.12) and Lemma 2.1, we get

$$nT(r, g) = T(r, L(g)) + O(1) \leq \overline{N}\left(r, \frac{1}{L(g)}\right) + \overline{N}\left(r, \frac{1}{L(f)}\right) + S(r, g) \leq$$

$$\leq \sum_{i=1}^{s} \overline{N}\left(r, \frac{1}{g - c_i}\right) + \sum_{j=1}^{s} \overline{N}\left(r, \frac{1}{f - c_j}\right) + S(r, g) \leq$$

$$\leq s\big[T(r, g) + T(r, f)\big] + S(r, g). \tag{3.13}$$

Because $n = l + l_2 + \ldots + l_s$, we get $n - l = l_2 + \ldots + l_s \geq s - 1$, i.e., $n - l \geq s - 1, n - s \geq l - 1$. From $5l > 4n + 5k + 7$, we have $l - 1 > 4(n - l) + 5k + 6 > 4(s - 1) + 5k + 6$, so $n - s \geq l - 1 >$ $> 4(s - 1) + 5k + 6$, i.e., $n - s > 4(s - 1) + 5k + 6$, thus, $s < \dfrac{n - 5k - 2}{5}$. Thus

$$nT(r, g) < \frac{n - 5k - 2}{5}\big[T(r, g) + T(r, f)\big] + S(r, g). \tag{3.14}$$

On the other hand, from (3.12) and Lemma 2.3, we see that $T(r, g) = T(r, f) + S(r, g)$. Substituting this into (3.14), we deduce that $\dfrac{3n + 10k + 4}{5}T(r, g) < S(r, g)$, which is a contradiction.

Thus $a = 1$, and so it follows from (3.12) that $L(f) = L(g)$.

Next we consider the case when f and g are polynomials. Suppose that $f - c$ and $g - c$ have u and v pairwise distinct zeros, respectively. Then $f - c$ and $g - c$ are of the forms

$$f - c = k_1(z - a_1)^{n_1}(z - a_2)^{n_2}\dots(z - a_u)^{n_u},$$

$$g - c = k_2(z - b_1)^{m_1}(z - b_2)^{m_2}\dots(z - b_v)^{m_v},$$

so that

$$[f - c]^l = k_1^l(z - a_1)^{ln_1}(z - a_2)^{ln_2}\dots(z - a_u)^{ln_u}, \tag{3.15}$$

$$[g - c]^l = k_2^l(z - b_1)^{lm_1}(z - b_2)^{lm_2}\dots(z - b_v)^{lm_v}, \tag{3.16}$$

where k_1 and k_2 are nonzero constants, $n_i l > 5k + 7$, $m_j l > 5k + 7$, n_i, m_j, $i = 1, 2, \dots u$, $j = 1, 2, \dots v$, are positive integers. Differentiating (3.12), we get

$$L(g)^{(k+1)} = aL(f)^{(k+1)}. \tag{3.17}$$

From (3.15), (3.16) and (3.17), we have

$$(z - a_1)^{ln_1-k-1}(z - a_2)^{ln_2-k-1}\dots(z - a_u)^{ln_u-k-1}\xi_1(z) =$$

$$= (z - b_1)^{lm_1-k-1}(z - b_2)^{lm_2-k-1}\dots(z - b_v)^{lm_v-k-1}\xi_2(z), \tag{3.18}$$

where $\xi_1(z)$ and $\xi_2(z)$ are polynomials $\deg\xi_1 = (n - l)\sum_{i=1}^{u} n_i + (u - 1)(k + 1)$ and $\deg\xi_2 = (n - l)\sum_{j=1}^{v} m_j + (v - 1)(k + 1)$. It follows that $5l > 4n + 5k + 7$, we have $2l - n > 3(n - l) + 5k + 7 > 5k + 7$.

Then

$$(2l - n)n_i > 5k + 7, \qquad (2l - n)m_j > 5k + 7, \qquad i = 1, 2, \dots, u, \quad j = 1, 2, \dots, v.$$

So that

$$\sum_{i=1}^{u}\big[n_i l - (k + 1)\big] - \sum_{i=1}^{u} n_i(n - l) = \sum_{i=1}^{u}\big[n_i(2l - n) - (k + 1)\big] > u(4k + 6) > (u - 1)(k + 1),$$

i.e.,

$$\sum_{i=1}^{u}\big[n_i l - (k + 1)\big] > (n - l)\sum_{i=1}^{u} n_i + (u - 1)(k + 1).$$

Similarly,

$$\sum_{j=1}^{v}\big[m_j l - (k + 1)\big] > (n - l)\sum_{j=1}^{v} m_j + (v - 1)(k + 1).$$

Thus from (3.18) we deduce that there exists z_0 such that $L(f(z_0)) = L(g(z_0)) = 0$, where z_0 has multiplicity greater than $5k + 7$. This together with (3.12), we deduce $p_1(z) \equiv 0$, which also prove the claim.

Therefore from (3.11) and (3.12) we get $a = 1$ and so $L(f) \equiv L(g)$. Hence, this completes the proof of Theorem1.2.

3.3 *Proof of Corollary* 1.1 (1.2). By using Lemma 2.9 (2.10) and the condition $5l > 4n + 5k + +7 (2l > n+2k+4)$, proceeding as in the proof of Theorem 1.2, we can similarly prove Corollary 1.1 (1.2). We omit the details here.

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