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SOLVABILITY OF BOUNDARY-VALUE PROBLEMS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS*

РОЗВ'ЯЗУВАНІСТЬ КРАЙОВИХ ЗАДАЧ ДЛЯ НЕЛІНІЙНИХ ДРОБОВИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ

We consider the existence of nontrivial solutions of boundary-value problem for the nonlinear fractional differential equation

$$\mathbf{D}^{\alpha}u(t)+\lambda[f(t,u(t))+q(t)]=0,\quad 0<t<1,$$
$$u(0)=0,\quad u(1)=\beta u(\eta),$$

where $\lambda>0$ is a parameter, $1<\alpha\leq 2$, $\eta\in(0,1)$, $\beta\in\mathbb{R}=(-\infty,+\infty)$, $\beta\eta^{\alpha-1}\neq 1$, $\mathbf{D}^{\alpha}$ is the Riemann – Liouville differential operator of order α, and $f\colon(0,1)\times\mathbb{R}\to\mathbb{R}$ is continuous, f may be singular at $t=0$ and/or $t=1$, $q(t)\colon[0,1]\to[0,+\infty)$ is continuous. We give some sufficient conditions for the existence of nontrivial solutions to the above boundary-value problems. Our approach is based on Leray – Schauder nonlinear alternative. Particularly, we do not use the nonnegative assumption and monotonicity of f which was essential for the technique used in almost all existed literature.

Розглянуто існування нетривіальних розв'язків крайової задачі для нелінійних дробових диференціальних рівнянь

$$\mathbf{D}^{\alpha}u(t)+\lambda[f(t,u(t))+q(t)]=0,\quad 0<t<1,$$
$$u(0)=0,\quad u(1)=\beta u(\eta),$$

де $\lambda>0$ — параметр, $1<\alpha\leq 2$, $\eta\in(0,1)$, $\beta\in\mathbb{R}=(-\infty,+\infty)$, $\beta\eta^{\alpha-1}\neq 1$, $\mathbf{D}^{\alpha}$ — диференціальний оператор Рімана – Ліувілля порядку α, функція $f\colon(0,1)\times\mathbb{R}\to\mathbb{R}$ неперервна, причому f може бути сингулярною при $t=0$ та (або) $t=1$, $q(t)\colon[0,1]\to[0,+\infty)$ неперервна. Наведено деякі достатні умови для існування нетривіальних розв'язків вказаних крайових задач. Застосований у дослідженнях підхід базується на нелінійній альтернативі Лереа – Шаудера. Зокрема, не використовується припущення про невід'ємність, а також монотонність функції f, що було істотним для методики, застосованої майже у всіх описаних у літературі дослідженнях.

1. Introduction. Fractional calculus has played a significant role in engineering, science, economy, and other fields. Many papers and books on fractional calculus, fractional differential equations have appeared recently, (see [1, 6 – 9]). It should be noted that most of papers and books on fractional calculus are devoted to the solvability of linear initial fractional differential equations in terms of special functions [5]. Recently, there are some papers deal with the existence and multiplicity of solutions (or positive solutions) of nonlinear initial fractional differential equations by the use of techniques of nonlinear analysis (fixed-point theorems, Leray – Shauder theory, etc.), see [7 – 10]. However, there are few papers consider the three-point problem for linear ordinary differential equations of fractional order, see [11, 12]. No contributions exist, as far as we know, concerning the existence and multiplicity of positive solutions of the following problem:

$$\mathbf{D}^{\alpha}u(t)+\lambda[f(t,u(t))+q(t)]=0,\quad 0<t<1,$$
$$u(0)=0,\quad u(1)=\beta u(\eta),\tag{1.1}$$

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where $\lambda > 0$ is a parameter, $1 < \alpha \leq 2$, $\eta \in (0,1)$, $\beta \in \mathbb{R} = (-\infty, +\infty)$ are real numbers, $\beta\eta \neq 1$, and $\mathbf{D}_{0+}^{\alpha}$ is the Riemann – Liouville differential operator of order α, and $f\colon (0,1) \times \mathbb{R} \to \mathbb{R}$ is continuous, f may be singular at $t = 0$ and/or $t = 1$, $q(t)\colon [0,1] \to [0, +\infty)$ is continuous. As far as we known, there has no paper which deal with the boundary-value problem for nonlinear fractional differential equation (1.1).

In [7], the authors consider the existence and multiplicity of positive solutions of nonlinear fractional differential equation boundary-value problem

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = u(1) = 0, \tag{1.2}$$

where $1 < \alpha \leq 2$ is a real number. D_{0+}^{α} is the standard Riemann – Liouville fractional derivative, and $f\colon [0,1] \times [0, +\infty) \to [0, +\infty)$ is continuous.

In [10], the authors consider the existence and multiplicity of positive solutions of nonlinear fractional differential equation boundary-value problem

$$D^{\alpha} u(t) + a(t) f(u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = u'(1) = 0, \tag{1.3}$$

where $1 < \alpha \leq 2$ is a real number. D^{α} is the Riemann – Liouville differential operator of order α, and $f\colon [0,1] \times [0, +\infty) \to [0, +\infty)$ is continuous, a is a positive and continuous function on $[0,1]$.

Motivated by the work mentioned above, in this paper, we establish serval sufficient conditions of the existence of nontrivial solutions for the above nonlinear fractional differential equations (1.1). Here, by a nontrivial solution of (1.1) we understand a function $u(t) \not\equiv 0$ which satisfies (1.1). Our results are new. Particularly, we do not use the nonnegative assumption and monotonicity which was essential for the technique used in almost all existed literature on f.

2. Preliminaries. For completeness, in this section, we will demonstrate and study the definitions and some fundamental facts of fractional order.

Definition 2.1 ([6], Definition 2.1)**.** *For a positive function $f(x)$ given in the interval $[0, \infty)$, the integral*

$$I^s f(x) = \frac{1}{\Gamma(s)} \int_0^x \frac{f(t)}{(x-t)^{1-s}} dt, \quad x > 0,$$

where $s > 0$, is called Riemann – Liouville fractional integral of order s.

Definition 2.2 [6, p. 36 – 37]**.** *For a positive function $f(x)$ given in the interval $[0, \infty)$, the expression*

$$D^s f(x) = \frac{1}{\Gamma(n-s)} \left(\frac{d}{dx}\right)^n \int_0^x \frac{f(t)}{(x-t)^{s-n+1}} dt,$$

where $n = [s] + 1, [s]$ denotes the integer part of number s, is called the Riemann – Liouville fractional derivative of order s.

Remark. If $f \in C[0,1]$, then $D^s I^s f(x) = f(x)$.

In order to rewrite (1.1), (1.2) as an integral equation, we need to know the action of the fractional integral operator I^s on $D^s f$ for a given function f. To this end, we first note that if $\lambda > -1$, then

$$D^s t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-s+1)} t^{\lambda-s},$$

$$D^s t^{s-k} = 0, \quad k = 1, 2, \ldots, n,$$

where $n = [s]$.

The following two lemmas, found in [7], are crucial in finding an integral representation of the boundary-value problem (1.1).

Lemma 2.1. *Let* $\alpha > 0$, $u \in C[0,1]$, *then the differential equation*

$$\mathbf{D}^\alpha u(t) = 0$$

has solutions $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$, $n = [\alpha] + 1$.

From the lemma above, we deduce the following statement.

Lemma 2.2. *Let* $\alpha > 0$, $u \in C[0,1]$, *then*

$$I^\alpha \mathbf{D}^\alpha u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$, $n = [\alpha] + 1$.

The following theorems will play major role in our next analysis.

Lemma 2.3 [3, 4]**.** *Let X be a real Banach space, Ω be a bounded open subset of X, $0 \in \Omega$, $T\colon \overline{\Omega} \to X$ is a completely continuous operator. Then, either there exists $x \in \partial\Omega$, $\mu > 1$ such that $T(x) = \mu x$, or there exists a fixed point $x^* \in \partial\overline{\Omega}$.*

3. Main results. In this section, we give our main results. First, we have the following lemma.

Lemma 3.1. *If* $1 < \alpha \leq 2$, $\beta\eta^{\alpha-1} \neq 1$, $u \in C[0,1]$. *Let* $h(t) \in C[0,1]$ *be a given function, then the boundary-value problem*

$$\mathbf{D}^\alpha u(t) + h(t) = 0, \quad 0 < t < 1,$$
$$u(0) = 0, \quad u(1) = \beta u(\eta), \tag{3.1}$$

has a unique solution

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + \frac{t^{\alpha-1}}{1-\beta\eta^{\alpha-1}} \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)\, ds -$$

$$- \frac{\beta t^{\alpha-1}}{1-\beta\eta^{\alpha-1}} \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} h(s)\, ds.$$

Proof. By the Lemma 2.2, we can reduce the equation of problem (3.1) to an equivalent integral equation

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}$$

for someconstants $c_1, c_2 \in \mathbb{R}$. As boundary conditions for problem (3.1), we have $c_2 = 0$ and

$$c_1 = \frac{1}{1-\beta\eta^{\alpha-1}} \frac{1}{\Gamma(\alpha)} \left(\int_0^1 (1-s)^{\alpha-1} h(s) ds - \beta \int_0^\eta (\eta - s)^{\alpha-1} h(s) ds \right).$$

Therefore, the unique solution of (3.1) is

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + \frac{t^{\alpha-1}}{1-\beta\eta^{\alpha-1}} \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)\, ds -$$

$$- \frac{\beta t^{\alpha-1}}{1-\beta\eta^{\alpha-1}} \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} h(s) ds$$

which completes the proof.

The lemma is proved.

Let $E = C[0,1]$ be endowed with the maximum norm $\|u\| = \max\limits_{0 \le t \le 1} |u(t)|$. Clearly, it follows that $(E, \|\cdot\|)$ is a Banach space.

Theorem 3.1. *Suppose that* $f(t,0) \not\equiv 0$, $t \in [0,1]$, $\beta\eta^{\alpha-1} \neq 1$, *and there exist nonnegative functions* $r \in C[0,1]$, $p \in C(0,1)$ (p *may be singular at* $t = 0$ *and/or* $t = 1$) *such that*

(H$_1$) $\displaystyle\int_0^1 (1-s)^{\alpha-1} p(s) ds < +\infty$;

(H$_2$) *the function* f *satisfies*

$$|f(t,u)| \le p(t)|u| + r(t), \quad \textit{a.e.} \quad (t,u) \in (0,1) \times \mathbb{R},$$

and there exists $t_0 \in [0,1]$ *such that* $p(t_0) \neq 0$.

Then there exists a constant $\lambda^* > 0$, *such that for any* $0 < \lambda \le \lambda^*$, *the boundary-value problem* (1.1) *has at least one nontrivial solution* $u^* \in C[0,1]$.

Proof. Let

$$A = \left(1 + \left|\frac{1}{1-\beta\eta^{\alpha-1}}\right|\right) \int_0^1 (1-s)^{\alpha-1} p(s) ds + \left|\frac{\beta}{1-\beta\eta^{\alpha-1}}\right| \int_0^\eta (\eta-s)^{\alpha-1} p(s) ds,$$

$$B = \left(1 + \left|\frac{1}{1-\beta\eta^{\alpha-1}}\right|\right) \int_0^1 (1-s)^{\alpha-1} k(s) ds + \left|\frac{\beta}{1-\beta\eta^{\alpha-1}}\right| \int_0^\eta (\eta-s)^{\alpha-1} k(s) ds,$$

where $k(s) = r(s) + q(s)$. By Lemma 3.1, problem (1.1) has a solution $u = u(t)$ if and only if u solves the operator equation

$$(Tu)(t) = -\frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big[f(s,u(s)) + q(s)\big]\, ds +$$

$$+ \frac{t^{\alpha-1}}{1-\beta\eta^{\alpha-1}} \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \big[f(s,u(s)) + q(s)\big]\, ds -$$

$$-\frac{\beta t^{\alpha-1}}{1-\beta\eta^{\alpha-1}}\frac{\lambda}{\Gamma(\alpha)}\int\limits_0^\eta(\eta-s)^{\alpha-1}\big[f(s,u(s))+q(s)\big]\,ds$$

in E. So we only need to seek a fixed point of T in E. In view of nonnegativeness and continuity of $(t-s)^{\alpha-1}$, $\dfrac{t^{\alpha-1}}{1-\beta\eta^{\alpha-1}}(1-s)^{\alpha-1}$, $\dfrac{\beta t^{\alpha-1}}{1-\beta\eta^{\alpha-1}}(\eta-s)^{\alpha-1}$ and continuity of $[f(t,u)+q(t)]$ and (H_1), by Ascoli – Arzela Theorem, it is well known that this operator $T\colon E\to E$ is a completely continuous operator.

Since $|f(t,0)|\leq r(t)$, a.e., $t\in[0,1]$, we know $\displaystyle\int_0^1[r(t)+q(t)]dt>0$. From $p(t_0)\neq 0$, we easily obtain $\displaystyle\int_0^1 p(s)ds>0$. Let

$$m=\frac{B}{A},\qquad \Omega=\big\{u\in C[0,1]\colon\|u\|<m\big\}$$

Suppose $u\in\partial\Omega$, $\mu>1$ such that $Tu=\mu u$. Then

$$\mu m=\mu\|u\|=\|Tu\|=\max_{0\leq t\leq 1}|(Tu)(t)|\leq$$

$$\leq\max_{0\leq t\leq 1}\frac{\lambda}{\Gamma(\alpha)}\int\limits_0^t(t-s)^{\alpha-1}|f(s,u(s))+q(s)|ds+$$

$$+\max_{0\leq t\leq 1}\frac{t^{\alpha-1}}{|1-\beta\eta^{\alpha-1}|}\frac{\lambda}{\Gamma(\alpha)}\int\limits_0^1(1-s)^{\alpha-1}|f(s,u(s))+q(s)|ds+$$

$$+\max_{0\leq t\leq 1}\frac{|\beta|t^{\alpha-1}}{|1-\beta\eta^{\alpha-1}|}\frac{\lambda}{\Gamma(\alpha)}\int\limits_0^\eta(\eta-s)^{\alpha-1}|f(s,u(s))+q(s)|ds\leq$$

$$\leq\frac{\lambda}{\Gamma(\alpha)}\int\limits_0^1(1-s)^{\alpha-1}(|f(s,u(s))|+q(s))ds+$$

$$+\frac{1}{|1-\beta\eta^{\alpha-1}|}\frac{\lambda}{\Gamma(\alpha)}\int\limits_0^1(1-s)^{\alpha-1}(|f(s,u(s))|+q(s))ds+$$

$$+\left|\frac{\beta}{1-\beta\eta^{\alpha-1}}\right|\frac{\lambda}{\Gamma(\alpha)}\int\limits_0^\eta(\eta-s)^{\alpha-1}(|f(s,u(s))|+q(s))ds\leq$$

$$\leq\left(1+\left|\frac{1}{1-\beta\eta^{\alpha-1}}\right|\right)\frac{\lambda}{\Gamma(\alpha)}\int\limits_0^1(1-s)^{\alpha-1}[p(s)|u(s)|+r(s)+q(s)]ds+$$

$$+\left|\frac{\beta}{1-\beta\eta^{\alpha-1}}\right|\frac{\lambda}{\Gamma(\alpha)}\int\limits_0^\eta(\eta-s)^{\alpha-1}[p(s)|u(s)|+r(s)+q(s)]ds\leq$$

$$\leq \frac{\lambda}{\Gamma(\alpha)}\Bigg[\left(1+\left|\frac{1}{1-\beta\eta^{\alpha-1}}\right|\right)\int\limits_0^1(1-s)^{\alpha-1}p(s)ds+$$

$$+\left|\frac{\beta}{1-\beta\eta^{\alpha-1}}\right|\int\limits_0^\eta(\eta-s)^{\alpha-1}p(s)ds\Bigg]\|u\|+$$

$$+\frac{\lambda}{\Gamma(\alpha)}\Bigg[\left(1+\left|\frac{1}{1-\beta\eta^{\alpha-1}}\right|\right)\int\limits_0^1(1-s)^{\alpha-1}[r(s)+q(s)]ds+$$

$$+\left|\frac{\beta}{1-\beta\eta^{\alpha-1}}\right|\int\limits_0^\eta(\eta-s)^{\alpha-1}[r(s)+q(s)]ds\Bigg].$$

Choose $\lambda^* = \dfrac{\Gamma(\alpha)}{2A}$. Then when $0 < \lambda \leq \lambda^*$, we have

$$\mu\|u\| \leq \frac{1}{2}\|u\| + \frac{B}{2A}.$$

Consequently,

$$\mu \leq \frac{1}{2} + \frac{B}{2mA} = 1.$$

This contradicts $\mu > 1$, by Lemma 2.3, T has a fixed point $u^* \in \overline{\Omega}$, since $f(t,0) \not\equiv 0$, then when $0 < \lambda \leq \lambda^*$, the boundary-value problem (1.1) has at least one nontrivial solution $u^* \in C[0,1]$.

Theprem 3.1 is proved.

Remark. Though the paper [13] devoted to a much more general case of multipoint problems for equations of arbitrary order $\alpha > 1$, our condition on f is obvious more general than [13]. For example, for Example 4.1, our results are not covered by Salem's results.

Theorem 3.2. *Suppose that* $f(t,0) \not\equiv 0$, $t \in [0,1]$, $\beta\eta^{\alpha-1} \neq 1$, *and there exist nonnegative functions* $p \in C(0,1)$ (p *may be singular at* $t = 0$ *and/or* $t = 1$) *such that*

(H$_1$) $\displaystyle\int_0^1 (1-s)^{\alpha-1}p(s)ds < +\infty$;

(H$_2$) *the function* f *satisfies*

$$\big|f(t,u_1) - f(t,u_2)\big| \leq p(t)|u_1 - u_2|, \quad a.e. \quad (t,u_i) \in (0,1)\times\mathbb{R}, \quad i = 1,2,$$

and there exists $t_0 \in [0,1]$ *such that* $p(t_0) \neq 0$.

Then there exists a constant $\lambda^* > 0$, *such that for any* $0 < \lambda \leq \lambda^*$, *the boundary-value problem* (1.1) *has an unique nontrivial solution* $u^* \in C[0,1]$.

Proof. In fact, if $u_2 = 0$, then we have $|f(t,u_1)| \leq p(t)|u_1| + |f(t,0)|$, a.e. $(t,u_1) \in [0,1]\times\mathbb{R}$. From Theorem 3.1, we know the boundary-value problem (1.1) has a nontrivial solution $u^* \in C[0,1]$.

But in this case, we prefer to concentrate on the uniqueness of nontrivial solutions for the boundary-value problem (1.1). Let T be given in Theorem 3.1, we shall show that T is a contraction. In fact,

$$\|Tu_1 - Tu_2\| = \max_{0\le t\le 1} \big|(Tu_1)(t) - (Tu_2)(t)\big| \le$$

$$\le \max_{0\le t\le 1} \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big|f(s,u_1(s)) - f(s,u_2(s))\big| ds +$$

$$+ \max_{0\le t\le 1} \frac{\lambda t}{|1-\beta\eta^{\alpha-1}|} \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \big|f(s,u_1(s)) - f(s,u_2(s))\big| ds +$$

$$+ \max_{0\le t\le 1} \frac{\lambda|\beta| t}{|1-\beta\eta^{\alpha-1}|} \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} \big|f(s,u_1(s)) - f(s,u_2(s))\big| ds \le$$

$$\le \max_{0\le t\le 1} \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)|u_1 - u_2| ds +$$

$$+ \max_{0\le t\le 1} \frac{\lambda t}{|1-\beta\eta^{\alpha-1}|} \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} p(s)|u_1 - u_2| ds +$$

$$+ \max_{0\le t\le 1} \frac{\lambda|\beta| t}{|1-\beta\eta^{\alpha-1}|} \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} p(s)|u_1 - u_2| ds \le$$

$$\le \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} p(s)|u_1 - u_2| ds +$$

$$+ \frac{\lambda}{|1-\beta\eta^{\alpha-1}|} \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} p(s)|u_1 - u_2| ds +$$

$$+ \frac{\lambda|\beta|}{|1-\beta\eta^{\alpha-1}|} \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} p(s)|u_1 - u_2| ds \le$$

$$\le \frac{\lambda}{\Gamma(\alpha)} \left[\int_0^1 (1-s)^{\alpha-1} p(s) ds + \frac{1}{|1-\beta\eta^{\alpha-1}|} \int_0^1 (1-s)^{\alpha-1} p(s) ds + \right.$$

$$\left. + \frac{|\beta|}{|1-\beta\eta^{\alpha-1}|} \int_0^\eta (\eta-s)^{\alpha-1} p(s) ds \right] \|u_1 - u_2\|.$$

If we choose $\lambda^* = \dfrac{\Gamma(\alpha)}{2A}$, where A as in the Theorem 3.1. Then when $0 < \lambda \le \lambda^*$, we have

$$\|Tu_1 - Tu_2\| \le \frac{1}{2} \|u_1 - u_2\|.$$

So T is indeed a contraction. Finally we use the Banach fixed point theorem to deduce the existence of an unique solution to the boundary-value problem (1.1).

***Corollary* 3.1.** *Suppose that* $f(t,0) \not\equiv 0$, *and*

$$0 \le M = \limsup_{|u|\to+\infty} \max_{0\le t\le 1} \frac{|f(t,u)|}{|u|} < +\infty,$$

$$\frac{M+1-\varepsilon}{\alpha\Gamma(\alpha)}\left[1+\frac{1}{|1-\beta\eta^{\alpha-1}|}+\frac{|\beta|\eta^{\alpha}}{|1-\beta\eta^{\alpha-1}|}\right] \le 1,$$

where $\varepsilon > 0$ *such that* $M+1-\varepsilon > 0$. *Then there exists a constant* $\lambda^* > 0$, *such that for any* $0 < \lambda \le \lambda^*$, *the boundary-value problem* (1.1) *has at least one nontrivial solution* $u^* \in C[0,1]$.

Proof. Let $\varepsilon > 0$ such that $M+1-\varepsilon > 0$. By (3.2), there exists $H > 0$ such that

$$|f(t,u)| \le (M+1-\varepsilon)|u|, \qquad |u| \ge H, \quad 0 \le t \le 1.$$

Let $N = \max_{t\in[0,1],|u|\le H} |f(t,u)|$. Then for any $(t,u) \in [0,1]\times\mathbb{R}$, we have

$$|f(t,u)| \le (M+1-\varepsilon)|u| + N.$$

From Theorem 3.2 we know the boundary-value problem (1.1) has at least one nontrivial solution $u^* \in C[0,1]$.

4. Examples.

***Example* 4.1.** Consider the following third-order three-point problem:

$$\mathbf{D}_{0+}^{3/2} y(t) = \lambda\left(y\frac{3t^2\sin t}{4\sqrt{1-t}} + t^3\right) + \lambda\cos t, \quad 0 < t < 1,$$
$$y(0) = 0, y(1) = 2\sqrt{2}y\left(\frac{1}{2}\right), \tag{4.1}$$

where $f(t,y) = y\dfrac{3t^2\sin t}{4\sqrt{1-t}} + t^3$, $q(t) = \cos t$. We choose $p(t) = \dfrac{1}{\sqrt{1-t}}$, $r(t) = t^3$, then

$$A = \left(1+\frac{\sqrt{2}}{\sqrt{2}}\right)\int_0^1 \sqrt{1-s}\frac{1}{\sqrt{1-s}}ds + \frac{4}{\sqrt{2}}\int_0^{1/2}\sqrt{1/2-s}\frac{1}{\sqrt{1-s}}ds =$$
$$= 2 + 2\sqrt{2}\left(\frac{1}{3}-\frac{\ln 3}{4}\right) = \frac{12+4\sqrt{2}-3\sqrt{2}\ln 3}{6},$$

and

$$\frac{\Gamma(3/2)}{2A} = \frac{3\sqrt{\pi}}{24+8\sqrt{2}-6\sqrt{2}\ln 3} \approx 0.204568.$$

Choose $\lambda^* = \dfrac{3\sqrt{\pi}}{24+8\sqrt{2}-6\sqrt{2}\ln 3} \approx 0.204568$, then by Theorem 3.1, (4.1) has a nontrivial solution $y^* \in C[0,1]$ for any $\lambda \in \left(0, \dfrac{3\sqrt{\pi}}{24+8\sqrt{2}-6\sqrt{2}\ln 3}\right] \approx$ $\approx (0, 0.204568]$.

***Example* 4.2.** Consider the following second-order boundary-value problem:

$$-\mathbf{D}_{0+}^{1.5}y(t) = \frac{1}{\sqrt{1-t}}(y - \cos y) + \lambda t e^{2t-1}, \qquad 0 < t < 1,$$
$$y(0) = y(1) = 0. \tag{4.2}$$

In this example $f(t, y(t)) = \frac{1}{\sqrt{1-t}}(y - \cos y)$, then

$$\big|f(t, y_1(t)) - f(t, y_2(t))\big| \le p(t)|y_1 - y_2|,$$

where $p(t) = \frac{1}{\sqrt{1-t}}$, by Computation, we get $\lambda^* = \frac{3\sqrt{\pi}}{24 + 8\sqrt{2} - 6\sqrt{2}\ln 3} \approx 0.204568$. Choose $\lambda^* = \frac{3\sqrt{\pi}}{24 + 8\sqrt{2} - 6\sqrt{2}\ln 3} \approx 0.204568$, then by Theorem 3.2, (4.2) has a nontrivial solution $y^* \in C[0,1]$ for any $\lambda \in \left(0, \frac{3\sqrt{\pi}}{24 + 8\sqrt{2} - 6\sqrt{2}\ln 3}\right] \approx (0, 0.204568]$.

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