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ON FUNDAMENTAL GROUP OF RIEMANNIAN MANIFOLDS WITH OMITTED FRACTAL SUBSETS

ПРО ФУНДАМЕНТАЛЬНУ ГРУПУ РІМАНОВИХ МНОГОВИДІВ З ПРОПУЩЕНИМИ ФРАКТАЛЬНИМИ ПІДМНОЖИНАМИ

We show that if K is a closed and bounded subset of a Riemannian manifold M of dimension $m > 3$, and the fractal dimension of K is less than $m - 3$, then the fundamental groups of M and $M - K$ are isomorphic.

Показано, що якщо K — замкнена й обмежена підмножина ріманового многовиду M розмірності $m > 3$, а фрактальна розмірність K менша за $m - 3$, то фундаментальні групи M і $M - K$ є ізоморфними.

1. Introduction. If K is a subset of a connected topological space M, it is interesting (but usually hard) to study, relations between fundamental groups of M and $M - K$. When the difference of the fractal dimensions (box dimension or Hausdorff dimension) of K and M is big enough, we expect that the fundamental groups of M and $M - K$ be isomorphic. It is proved in [1] that if $M = R^m$ or $M = S^m$, $m \geq 2$ and F is a compact subset of M and the Hausdorff dimension of F is strictly less than $m - k - 1$, then $M - F$ is k-connected (i.e., its homotopy groups π_i vanish for $i \leq k$). Consequently if $\dim_H(F) < m - 2$ then $R^n - F$ and $S^n - F$ are simply connected. In this paper, we consider a more general case when M is a Riemannian manifold then we prove the following theorem.

Theorem 1.1. *Let M^m be a Riemannian manifold of dimension $m > 3$, and K be a bounded and closed subset of M such that $\overline{\dim}_B(K) < m - 3$. Then $\pi_1(M)$ is isomorphic to $\pi_1(M - K)$.*

Before giving the proof of the theorem, we mention some preliminaries. Let A be a subset of a metric space (M, d). We denote by $\dim A$ the topological dimension of A. Let ϵ be a positive number and put

$$B_\epsilon(A) = \big\{x \in M \colon d(x, a) < \epsilon \ \text{ for some } \ a \in A\big\}.$$

If A is bounded then the upper box dimension of A is defined by

$$\overline{\dim}_B A = \limsup_{\delta \to 0} \frac{\log(m_\delta A)}{-\log \delta},$$

where, $m_\delta A$ is the maximum number of disjoint balls of radius δ, with centers contained in A. The lower box dimension $\underline{\dim}_B(A)$ is defined in similar way. Another definition for dimension, which is widely used in fractal geometry is Hausdorff dimension (see [2]). We use the upper box dimension in our theorem. But a similar result is true for lower box dimension and also for Hausdorff dimension.

***Remark* 1.1.** (a) If A is a submanifold of a Riemannian manifold M, then

$$\overline{\dim}_B(A) = \dim(A).$$

(b) If (M, d) and (N, d') are metric spaces and $f \colon M \to N$ is a map such that for some positive number $c > 0$, $d'(f(x), f(y)) \leq c d(x, y)$ (f is Lipschitz), then

$$\overline{\dim}_B(f(A)) \le \overline{\dim}_B(A).$$

(c) If A_1 and A_2 are bounded subsets of M, then

$$\overline{\dim}_B(A_1 \times A_2) \le \overline{\dim}_B(A_1) + \overline{\dim}_B(A_2).$$

***Remark* 1.2.** In the followings, for each positive number r, we denote by $S^{n-1}(r)$ the sphere of radius r and center at the origin of R^n. Let D be a closed $(n-1)$-disc in R^n and let a be a point outside of D. The set $C = \{ta + (1-t)d\colon d \in D,\ 0 \le t \le 1\}$ is called a cone with vertex a, over D. The following map is called a radial projection

$$f\colon\ C \to D\colon\ f(ta + (1-t)d) = d.$$

If $x_1, x_2 \in C$ and $x_1 \to a$, $x_2 \to a$ then $|x_2 - x_1| \to 0$. Thus f is not Lipschitz (because $|f(x_1) - f(x_2)|$ is bounded). But, if W is an open neighborhood of a in R^n, the map $f\colon (C - W) \to D$ is a Lipschitz map.

2. Proof of Theorem 1.1.

***Step* 1.** *Let* $0 < r_2 < r_1$, $A(r_1, r_2) = \{x \in R^n\colon r_2 \le |x| \le r_1\}$, $n > 2$, *and let* K *be a closed subset of* $A(r_1, r_2)$, *such that* $\overline{\dim}_B(K) < n-1$. *Then there are points* $a_1 \in S^{n-1}(r_1)$ *and* $a_2 \in S^{n-1}(r_2)$ *such that the line segment* a_2a_1, *joining two points* a_1 *and* a_2, *does not intersect* K.

Proof. Since $\overline{\dim}_B(K) < n-1$, then $S^{n-1}(r_1) - K \ne \varnothing$. Let $a_1 \in S^{n-1}(r_1) - K$ and let o be the origin of R^n. Denote by oa_1 the line segment joining o to a_1. Put $b = oa_1 \cap S^{n-1}(r_2)$ and let c be the mid point of ob and consider the $(n-1)$-disc D, with the center at c and boundary on $S^{n-1}(r_2)$, which is perpendicular to ob at the point c. Since K is closed, there is an open neighborhood W of a_1, such that $K \cap W = \varnothing$. Let C be the cone over D with the vertex a_1, and consider the radial projection map $f : (C - W) \to D$. f is a Lipschitz map. Thus

$$\overline{\dim}_B(f(K \cap (C - W))) \le \overline{\dim}_B(K \cap (C - W)) < n - 1.$$

Thus, $f(K \cap (C - W))$ does not cover D. If $d \in (D - f((C - W) \cap K))$ then the line segment a_1d does not intersect K. If $a_2 = a_1d \cap S^{n-1}(r_2)$, then a_1a_2 is the desired line segment.

***Step* 2.** *If* $K \subset R^n$, $n > 2$, *and* $\overline{\dim}_B(K) < n-1$, *then there is a path* $\sigma : [0,1] \to R^n$ *such that* $\sigma(0) = o$ *and for each* $t \in (0,1]$, $\sigma(t) \notin K$.

Proof. Consider the spheres $S^{n-1}\left(\frac{1}{m}\right)$, $m \in N$. Since $\overline{\dim}_B(K) < n-1$, then for each $r > 0$, $S^{n-1}(r) - K \ne \varnothing$. Let $a_1 \in (S^{n-1}(1) - K)$. By Step 1, there is point $a_2 \in S^{n-1}\left(\frac{1}{2}\right)$, such that $a_1a_2 \cap K = \varnothing$. Let $\sigma_1\colon \left[\frac{1}{2}, 1\right] \to R^n$ be a path from a_2 to a_1 along the line segment a_2a_1. Now, by induction, we can find the points $a_m \in S^{n-1}\left(\frac{1}{m}\right)$, $m > 1$, and the paths $\sigma_{m-1}\colon \left[\frac{1}{m}, \frac{1}{m-1}\right] \to R^n$, along the line segments a_ma_{m-1}, such that $a_{m-1}a_m \cap K = \varnothing$. The following path is the desired path

$$\sigma\colon [0,1] \to R^n, \quad \sigma(0) = 0, \quad \text{and} \quad \sigma(t) = \sigma_m(t) \quad \text{if} \quad t \in \left[\frac{1}{m}, \frac{1}{m-1}\right], \quad m > 1.$$

Let $\alpha, \beta\colon I = [0,1] \to M$ be two continuous paths in M with the same end-points. We recall that a continuous map $F\colon [0,1] \times [0,1] \to M$ with the following properties, is called a homotopy equivalence between α and β

$$F(s,0)=\alpha(s), \qquad F(s,1)=\beta(s), \quad s\in I,$$

$$F(0,t)=\alpha(0)=\beta(0), \qquad F(1,t)=\alpha(1)=\beta(1), \quad t\in I.$$

***Step* 3.** *Let E be a closed and bounded subset of R^n, $n>3$, such that $\overline{\dim}_B(E)<$ $< n-3$. Let $\alpha,\beta\colon I\to(R^n-E)$ be two loops at the point $x_0\in(R^n-E)$ and $F\colon I\times I\to R^n$ be a differentiable homotopy equivalence between α and β (in R^n). If $\epsilon>0$ then there is a homotopy equivalence $G\colon I\times I\to(R^n-E)$ (homotopy equivalence in (R^n-E)) between α and β such that*

$$\max\left\{|F(s,t)-G(s,t)|\colon (s,t)\in I\times I\right\}<\epsilon.$$

Proof. Put $N=F(I\times I)$ and let

$$\phi\colon N\times R^n\to R^n, \quad \phi(x,y)=y-x.$$

Consider the following metric on $N\times R^n$:

$$d((x_1,y_1),(x_2,y_2))=|x_1-x_2|+|y_1-y_2|.$$

Put $K=\phi(N\times E)$. ϕ is a Lipschitz map, so

$$\overline{\dim}_B(K)=\overline{\dim}_B\phi(N\times E)\le\overline{\dim}_B(N\times E)\le$$

$$\le\overline{\dim}_B(N)+\overline{\dim}_B(E)<2+n-3=n-1.$$

By Step 2, there is a path $\sigma\colon[0,1]\to R^n$, such that $\sigma(0)=o$ and for each $t\in(0,1]$, $\sigma(t)\in(R^n-K)$. Let $\theta\colon I\times I\to[0,1]$ be a continuous function such that

$$\theta(s,t)=0 \quad \text{if} \quad \text{and only if} \quad (s,t) \quad \text{belongs to the boundary of} \quad I\times I.$$

Since σ is continuous, there is a $\delta>0$ such that

$$|\sigma(\delta\theta(s,t))|<\epsilon, \quad (s,t)\in I\times I.$$

Now, put

$$G\colon I\times I\to R^n, \qquad G(s,t)=F(s,t)+\sigma(\delta\theta(s,t)).$$

We have

$$G(s,0)=F(s,0)=\alpha(s), \qquad G(s,1)=F(s,1)=\beta(s), \quad s\in I,$$

in similar way

$$G(0,t)=G(1,t)=x_0, \quad t\in I.$$

Thus, G is a homotopy equivalence between α and β. Also we obtain

$$G(s,t)\notin E, \quad (s,t)\in I\times I.$$

Because, if $G(s,t)\in E$ then

$$(F(s,t),F(s,t)+\sigma(\delta\theta(s,t))\in N\times E\Rightarrow(F(s,t)+\sigma(\delta(\theta(s,t)))-F(s,t))\in K.$$

Therefore, $\sigma(\delta\theta(s,t))\in K$, which is contradiction. This means that $G\colon I\times I\to(R^n-$ $-E)$ is a homotopy equivalence between α and β in (R^n-E). Also we have

$$|G(s,t)-F(s,t)|=|\sigma(\delta\theta(s,t))|<\epsilon.$$

***Step* 4.** *Let U be an open subset of R^n, $n > 3$, $E \subset U$ and $\overline{\dim}_B(E) < n - 3$. Then $\pi_1(U)$ is isomorphic to $\pi_1(U - E)$.*

Proof. Let $x_0 \in (U - E)$ and for each loop $\alpha\colon I \to (U - E)$ at x_0, denote by $[\alpha]_1$ and $[\alpha]_2$ the elements of $\pi_1(U - E, x_0)$ and $\pi_1(U, x_0)$ generated by α. Put

$$\phi\colon \pi_1(U - E) \to \pi_1(U), \qquad \phi([\alpha]_1) = [\alpha]_2.$$

We show that ϕ is one to one and onto. Let $[\alpha]_1, [\beta]_1 \in \pi_1(U - E)$. If $[\alpha]_2 = [\beta]_2$ then there is a differentiable homotopy equivalence $F\colon I \times I \to U$ between α and β in U. By Step 3, for each $\epsilon > 0$, there is a homotopy equivalence $G : I \times I \to (R^n - E)$ between α and β such that

$$|G(s,t) - F(s,t)| < \epsilon, \quad (s,t) \in I \times I.$$

Since for each (s,t), $F(s,t) \in U$, we can choose ϵ sufficiently small, such that $G(s,t) \in$ $\in U$ (i.e., $G(s,t) \in U - E$). Thus G will be a homotopy equivalence between α and β in $U - E$. Then $[\alpha]_1 = [\beta]_1$ and consequently ϕ is one to one.

Now, we show that ϕ is onto. let $[\gamma] \in \pi_1(U, x_0)$ and suppose that γ is a differentiable representative of $[\gamma]$ and let $L = \{\gamma(t)\colon t \in [0,1]\}$. Consider the following metric on $L \times R^n$:

$$d\big((x_1,y_1),(x_2,y_2)\big) = |x_1 - x_2| + |y_1 - y_2|.$$

Put $\phi\colon L \times R^n \to R^n$, $\phi(x,y) = y - x$ and let $K = \phi(L \times E)$. ϕ is Lipschitz, so

$$\overline{\dim}_B K \le \overline{\dim}_B(L \times E) \le \overline{\dim}_B L + \overline{\dim}_B E < 1 + n - 3 = n - 2.$$

Thus, as like as the proof of Step 2, we can find a path $\sigma\colon [0,1] \to R^n$ such that $\sigma(0) = o$ and

$$\sigma(t) \notin K, \quad t \in (0,1].$$

Let $\theta\colon [0,1] \to [0,1]$ be a continuous function such that

$$\theta(s) = 0 \quad \text{if} \qquad \text{and only if} \quad s \in \{0,1\}.$$

For each $\epsilon > 0$, there is a $\delta > 0$ such that

$$|\sigma(\delta\theta(s))| < \epsilon, \quad s \in [0,1].$$

Put

$$\alpha\colon [0,1] \to R^n, \qquad \alpha(s) = \gamma(s) + \sigma(\delta\theta(s))$$

and let

$$H(s,t) = \gamma(s) + \sigma(\delta t\theta(s)).$$

Sine for each $s \in [0,1]$, $\gamma(s) \in U$, we can choose the number ϵ, so small that

$$\alpha(s) \in U, \quad H(s,t) \in U.$$

Also we have $\alpha(s) \notin E$ (because, if $\alpha(s) \in E$ then $(\gamma(s), \alpha(s)) \in L \times E$, so $\alpha(s) -$ $- \gamma(s) \in K$, then $\sigma(\delta\theta(s)) \in K$, which is contradiction). Since $H\colon I \times I \to U$, is a homotopy equivalence between γ and α in U, we get that

$$\phi[\alpha]_1 = [\alpha]_2 = [\gamma].$$

Thus ϕ is onto.

***Step* 5.** *By Nash's embedding theorem, M^m can be embedded in R^n for sufficiently large n. Consider the normal vector bundle $M \to TM^{\perp}: p \to (T_pM)^{\perp}$ over the submanifold M of R^n (i.e., $TM^{\perp} = \{(p, v): p \in M, v \in T_pM^{\perp}\}$). There exists a neighborhood U_0 of the null section O_M in $(TM)^{\perp}$ such that the map* exp (*see* [3] *for definition of* exp) *is a diffeomorphism of U_0 on to an open subset $U \subset R^n$ (U is called a tubular neighborhood of M in R^n)*

$$\exp: U_0 \to U, \qquad \exp(p, v) = \exp_p(v).$$

The following map Ψ is a deformation retract of U_0 on to O_M:

$$\Psi: U_0 \times I \to U_0,$$

$$\Psi((p, v), t) = (p, (1 - t)v).$$

Thus, the following map is a deformation retract of U on to M (i.e., $\pi_1(M)$ is isomorphic to $\pi_1(U)$).

$$\Phi: U \times I \to U, \qquad \Phi(x, t) = \exp(\Psi(\exp^{-1}(x), t)).$$

Consider the map $\varsigma: U \to M$ defined by $\varsigma(x) = \Phi(x, 1)$ and put $\hat{K} = \varsigma^{-1}(K)$. It easy to show that

$$\dim_B(\hat{K}) \leq \dim_B(K) + (n - m) < (m - 3) + (n - m) < n - 3.$$

Now, we can use Step 4, to get that $\pi_1(U)$ is isomorphic to $\pi_1(U - \hat{K})$. Since M is a deformation retract of U, it is easy to show that $M - K$ is a deformation retract of $U - \hat{K}$. Thus $\pi_1(U - \hat{K})$ is isomorphic to $\pi_1(M - K)$. Therefore, $\pi_1(M - K)$ is isomorphic to $\pi_1(M)$.

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