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ON THE DIRICHLET KERNELS WITH RESPECT TO SOME SPECIAL REPRESENTATIVE PRODUCT SYSTEMS*

ПРО ЯДРА ДІРІХЛЕ ВІДНОСНО ДЕЯКИХ СПЕЦІАЛЬНИХ РЕПРЕЗЕНТАТИВНИХ СИСТЕМ ДОБУТКІВ

The Fourier analysis uses the calculations with kernel functions from the beginning. The maximal values of the nth Dirichlet kernels divided by n for the Walsh – Paley, for the "classical" Vilenkin, and some other systems are 1. In the paper we deal with some more general systems and, from the results, we develop methods aimed at assigning the properties of specific systems. In these cases, the situation with $\dfrac{D_n}{n}$ can be different.

Аналіз Фур'є використовує розрахунки з ядерними функціями з самого початку. Максимальні значення n-х ядер Діріхле, поділених на n, для систем Уолша – Пейлі, „класичних" систем Віленкіна та деяких інших систем дорівнюють 1. Ми розглядаємо більш загальні системи і, використовуючи результати, що отримані, розробляємо методи, призначені для визначення властивостей конкретних систем. У цих випадках ситуація з відношенням $\dfrac{D_n}{n}$ може бути іншою.

1. Introduction. Let $m := (m_0, m_1, \dots)$ be a sequence of positive integers not less than 2. Denote $\mathbb{N}$ by the set of nonnegative integers and $\mathbb{P}$ by the set of positive ones. Denote by G_k a finite group (not necessarily Abelian) with order m_k, $k \in \mathbb{N}$. Define the measure on G_k as follows:

$$\mu_k(\{j\}) := \frac{1}{m_k}, \quad j \in G_k, \quad k \in \mathbb{N}.$$

Let G be the complete direct product of the sets G_k, with the product of the topologies and measures (denoted by μ). This product measure is a regular Borel one on G with $\mu(G) = 1$. If the sequence m is bounded, then G compact totally disconnected group is called a bounded group, otherwise it is an unbounded one. The elements of G can be represented by sequences $x := (x_0, x_1, \dots)$. It is easy to give a neighbourhood base of G :

$$I_0(x) := G,$$

$$I_n(x) := \{y \in G | y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}$$

for $x \in G$, $n \in \mathbb{P}$. Define the well-known generalized number system in the usual way. If $M_0 := 1$, $M_{k+1} := m_k M_k$, $k \in \mathbb{N}$, then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $0 \le n_j < m_j$, $j \in \mathbb{N}$, and only a finite number of n_j's differ from zero. Let $|n| := \max\{k \in \mathbb{N} : n_k \ne 0\}$ (that is, $M_{|n|} \le n < M_{|n|+1}$) if $n \in \mathbb{P}$, and $|0| := 0$. Let $n^{(k)} := \sum_{j=k}^{\infty} n_j M_j$.

Denote by Σ_k the dual object of the group G_k, $k \in \mathbb{N}$. So each $\sigma \in \Sigma_k$ is a set of continuous irreducible unitary representations of G_k which are equivalent to some fixed representation $U^{(\sigma)}$.

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Let d_σ be the dimension of its representation space and let $\{\zeta_1, \ldots, \zeta_{d_\sigma}\}$ be a fixed, but arbitrary orthonormal basis in the representation space. The functions

$$u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \zeta_i, \zeta_j \rangle, \quad i, j \in \{1, \ldots, d_\sigma\}, \ x \in G,$$

are called the coordinate functions of $U^{(\sigma)}$ and $\{\zeta_1, \ldots, \zeta_{d_\sigma}\}$ is the basis. Thus for each $\sigma \in \Sigma_k$ we get d_σ^2 coordinate functions, m_k number of functions for the whole dual object of G_k in all.

Let $\{\varphi_k^s \colon 0 \leq s < m_k\}$ be a system of all normalized coordinate functions of the group G_k. We do not decide the order of the system φ now, but we suppose that φ_k^0 is always the character 1. So for every $0 \leq s < m_k$ there exists a $\sigma \in \Sigma_k$, $i, j \in \{1, \ldots, d_\sigma\}$, such that

$$\varphi_k^s(x) = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x), \quad x \in G.$$

Let ψ be the product system of φ_k^s, namely

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k), \quad x \in G, \quad n \in \mathbb{N}.$$

We say that ψ is the representative product system of φ. The Weyl – Peter's theorem (see [3]) ensures that the system ψ is orthonormal and complete on $L^2(G)$.

Let $f \colon G \to \mathbb{C}$ an integrable function. Let us define the Fourier coefficients and partial sums by

$$\widehat{f}_k := \int_G f \overline{\psi}_k, \quad k \in \mathbb{N}, \quad S_n f := \sum_{k=0}^{n-1} \widehat{f}_k \psi_k, \quad n \in \mathbb{P}.$$

We define the Dirichlet kernels in this way

$$D_n(y, x) := \sum_{k=0}^{n-1} \psi_k(y) \overline{\psi}_k(x), \qquad n \in \mathbb{P}, \quad D_0 := 0.$$

We notice that in most of restricted systems (denoted by ϑ now) Dirichlet kernel functions depend only on one element of the domain. The "one way" connection between the two conceptions is $D_n(y, x) = D_n^\vartheta(y - x)$.

It is easy to see that

$$S_n f(x) = \int_G f(y) D_n(x, y) d\mu(y).$$

This shows the importance of the Dirichlet kernels in the study of the convergence of Fourier series. For more on representative product systems see, e.g., [2].

The representative product systems are generalizations of the known Walsh – Paley and Vilenkin systems. Indeed, we obtain the Vilenkin systems (which are generalizations of the Walsh – Paley system) if the sequence m is an arbitrary sequence of integers greater than 1 and $G_k = \mathcal{Z}_{m_k}$, the cyclic group of order m_k for all $k \in \mathbb{N}$. The characters of $\mathcal{Z}_{m_k}$ are the generalized Rademacher functions:

$$\varphi_k^s(x) = \exp(2\pi \imath s x / m_k), \qquad s \in \{0, \ldots, m_k - 1\}, \quad x \in G, \quad \imath^2 = -1.$$

Finally, let us define the maximal value sequence of Dirichlet kernels in the following way

$$\mathbb{D}_n := \sup_{x,y\in G} |D_n(y,x)|, \quad n \in \mathbb{N}.$$

In the cases of the original (commutative) Vilenkin systems (and Walsh–Paley system) $n = \mathbb{D}_n$ holds for every $n \in \mathbb{N}$, because of $n = D_n(0) \geq |D_n(x)|$ for all $x \in G$ and $n \in \mathbb{N}$ in those systems. In the case of general representative product systems the situation can be different.

2. Auxiliary results.

Lemma 1 [2]. *Let $x, y \in G,\ n \in \mathbb{N}$. Then*

$$D_{M_n}(y,x) = \begin{cases} M_n & \text{if} \quad y \in I_n(x), \\ 0 & \text{if} \quad y \notin I_n(x). \end{cases}$$

Lemma 2 [2]. *Let $x, y \in G,\ n \in \mathbb{N}$. Then*

$$D_n(y,x) = \sum_{i=0}^{\infty} D_{M_i}(y,x) \left(\sum_{j=0}^{n_i-1} \varphi_i^j(y_i) \bar{\varphi}_i^j(x_i) \right) \psi_{n^{(i+1)}}(y) \bar{\psi}_{n^{(i+1)}}(x).$$

The next result helps us in the counting of the Dirichlet kernels.

Lemma 3 [4]. *Let $n \in \mathbb{N}$. Then*

$$\mathbb{D}_n = \sup_{x\in G} D_n(x,x).$$

Lemma 4 [1]. *Let $n \in \mathbb{P}$. Then*

$$1 \leq \frac{\mathbb{D}_n}{n} \leq m_{|n|}.$$

The study of quotients $\dfrac{\mathbb{D}_n}{n}$ is important in order to estimate the Dirichlet kernels. As it was mentioned these quotients equal 1 in the commutative cases, but they can be unbounded if the dimensions of the representations appeared in the finite groups are also unbounded (for more see [4]). On the other hand the order of the systems plays an important role in the value of the quotient $\dfrac{\mathbb{D}_n}{n}$.

The statement of the following lemma is necessary to the proof of one of my new result.

Lemma 5 [1]. *Sequence $\mathbb{D}_n$ is monotonically increasing.*

3. Results on special systems. If the system is based on the product of same groups, and the generalized Rademacher functions are also the same on them, we get an interesting property of the Dirichlet kernel functions.

Lemma 6. *Let $x \in I_{|n|}(y) \backslash I_{|n|+1}(y)$, where $x, y \in G,\ n \in \mathbb{N}$ and let $\breve{z} := (z_1, z_2, \ldots)$ for any $z \in G$. If $m_k = p$ and $\varphi_k^s(x) = \varphi^s(x)$ for all $x \in G,\ k \in \mathbb{N},\ s \in \{0, \ldots, p-1\}$, where $2 \leq p \in \mathbb{N}$ is fixed, then*

$$D_{pn}(y,x) = pD_n(\breve{y}, \breve{x}).$$

Proof. From Lemmas 1 and 2 we obtain, that if $x \in I_{|n|}(y) \backslash I_{|n|+1}(y)$, where $x, y \in G$ and $n \in \mathbb{N}$, then

$$D_n(y,x) = \sum_{i=0}^{|n|} M_i \left(\sum_{j=0}^{n_i-1} \varphi_i^j(y_i) \bar{\varphi}_i^j(x_i) \right) \psi_{n^{(i+1)}}(y) \bar{\psi}_{n^{(i+1)}}(x)$$

holds.

In this way, because of $|pn| = |n| + 1$ and $(pn)_i = n_{i-1}$ we get

$$D_{pn}(y,x) = \sum_{i=0}^{|pn|} M_i \left(\sum_{j=0}^{(pn)_i - 1} \varphi^j(y_i)\bar{\varphi}^j(x_i) \right) \psi_{(pn)^{(i+1)}}(y)\bar{\psi}_{(pn)^{(i+1)}}(x) =$$

$$= \sum_{i=0}^{|n|+1} M_i \left(\sum_{j=0}^{n_{i-1}-1} \varphi^j(y_i)\bar{\varphi}^j(x_i) \right) \psi_{(pn)^{(i+1)}}(y)\bar{\psi}_{(pn)^{(i+1)}}(x) = (*).$$

If $|(pn)^{(i+1)}| = 0$ then $|n^{(i)}| = 0$. Otherwise $|(pn)^{(i+1)}| = |pn|$. Let $k^- := k - 1$. Using $(pn)^{(i+1)} = pn^{(i)}$ and $z_k = \breve{z}_{k^-}$ it is easy to see that

$$\psi_{(pn)^{(i+1)}}(z) = \prod_{k=0}^{|pn|} \varphi^{((pn)^{(i+1)})_k}(z_k) = \prod_{k=0}^{|n|+1} \varphi^{(pn^{(i)})_k}(z_k) =$$

$$= \prod_{k=1}^{|n|+1} \varphi^{(n^{(i)})_{k-1}}(z_k) = \prod_{k^-=0}^{|n|} \varphi^{(n^{(i)})_{k^-}}(\breve{z}_{k^-}) = \psi_{n^{(i)}}(\breve{z}).$$

So

$$(*) = \sum_{i^-=0}^{|n|} M_{i^-+1} \left(\sum_{j=0}^{n_{i^-}-1} \varphi^j(y_{i^-+1})\bar{\varphi}^{j^-+1}(x_{i^-+1}) \right) \psi_{n^{(i^-+1)}}(\breve{y})\bar{\psi}_{n^{(i^-+1)}}(\breve{x}) =$$

$$= p \sum_{i^-=0}^{|n|} M_{i^-} \left(\sum_{j=0}^{n_{i^-}-1} \varphi^j(\breve{y}_{i^-})\bar{\varphi}^j(\breve{x}_{i^-}) \right) \psi_{n^{(i^-+1)}}(\breve{y})\bar{\psi}_{n^{(i^-+1)}}(\breve{x}) = pD_n(\breve{y},\breve{x}).$$

Lemma 6 is proved.

***Corollary* 1.** *If $m_k = p$ and $\varphi_k^s(x) = \varphi^s(x)$ for all $x \in G,\ k \in \mathbb{N},\ s \in \{0,\ldots,p-1\}$, where $2 \le p \in \mathbb{N}$ is fixed, then*

$$D_{pn} = pD_n$$

holds.

Proof. Considering the definition of D_n, it is an obvious consequence of Lemmas 6 and 3.

This Corollary 1 can account for the fractal-like, self-similar structure of the graph of D_n in the case of any regular order of the system for $\mathcal{S}_3$ (see, e.g., Fig. 1 or [4]) and in the case of other systems, as for $\mathcal{Q}_2$ or for $\mathcal{U}_4$. For more see [4].

On the other hand, Corollary 1 could help us counting preciser estimates to expressions containing D_n. For example Lemma 4 gives us a rough upper estimate as $\dfrac{D_n}{n} \le p$ in this special case, but using Corollary 1 we can verify the next theorem, which is a good tool to get a better estimation.

Theorem 1. *If $m_k = p$ and $\varphi_k^s(x) = \varphi^s(x)$ for all $x \in G,\ k \in \mathbb{N},\ s \in \{0,\ldots,p-1\}$, where $2 \le p \in \mathbb{N}$ is fixed, then*

$$\frac{D_n}{n} < e^{\frac{1}{(p-1)p^{r-1}}} \max_{k \in \{p^{r-1}+1,\ldots,p^r\}} \frac{D_k}{k}$$

for all $r, n \in \mathbb{P}$.

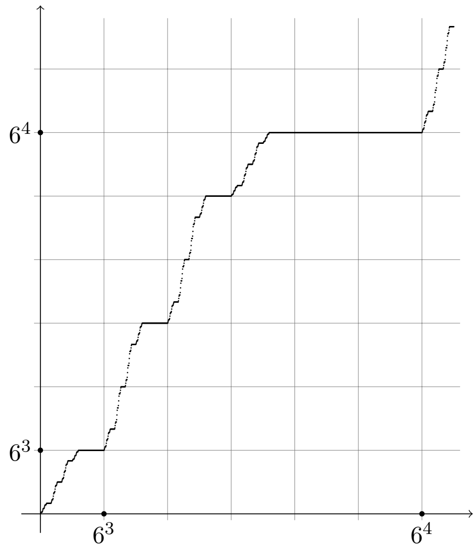

Fig. 1. D_n on the complete product of $\mathcal{S}_3$.

Proof. If $n < p$, then the proof is trivial. Otherwise let us define sequences $n[k]$ and $l[k]$ in the following way. Let $n[0] := n$ and $n[k] := p(n[k+1]-1) + l[k]$, where $l[k] \in \{0, \ldots, p-1\}$ and $n[k] \in \{p^{|n|-k}+1, \ldots, p^{|n|+1-k}\}$. Using Corollary 1 and Lemma 5 we have

$$\frac{D_{n[j]}}{n[j]} = \frac{D_{p(n[j+1]-1)+l[j])}}{p(n[j+1]-1)+l[j]} \le \frac{D_{pn[j+1]}}{p(n[j+1]-1)} = \frac{D_{n[j+1]}}{n[j+1]-1} =$$

$$= \frac{D_{n[j+1]}}{n[j+1]} \frac{n[j+1]}{n[j+1]-1} \le \frac{D_{n[j+1]}}{n[j+1]} \left(1 + \frac{1}{p^{|n|-j-1}}\right).$$

From this inequality we obtain

$$\frac{D_n}{n} \le \max_{k \in \{p^{r-1}+1, \ldots, p^r\}} \frac{D_k}{k} \prod_{j=0}^{|n|-r} \left(1 + \frac{1}{p^{|n|-j-1}}\right) < \max_{k \in \{p^{r-1}+1, \ldots, p^r\}} \frac{D_k}{k} \prod_{i=0}^{\infty} \left(1 + \frac{1}{p^{r-1+i}}\right),$$

and from the arithmetic-geometric mean inequality

$$\prod_{i=0}^{n} \left(1 + \frac{1}{p^{r-1+i}}\right) \le \left(\frac{n + \sum_{i=0}^{n} \frac{1}{p^{r-1+i}}}{n}\right)^n <$$

$$< \left(1 + \frac{\sum_{i=0}^{\infty} \frac{1}{p^{r-1+i}}}{n}\right)^n \xrightarrow{n \to \infty} e^{\frac{1}{(p-1)p^{r-1}}}.$$

Theorem 1 is proved.

Table 1. A possible system for $\mathcal{S}_3$

	e	(12)	(13)	(23)	(123)	(132)
φ^0	1	1	1	1	1	1
φ^1	$\sqrt{2}$	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
φ^2	$\sqrt{2}$	$-\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
φ^3	1	-1	-1	-1	1	1
φ^4	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$
φ^5	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$

Since $\lim_{r\to\infty} e^{\frac{1}{(p-1)p^{r-1}}} = 1$, Theorem 1 give us a relative error easily. In this way we can approximate $\sup_{n\in\mathbb{P}} \dfrac{D_n}{n}$ arbitrary.

Now let us investigate a concrete system. Namely, let us see the complete product of $\mathcal{S}_3$, which is the symmetric group on 3 elements. It means that $m_k = 6$ holds for all $k \in \mathbb{N}$. $\mathcal{S}_3$ has two characters and a 2-dimensional representation. The values of the system φ obtained from the 2-dimensional representation depend on the chosen basis. Table 1 contains the values of a possible system φ (for details see [4]). You can see a part of D_n sequence from this system in Fig. 1.

***Corollary* 2.** *Let the system be the described one in Table* 1 *for* $\mathcal{S}_3$. *In this case*

$$\frac{D_n}{n} < 2.04$$

for all $n \in \mathbb{P}$.

Proof. With some manual counting we verified that

$$\max_{k\in\{2,\ldots,6\}} \frac{D_k}{k} = \frac{D_3}{3} = \frac{5}{3}.$$

Using Lemma 6 with $p = 6$ and $r = 1$ we have

$$\frac{D_n}{n} < \frac{5}{3} e^{\frac{1}{5}} < 2.04.$$

In this situation this estimate is definitely better than the 6, what we got from Lemma 4. Enlarging value of r we can obtain even better upper estimates.

***Corollary* 3.** *Let the system be the described one in Table* 1 *for* $\mathcal{S}_3$. *In this case*

$$1.92303 < \sup_{n\in\mathbb{P}} \frac{D_n}{n} < 1.92309.$$

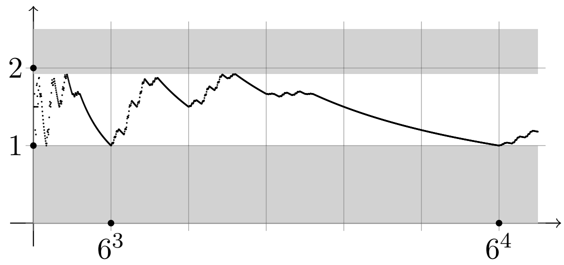

Fig. 2. $1 \leq \frac{D_n}{n} < 1.92309$ on the complete product of $\mathcal{S}_3$.

Proof. With similar method and using a computer program to the mechanical counting it is easy to show that the exact upper limit is between $1.92303 < \dfrac{38880}{20218}$ and $e^{\frac{1}{38880}}\dfrac{38880}{20218} < 1.92309$ (see Fig. 2).

Of course we can use this method to estimate for other systems, too.

In the end, maximizing $\dfrac{D_k}{k}$ (with the help of a computer program) on the set $\{6^{r-1}+1,\ldots,6^r\}$, where $r \in \{1,\ldots,6\}$ we get $\dfrac{5}{3}, \dfrac{30}{16}, \dfrac{180}{94}, \dfrac{1080}{562}, \dfrac{6480}{3370}, \dfrac{38880}{20218}$, respectively. It is easy to find a formula for this finite sequence, it is $\dfrac{5\cdot 6^{r-1}}{b_r}$, where $b_1 = 3$ and $b_r = 6b_{r-1} - 2$. If this idea also worked for the following members of the sequence, we would find the exact upper limit easily.

Conjecture. *Let the system be the described one in Table* 1 *for* $\mathcal{S}_3$. *In this case*

$$\sup_{n\in\mathbb{P}} \frac{D_n}{n} = \frac{25}{13}.$$

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