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SEMIPERFECT IPRI-RINGS AND RIGHT BÉZOUT RINGS*
НАПІВДОСКОНАЛІ ІПРІ-КІЛЬЦЯ ТА ПРАВІ КІЛЬЦЯ БЕЗУ

We present a survey of some results on ipri-rings and right Bézout rings. All these rings are generalizations of principal ideal rings. From the general point of view, decomposition theorems for semiperfect ipri-rings and for right Bézout rings are proved.

Наведено огляд результатів з теорії іпрі-кілець та правих кілець Безу. Ці кільця є узагальненням кілець головних ідеалів. Із загальної точки зору доведено теореми розкладу для напівдосконалих іпрі-кілець та правих кілець Безу.

1. Introduction. Recall that a *principal right ideal ring* is a ring with identity $1 \neq 0$ in which every right ideal is principal (A. W. Goldie [8] called it a *pri-ring*). A *principal left ideal ring* (a pli-ring) can be defined analogously. Properties of pli-rings were considered in [15].

A *principal ideal ring* is a ring which is both a principal right and principal left ideal ring. A ring A with the Jacobson radical R is a *primary ring* if A/R is a simple Artinian ring.

One of the main examples of a principal ideal ring is the ring $\mathbb{Z}$ of all integers. It is well-known that every commutative principal ideal ring is a finite direct sum of rings which are either integral domains or are completely primary (see [33], Chapter 4). Analogous theorem was proved by K. Asano for the case of non-commutative Artinian rings. He proved in [1] that each such a ring is a finite direct sum of primary rings.

A. W. Goldie considered the structure of pri-rings. He proved in [8] the following main theorems.

Theorem A. *A pri-ring with no nilpotent ideals is a finite direct sum of prime pri-rings.*

Theorem B. *A prime pri-ring is a complete matrix ring K_n, where K is a right Noetherian integral domain.*

Theorem C. *A pri-ring, which is left Noetherian, is a finite direct sum of pri-rings, each being either a prime ring or a primary ring.*

J. C. Robson [25] considered a wider class of pri-rings which he called *ipri-rings* and *ipli-rings*. An ipri-ring (ipli-ring) is a ring in which every two-sided ideal is a principal right (left) ideal. A ring A with the nilpotent radical W is called W-simple if A/W is a simple ring. J.C. Robson extended several results concerning pri-rings which were proved by A.W. Goldie. In particular, he proved the following theorems.

Theorem 1 [25]. *A Noetherian ipri-ring is a finite direct sum of ideals each of which is a Noetherian ipri-ring and is either prime or W-simple. A Noetherian ipri-ring has a* (*right and left*) *quotient ring which is an Artinian pri-ring.*

Theorem 2 [25]. *If A is a Noetherian ipri- and ipli-ring then multiplication of ideals in A is commutative; A is a direct sum of rings each of which is prime or W-*

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simple, and in each of these rings every proper ideal is a unique product of maximal ideals.

Note that one-sided Artinian ipri- and ipli-rings were considered earlier by N. Jacobson in [14].

Theorem I ([14], Theorem 37)**.** *If A is a ring with an identity satisfying the descending chain condition on one-sided ideals, and every two-sided ideal of A is a principal right ideal and a principal left ideal, then A is a direct sum of two-sided ideals which are primary rings having these properties.*

In [9] (§ 12.2) an analogous theorem was proved for semiperfect rings.

Definition 1.1. *A ring $\mathcal{O}$ (not necessary commutative) is called a principal ideal domain if it has no zero divisors and all its right and left ideals are principal.*

Theorem 1.1 ([9], § 12.2)**.** *Let A be a semiperfect ring such that every two-sided ideal in A is both a right principal ideal and a left principal ideal. Then A is a principal ideal ring isomorphic to a direct product of a finite number of full matrix rings over Artinian uniserial rings and local principal ideal domains. Conversely, all such rings are semiperfect principal ideal rings.*

Another generalization of pri-rings are right Bézout rings.

Definition 1.2. *A ring is said to be right (resp. left) Bézout ring if its every finitely generated right (left) ideal is principal. A ring which is a right and left Bézout ring is called a Bézout ring. A Bézout domain is an integral domain in which every finitely generated ideal is principal.*

A pri-ring is obviously a right Bézout ring. In a certain sense, a right Bézout ring is a non-Noetherian analog of a pri-ring. On the other hand from the fact that any right ideal in a right Noetherian ring is finitely generated it immediately follows the following statement.

Proposition 1.1. *A right Noetherian ring is a right Bézout ring if and only if it is a pri-ring.*

The main examples of commutative Bézout domains which are not principal ideal domains (PID) and not Noetherian are:

1. The ring $\mathcal{O}(D)$ of all functions in single complex variable holomorphic in a domain D of the complex plane $\mathbb{C}$.
2. The ring of holomorphic functions given on the entire complex plane $\mathbb{C}$.
3. The ring of all algebraic integers.

First the properties of the ring $\mathcal{O}(D)$ of all functions in single complex variable holomorphic in a domain D of the complex plane $\mathbb{C}$ was considered by J. H. M. Wedderburn in 1915 [29]. In this paper he considered the problem of reducing the matrix whose coefficients are functions from the ring $\mathcal{O}(D)$ to an equivalent diagonal matrix. In particular, he proved the main lemma:

Lemma 1.1 (Wedderburn [29])**.** *Let $f, g \in \mathcal{O}(D)$ be two holomorphic functions which are holomorphic in a domain $D \subset \mathbb{C}$ and which are relatively prime, i.e., have no common zeros in D. Then there exist two functions $p, q \in \mathcal{O}(D)$ holomorphic in D such that*

$$pf + qg = 1.$$

Wedderburn's lemma together with Mittag-Leffler series and Weierstrass products makes it possible to prove that in the ring $\mathcal{O}(D)$ any finitely generated ideal is principal.

Below an example of ideals in $\mathcal{O}(D)$ that are not finitely generated is given. This example and the historical notes about ideal theory in rings of holomorphic functions can be found in the very interesting book on complex function theory [24].

Let S be an infinite locally finite set in a domain $D \subset \mathbb{C}$. The set

$$\mathcal{I} = \left\{ f \in \mathcal{O}(D) \colon \; f \text{ vanishes almost everywhere on } S \right\}$$

is an ideal in $\mathcal{O}(D)$ which is not finitely generated.

From this example the following statement follows.

Proposition 1.2 [24, p. 136]**.** *No ring $\mathcal{O}(D)$ is Noetherian; in particular, $\mathcal{O}(D)$ is never a principal ideal ring.*

It worth to say that even a great algebraist like J. H. M. Wedderburn in his paper [29] with famous Lemma 1.1 which is a basis for building the ideal theory of functions holomorphic in an arbitrary domain $D \subset \mathbb{C}$ wrote nothing about ideal theory. O. Helmer was the first who considered the ideal theory of holomorphic functions. In paper [12] in 1940 he extensively investigated divisibility properties of the ring $\mathcal{O}(K) = K\langle z \rangle$ of integral functions on K, i.e., functions in single variable z holomorphic on K, where K is an arbitrary subfield of $\mathbb{C}$. In this paper O. Helmer independently proved the Wedderburn lemma for $D = K$. His main result was that any finitely generated ideal in $\mathcal{O}(K)$ is principal. In his next paper [11] O.Helmer studied the algebraic structure of an abstract commutative ring A in which every finitely generated ideal is principal, and which further satisfies the following conditions: for any $a, c \in A$ with $a \neq 0$ one can write $a = rs$ with $(r, c) = 1$ and $(s_1, c) \neq 1$ for any non-unit divisor s_1 of s. O. Helmer called this ring an *adequate ring*. The ring of integral functions forms an excellent example of such a ring. In paper [12], Theorem 8, O. Helmer also gives the following example of an ideal S in $K\langle z \rangle$ which is not principal:

$$S = \left(\sin z, \sin \frac{1}{2} z, \sin \frac{1}{4} z, \ldots \right).$$

The ideal structure of the ring of entire functions, also called integral functions, which are complex-valued functions that is holomorphic over the whole complex plane $\mathbb{C}$, were studied also by M. Henrikson and L. Gilman (see [6, 7, 13]). The particular attention in these papers were paid to maximal and prime ideals. An abstract ring which all finitely generated ideals are principal they called an *F-ring*.

An example of a non-commutative Bézout ring is the (right) ring of skew formal series $K[[x, \sigma]]$, where K is a field, σ is a nontrivial automorphism of K, with multiplication defined by the rule $ax^i = x^i \sigma^i(a)$ for any $a \in K$ and $i \in N$. Indeed this ring is a noncommutative pri-ring.

The theory of principal ideal rings and their generalizations has a long and very interesting history (see [2 – 6, 11, 12, 15, 16, 18, 21, 22, 24 – 32] and many other papers).

The description of left Bézout semiperfect rings was obtained by R. B. Warfield, Jr. in [27].

Theorem 1.2 (Warfield [27])**.** *A semiperfect ring A is left Bézout if and only if A is a direct product of a finite number of full matrix rings over left uniserial rings.*

For generalizations of this theorem see [28], and see also [9] (Chapter 12) for further reading and references.

In this paper we consider semiperfect pri-rings, ipri-rings and right Bézout rings. The Annihilation lemma (see Lemma 2.1) and Corollary 2.2 from the Generator lemma (see Lemma 2.2) are used to prove from the general point of view the decomposition theorems: all these rings are direct products of a finite number of full matrix rings over uniserial rings. In particular, it is given another proof of the first part of Theorem 1.2.

All rings in this paper are assumed to be associative (but not necessary commutative) with $1 \neq 0$ and all modules are assumed to be unitary.

2. Semiperfect rings. Let I be a two-sided ideal of a ring A. One says that *idempotents can be lifted modulo an ideal* I of A if from the fact that $g^2 - g \in I$, where $g \in A$, it follows that there exists an idempotent $e^2 = e \in A$ such that $e - g \in I$.

Proposition 2.1 ([9], § 10.3)**.** *Idempotents can be lifted modulo any nil-ideal I of a ring A.*

***Corollary* 2.1** ([9], § 10.3)**.** *Idempotents can be lifted modulo the radical of an Artinian ring.*

Let $\operatorname{rad} A = R$ be the Jacobson radical of a ring A. Recall that a ring A is called *semilocal* if A/R is a right Artinian ring, or equivalently, if A/R is a semisimple ring. Any local ring is a semilocal and any right (or left) Artinian ring is semilocal.

Let M be an arbitrary A-module. Denote by $\operatorname{rad} M$ the intersection of all its maximal submodules. By convention, if M does not have maximal submodules, then $\operatorname{rad} M = M$. This submodule is called the radical of the module M.

Proposition 2.2 ([9], § 5.1)**.** *Let A be a ring with the Jacobson radical R. If P is a nonzero projective A-module, then* $\operatorname{rad} P = PR \neq P$.

If A is a semilocal ring then one has the analogous proposition for any A-module M.

Proposition 2.3 ([20], Proposition 24.4)**.** *If M is a nonzero right A-module over a semilocal ring A then* $\operatorname{rad} M = MR$.

Recall some definitions and main facts on semiperfect rings (see [9], Chapters 10, 11). Let R be the Jacobson radical of a ring A.

Definition 2.1. *A ring A is called semiperfect if A is semilocal and idempotents can be lifted modulo the Jacobson radical R of A.*

Right Artinian rings and local rings are examples of semiperfect rings.

Let A be a ring and $e^2 = e \in A$ be a nonzero idempotent of A. An idempotent $e \in A$ is called *local* if the ring eAe is local.

A submodule N of a module M is called *small* if the equality $N + X = M$ implies $X = M$ for any submodule X of the module M.

Definition 2.2. *A projective module P is called a projective cover of a module M and it is denoted by $P(M)$ if there is an epimorphism $\varphi\colon P \to M$ such that* $\operatorname{Ker}\varphi$ *is a small submodule in P.*

If a simple module U has a projective cover $P(U)$ then $P(U)$ has exactly one maximal submodule $\operatorname{rad} P(U)$ and $P(U)/\operatorname{rad} P(U) \cong U$.

If a module M has a projective cover $P(M)$, then the projective cover is unique up to isomorphism. The projective cover $P(M)$ of M, where $M = M_1 \oplus M_2$, is equal to $P(M_1) \oplus P(M_2)$.

The following four theorems are the main theorems in the theory of semiperfect rings (see [9], Chapter 10).

Theorem 2.1. *A ring A is semiperfect if and only if it can be decomposed into a direct sum of right ideals each of which has exactly one maximal submodule.*

Theorem 2.2 (B. J. Müller). *A ring A is semiperfect if and only if $1 \in A$ can be decomposed into a sum of a finite number of pairwise orthogonal local idempotents.*

Theorem 2.3 (H. Bass). *The following conditions are equivalent for a ring A*:

(a) *A is semiperfect*;

(b) *any finitely generated right A-module has a projective cover*;

(c) *any cyclic right A-module has a projective cover.*

Theorem 2.4. *Any indecomposable projective module over a semiperfect ring A is finitely generated, it is a projective cover of a simple A-module and has exactly one maximal submodule. There is a one-to-one correspondence between mutually nonisomorphic indecomposable projective A-modules $P_1, \ldots, P_s$ and mutually nonisomorphic simple A-modules which is given by the following correspondences: $P_i \mapsto P_i/P_iR = U_i$ and $U_i \mapsto P(U_i)$.*

Definition 2.3. *An indecomposable projective right module over a semiperfect ring A is called a principal right module. A principal left module is an indecomposable projective left A-module.*

Note that any principal right (resp. left) A-module is exactly a cyclic indecomposable projective module, and it has the form eA (resp. Ae), where e is a local idempotent.

Write $X^n = \underbrace{X \oplus \ldots \oplus X}_{n \text{ times}}$ for any right A-module X and $X^0 = 0$.

Let A be a semiperfect ring with the Jacobson radical R.

Proposition 2.4 ([9], Proposition 11.1.1). *Let $A = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$ be the decomposition of a semiperfect ring A into a direct sum of principal right A-modules and let $1 = f_1 + \ldots + f_s$ be a corresponding decomposition of the identity of A into a sum of pairwise orthogonal idempotents, i.e., $f_iA = P_i^{n_i}$. Then the Jacobson radical of the ring A has a two-sided Peirce decomposition of the following form*:

$$R = \begin{pmatrix} R_{11} & A_{12} & \ldots & A_{1n} \\ A_{21} & R_{22} & \ldots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \ldots & R_{nn} \end{pmatrix},$$

where $R_{ii} = \operatorname{rad}(f_iAf_i)$, $A_{ij} = f_iAf_j$ for $i, j = 1, \ldots, n$.

The ring f_iAf_i is isomorphic to $\operatorname{End}_A(P_i^{n_i}) \simeq M_{n_i}(\operatorname{End}(P_i))$, where $\operatorname{End}_A(P_i) = \mathcal{O}$ is a local ring by [9] (Theorem 10.3.8). By [9] (Proposition 3.4.10), $\operatorname{rad} M_{n_i}(\mathcal{O}) = M_{n_i}(\operatorname{rad} \mathcal{O}_i)$.

Set $U_i = P_i/P_iR$. Since $\bar{A} = A/R = U_1^{n_1} \oplus \ldots \oplus U_s^{n_s}$, the idempotents $f_1, \ldots, f_s$ are central modulo the radical and all simple right A-modules are exhausted by the modules $U_1, \ldots, U_s$. Analogously, if $V_i = Q_i/RQ_i$, then all simple left A-modules are exhausted by the modules $V_1, \ldots, V_s$.

Lemma 2.1 (Annihilation lemma). *Let $1 = f_1 + \ldots + f_s$ be a canonical decomposition of the identity of a semiperfect ring A. For every simple right A-module U_i and for each f_j, $U_if_j = \delta_{ij}U_i$, $i, j = 1, \ldots, s$. Similarly, for every simple left A-module V_i and for each f_j, $f_jV_i = \delta_{ij}V_i$, $i, j = 1, \ldots, s$.*

The following lemma allows to compute the minimal number of generators $\mu_A(X)$ of a finitely generated module X over a semiperfect ring A.

Lemma 2.2 (Generator lemma). *Let $A = \bigoplus_{i=1}^{s} P_i^{n_i}$ be the decomposition of a semiperfect ring A into a direct sum of principal right A-modules, X be a finitely generated right A-module and $P(X) = \bigoplus_{i=1}^{s} P_i^{m_i}$ be a projective cover of X. If $m = \max \frac{m_i}{n_i}$ is an integer, then $\mu_A(X) = m$. Otherwise, $\mu_A(X) = [m] + 1$.*

Here $[m]$ is the integral part of m, i.e., the largest integer $\leq m$.

At first this lemma was proved in [17], see also [9] (Lemma 11.1.8). From this lemma it immediately results the following main corollary.

***Corollary* 2.2.** *Let M be a finitely generated right A-module and $P(M) = P_1^{m_1} \oplus \ldots \oplus P_s^{m_s}$ be a projective cover of M. A module M is cyclic if and only if $m_1 \leq n_1, \ldots, m_s \leq n_s$.*

Note that this corollary is the same as Lemma 1.8 in [27].

Recall definitions of serial modules and rings (see [9], § 12.1).

Definition 2.4. *A module is called uniserial if the lattice of its submodules is a chain, i.e., the set of all its submodules is linearly ordered by inclusion. A module is called serial if it decomposes into a direct sum of uniserial submodules.*

Definition 2.5. *A ring is called right* (*resp. left*) *uniserial if it is a right* (*resp. left*) *uniserial module over itself, i.e., the lattice of right ideals is linearly ordered. A ring is called right* (*resp. left*) *serial if it is a right* (*resp. left*) *serial module over itself. A ring which is both a right and left serial ring is called a serial ring.*

Note that a right uniserial ring is local, and a right serial ring is semiperfect.

For a notion of a quiver $Q(A)$ of a semiperfect right Noetherian ring A see [9] (Chapter 11).

Let $Q = (VQ, AQ, s, e)$ be a quiver (directed graph, or digrapf), which is given by two sets VQ, AQ and two mappings $s, e\colon AQ \to VQ$. The elements of VQ are called vertices or points, and those of AQ arrows. Usually the vertices of Q will be denoted by numbers $1, 2, \ldots, s$. If an arrow $\sigma \in AQ$ connects the vertex $i \in VQ$ with the vertex $j \in VQ$, then $i = s(\sigma)$ is called its *start vertex* (or *source vertex*) and $j = e(\sigma)$ is called its *end vertex* (or *target vertex*). This will be denoted as $\sigma\colon s(\sigma) \to e(\sigma)$, or shortly $\sigma\colon i \to j$.

A *path of a quiver* Q from the vertex i to the vertex j is an ordered set of k arrows $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ such that the start vertex of each arrow σ_m coincides with the end vertex of the previous one σ_{m-1} for $1 < m \leq k$, and moreover, the vertex i is the start vertex of σ_1, while the vertex j is the end vertex of σ_k. The number k of arrows is called the *length of the path*.

The start vertex i of the arrow σ_1 is called the *start of the path* and the end j of the arrow σ_k is called the *end of the path*. The path is said to connect the vertex i with the vertex j and this is denoted by $\sigma_1\sigma_2 \ldots \sigma_k\colon i \to j$.

By convention it is considered that the path ε_i of length zero connects vertex i with itself without any arrow.

Definition 2.6. *A path, connecting a vertex of a quiver with itself and of length not equal to zero, is called an oriented cycle. An oriented cycle of the length 1 is called a one-pointed cycle or a loop. A quiver without multiple arrows and multiple loops is called a simply laced quiver.*

***Remark* 2.1.** Note that first the notion of quiver was introduced by P. Gabriel for finite directed graphs when both sets VQ and AQ are finite.

Definition 2.7. *A serial ring is called a primary decomposable serial ring if it is isomorphic to a finite direct product of primary rings.*

Theorem 2.5 ([9], § 12.4)**.** *For a semiperfect two-sided Noetherian ring A the following conditions are equivalent*:

(a) *A is a principal ideal ring*;

(b) *A is a primary decomposable serial ring*;

(c) *both the right and left quiver of A is a disconnected union of points and one-pointed cycles.*

Recall that a ring A is called *quasi-Frobenius* if it is a self-injective two-sided Artinian ring.

Theorem 2.6. *The following conditions are equivalent for a two-sided Artinian indecomposable ring A.*

1. *A is a principal ideal ring.*
2. *A is a quasi-Frobenius ring and A has a minimal* (*by inclusion*) *two-sided ideal $\mathcal{I}$ such that $A/\mathcal{I}$ is quasi-Frobenius.*
3. *A ring A and all its factor rings $A/\mathcal{I}$ for each ideal $\mathcal{I}$ of A are quasi-Frobenius.*

Proof. $1 \Longleftrightarrow 2$. This is Theorem 4.15.5 [10].

$3 \Longrightarrow 2$. It is trivial.

$1 \Longrightarrow 3$. By Theorem 2.5, A is a primary serial ring which is a full matrix ring $M_n(B)$, where B is an Artinian uniserial ring. Let $\mathcal{M}$ be the Jacobson radical of B. Then $R = M_n(\mathcal{M})$ is the Jacobson radical of A and any two-sided ideal of A has the form $R^i = M_n(\mathcal{M}^i)$. Therefore each ring A/R^i is quasi-Frobenius.

3. Decomposition theorems. Right Bézout rings and ipri-rings are natural generalization of pri-rings. Clearly, every pri-ring is a right Noetherian right Bézout ring and a right Noetherian ipri-ring. In the general case there are two strict inclusions:

$$\text{pri-rings} \subset \text{ipri-rings},$$

$$\text{pri-rings} \subset \text{right Bézout rings}.$$

Example **3.1.** Let p be a prime integer, $\mathbf{Q}$ be the field of rational numbers,

$$\mathbf{Z}_p = \left\{ \frac{m}{n} \in \mathbf{Q} \,\middle|\, (n,p) = 1 \right\}.$$

Consider the following ring

$$\mathcal{O} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \,\middle|\, \alpha \in \mathbf{Z}_p,\ \beta \in \mathbf{Q} \right\}.$$

The unique maximal ideal of the ring $\mathcal{O}$ is the following ideal

$$\mathcal{M} = \left\{ \begin{pmatrix} p\alpha & \beta \\ 0 & p\alpha \end{pmatrix} \,\middle|\, \alpha \in \mathbf{Z}_p, \beta \in \mathbf{Q} \right\}$$

and

$$\bigcap_{k=1}^{\infty} \mathcal{M}^k = \left\{ \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \,\middle|\, \beta \in \mathbf{Q} \right\} = X,$$

which is a uniserial A-module.

The ring $\mathcal{O}$ is a commutative uniserial ring and therefore $\mathcal{O}$ is a right Bézout ring. But $\mathcal{O}$ is neither a pri-ring, nor ipri-ring, since X is a principal ideal.

In this section decomposition theorems are given under the additional condition that they are all semiperfect.

Recall that a module M is called *local* if M has the largest proper submodule. Equivalently, a module M is local if it is cyclic, non-zero and has the unique maximal proper submodule.

Proposition 3.1. *Let A be a local ring with the unique maximal ideal R. A right A-module M is local if and only if $M = mA$ for some $m \in M$.*

Proof. If M is a local module, then M is cyclic, which follows from the definition.

Inversely, assume that M is a cyclic module. Since A is a local ring, from Proposition 2.3 it follows that $\mathrm{rad}M = MR$. Therefore M/MR is a semisimple module.

If M/MR is not simple, then M contains at least two maximal submodules. Then the projective cover $P(M) \cong A^m$ where $m \geq 2$. This implies, by Corollary 2.2, that M is not cyclic.

Therefore M/MR is simple, which means that $MR = \mathrm{rad}M$ is a maximal submodule of M. Since M is cyclic, MR is the unique maximal submodule of M, by Nakayama's lemma. So that M is a local module.

Lemma 3.1. *Let A be a local ring with a unique maximal ideal R. Then any principal right ideal I of A has a unique maximal submodule which has the form IR. Moreover, if $I = L_1 + \ldots + L_m$, where L_i are right ideals of A, then there exists i such that $I = L_i$.*

Proof. Let I be a principal right ideal of A. By Proposition 2.3, IR is the intersection of all maximal submodules (right ideals) of I. If I contains at least, two maximal submodules, then the projective cover $P(I) \cong A^m$ where $m \geq 2$, and so I is not principal, by Corollary 2.2. Consequently, a principal right ideal I is a local module.

If $I = L_1 + \ldots + L_m$, where L_i are right ideals of A, then either $L_i \subset IR$, or $L_i = I$. If all $L_i \subset IR$, then $I \subset IR$. It is a contradiction.

Theorem 3.1. *A local ipri-ring $\mathcal{O}$ is right uniserial.*

Proof. Let $\mathcal{M}$ be the unique maximal ideal of $\mathcal{O}$. For any $n > 0$ an ideal $\mathcal{M}^n$ is a two-sided ideal, and so it is a principal right ideal. By Lemma 3.1, $\mathcal{M}^n$ has the unique maximal submodule $\mathcal{M}^n\mathcal{M} = \mathcal{M}^{n+1}$. So one has the chain of principal right ideals

$$\mathcal{O} \supset \mathcal{M} \supset \mathcal{M}^2 \supset \ldots \supset \mathcal{M}^n \supset \ldots$$

with simple factors. Since $\mathcal{O}$ is an ipri-ring, $\mathcal{M}^\omega = \bigcap\limits_{n=1}^{\infty} \mathcal{M}^n$ is a two-sided ideal which is a principal right ideal.

Let I be a two-sided ideal of a ring A. Define the transfinite powers of I following to [19]:

$$I^1 = I,$$

$$I^{\beta+1} = I^\beta \cdot I,$$

$$I^\alpha = \bigcap_{\beta<\alpha} I^\beta, \quad \text{for any limit ordinal } \alpha.$$

All transfinite powers of $\mathcal{M}$ are two-sided ideals, and, since $\mathcal{O}$ is an ipri-ring, they are all principal right ideals. Therefore taking into account this definition and Lemma 3.1, we obtain the following chain of principal right ideals of $\mathcal{O}$:

$$\mathcal{O} \supset \mathcal{M} \supset \mathcal{M}^2 \supset \ldots \supset \mathcal{M}^n \supset \ldots \supset \mathcal{M}^\omega \supset \mathcal{M}^{\omega+1} \supset \mathcal{M}^{\omega+2} \supset \ldots$$

with simple factors, which implies that $\mathcal{O}$ is a right uniserial ring.

Theorem 3.2. *Let A be a semiperfect ipri-ring. Then A is a finite direct product of full matrix rings over local rings.*

Proof. Let $A = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$ be a decomposition of a semiperfect ring A into a direct sum of principal A-modules, and $1 = f_1 + \ldots + f_s$ be a corresponding canonical decomposition of the identity of A into a sum of pairwise orthogonal idempotents, i.e., $f_i f_j = \delta_{ij} f_j$ $(i, j = 1, \ldots, s)$ and $f_i A = P_i^{n_i}$, $i = 1, \ldots, s$.

Let $1 = h_1 + h_2$, where $h_1 = f_1$ and $h_2 = 1 - f_1$ are idempotents of A. Then $R = \begin{pmatrix} R_1 & X \\ Y & R_2 \end{pmatrix}$, where $A_1 = h_1 A h_1$, $A_2 = h_2 A h_2$, $X = h_1 A h_2$ and $Y = h_1 A h_2$. By Proposition 2.4, $R = \begin{pmatrix} R_1 & X \\ Y & R_2 \end{pmatrix}$, where R_i is the Jacobson radical of A_i, $i = 1, 2$.

Consider the following two-sided ideal I of A: $I = \begin{pmatrix} A_1 & X \\ Y & YX \end{pmatrix}$. Obviously, $I^2 = I$ and $I = gA$ for some $g \in I$ since A is an ipri-ring. Then $I = h_1 gA \oplus h_2 gA$. Therefore, $h_1 I = h_1 gA = (A_1, X)$ and $h_2 I = h_2 gA = (Y, YX)$.

One has

$$IR = \begin{pmatrix} R_1 & X \\ YR_1 & YX \end{pmatrix}.$$

Suppose, $Y \neq 0$. By Nakayama's lemma, $(R_1, X) \neq (A_1, X)$ and $(YR_1, YX) \neq$ $\neq (Y, YX)$. Consider the semisimple A-module I/IR. We have that $(I/IR) f_1 =$ $= I/IR$ and the Annihilation lemma $I/IR = U_1^m$, where $m > n_1$ (note that $Y \neq 0$). Therefore, $P(I) = P_1^m$. By Corollary 2.2, $m \leq n_1$. This contradiction shows that $Y = 0$ and $A = \begin{pmatrix} A_1 & X \\ 0 & A_2 \end{pmatrix}$. Obviously, $K = \begin{pmatrix} 0 & X \\ 0 & A_2 \end{pmatrix}$ is a two-sided ideal in A. Hence, $K = h_1 tA \oplus h_2 tA$, where $K = tA$. Let $X \neq 0$. Then for the semisimple A-module K/KR we have $(K/KR) f_1 = 0$. Therefore, the Annihilation lemma $K/KR =$ $= A_2/R_2 \oplus X/XR_2$ is a direct sum of simple A-modules $U_2, \ldots, U_s$. A module X/XR_2 is nonzero by Nakayama's lemma. Let U_k be a direct summand of X/XR_2, $2 \leq k \leq s$. Then $P(K)$ contains a direct summand $P_k^{n_k+1}$ and by Corollary 2.2 K is not a principal right ideal. As above $X = 0$.

Consequently, $A = M_{n_1}(\operatorname{End}_A P_1) \times \operatorname{End}_A(P_2^{n_2} \oplus \ldots \oplus P_s^{n_s})$. Applying the induction by s we obtain $\operatorname{End}_A(P_2^{n_2} \oplus \ldots \oplus P_s^{n_s}) = M_{n_2}(\operatorname{End}_A P_2) \times \ldots \times M_{n_s}(\operatorname{End}_A P_s)$.

Since A is a semiperfect ring, all rings $\mathcal{O}_1 = \operatorname{End}_A P_1$, $\mathcal{O}_2 = \operatorname{End}_A P_2, \ldots, \mathcal{O}_s =$ $= \operatorname{End}_A P_s$ are local.

The theorem is proved.

From Theorems 3.1 and 3.2 it follows the following main theorem:

Theorem 3.3 (Decomposition theorem for semiperfect ipri-rings)**.** *Let A be a semiperfect ipri-ring. Then A is a finite direct product of full matrix rings over right uniserial rings.*

Since any pri-ring is a right Noetherian ipri-ring, we immediately have the following corollary.

***Corollary* 3.1.** *A semiperfect pri-ring is a finite direct product of full matrix rings over right uniserial rings.*

From Theorem 3.3 and Warfield's Theorem 1.2 there results the following statement.

***Corollary* 3.2.** *A semiperfect ipri-ring is a right Bézout ring.*

Thus in the case of semiperfect rings one has the following chain of rings:

$$\text{pri-rings} \subset \text{ipri-rings} \subset \text{right Bézout rings.}$$

Lemma 3.2. *Let $A = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$ be a decomposition of a semiperfect ring A into a direct sum of principal right A-modules. Let $\phi\colon P_i \to P_j$ be a nonzero homomorphism $(i \neq j)$. (One can assume that $i < j$.) The right ideal of the form*

$$J_\phi = P_1^{n_1} \oplus \ldots \oplus P_i^{n_i} \oplus \ldots \oplus P_j^{n_j - 1} \oplus \operatorname{Im}\phi \oplus P_{j+1}^{n_{j+1}} \oplus \ldots \oplus P_s^{n_s}$$

is not cyclic.

Proof. Note that J_ϕ is a finitely generated ideal. Let $P(J_\phi)$ be the projective cover of J_ϕ. Then $P(J_\phi) = P_1^{n_1} \oplus \ldots \oplus P_i^{n_i+1} \oplus \ldots \oplus P_j^{n_j-1} \oplus \ldots \oplus P_s^{n_s}$. By Corollary 2.2, J_ϕ is not cyclic.

***Corollary* 3.3.** *Let A be a semiperfect right Bézout ring with decomposition $A = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$ into a direct sum of principal right A-modules. Then*

$$\operatorname{Hom}_A\left(P_i, \sum_{k \neq i} \oplus P_k^{n_k}\right) = 0.$$

The following theorem is a part of Theorem 1.14 [27] proved by R. B. Warfield, Jr. Here it is given some other proof of this theorem which shows its close connection with the proof of Theorem 3.2.

Theorem 3.4 (Decomposition theorem for right Bézout semiperfect rings)**.** *A right Bézout semiperfect ring A is a finite direct product of full matrix rings over local rings.*

Proof. Let $A_A = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$ be a decomposition as above, and let $s \geq 2$. Let $f_1 A = P_1^{n_1}$, where $f_1^2 = f_1$ and $e = 1 - f_1$. Consider the two-sided Peirce decomposition of A with respect to the decomposition of $1 = f_1 + e$:

$$A = \begin{pmatrix} A_1 & X \\ Y & A_2 \end{pmatrix}$$

where $A_1 = f_1 A f_1$, $A_2 = eAe$, $X = f_1 A e$ and $Y = eAf_1$:

By Corollary 3.3, $\operatorname{Hom}_A(P_1^{n_1}, P_2^{n_2} \oplus \ldots \oplus P_s^{n_s}) = 0$, whence $Y = eAf_1 = 0$.

Suppose $X \neq 0$ and $x \in X$, $x \neq 0$. We have $x = xf_2 + \ldots + xf_s$. So, there exists an i, such that $(2 \leq i \leq s)$ and $xf_i \neq 0$. Therefore, $f_1 x f_i$ defines a nonzero homomorphism $\theta : P_i^{n_i} \to P_1^{n_1}$ by the formula

$$\theta(f_i a) = f_1 x f_i a.$$

By Corollary 3.3, we obtain a contradiction.

Therefore $X = 0$. Consequently, $A = M_{n_1}(\operatorname{End}_A P_1) \times \operatorname{End}_A(P_2^{n_2} \oplus \ldots \oplus P_s^{n_s})$. Applying the induction by s we have $\operatorname{End}_A(P_2^{n_2} \oplus \ldots \oplus P_s^{n_s}) = M_{n_2}(\operatorname{End}_A P_2) \times \ldots \times M_{n_s}(\operatorname{End}_A P_s)$.

The theorem is proved.

Theorem 3.5. *A local ring $\mathcal{O}$ is right Bézout if and only if $\mathcal{O}$ is right uniserial.*

Proof. Let $\mathcal{O}$ be a right uniserial, and let $J = x_1\mathcal{O} + \ldots + x_n\mathcal{O}$ be a finitely generated right ideal of $\mathcal{O}$. Assume that $x_1, \ldots, x_n$ is a minimal system of generators of J and $n \geq 2$. Consider the right ideals $x_1\mathcal{O}$ and $x_2\mathcal{O}$. Since $\mathcal{O}$ is right uniserial, one can assume that $x_1\mathcal{O} \subseteq x_2\mathcal{O}$ and $J = x_1\mathcal{O} + x_3\mathcal{O} + \ldots + x_n\mathcal{O}$. Therefore, $x_1, \ldots, x_n$ is not a minimal system of generators of J. This contradiction shows that $n = 1$, and so $\mathcal{O}$ is a right Bézout ring.

Conversely, let $\mathcal{O}$ be a right Bézout local ring, but $\mathcal{O}$ is not a right uniserial ring. Then there exist right ideals X and Y such that $X + Y$ strongly contains X and Y. Moreover, $X \supsetneqq X \cap Y$, and $Y \supsetneqq X \cap Y$. Consequently, there exist x and y such that $x \in X$ and $x \notin Y$ and $y \in Y$, $y \notin X$. Denote by $K = x\mathcal{O} + y\mathcal{O}$ a two-generated right ideal of $\mathcal{O}$. Since $\mathcal{O}$ is right Bézout, $K = z\mathcal{O}$.

By Lemma 3.1, K is a local module, and either $K = x\mathcal{O}$ or $K = y\mathcal{O}$. Consequently, a right Bézout local ring is right uniserial.

The theorem is proved.

From Theorems 3.4 and 3.5 the main decomposition theorem for right Bézout semiperfect rings follows.

Theorem 3.6 (Decomposition theorem for right Bézout semiperfect rings)**.** *A right Bézout semiperfect ring A is a finite direct product of full matrix rings over uniserial rings.*

It is easy to see that the properties (1) ipri; (2) pri; (3) right Bézout hold if and only if these properties hold for finite direct products and direct summands of rings with these properties.

Proposition 3.2. *Let $\mathcal{O}$ be a local ring. Then $M_n(\mathcal{O})$ is a right Bézout ring if and only if $\mathcal{O}$ is a right Bézout ring.*

Proof. Let $A = M_n(\mathcal{O})$ be a right Bézout ring with the maximal ideal $\mathcal{M}$, then $\operatorname{rad} A = M_n(\mathcal{M})$. Suppose that L is a finitely generated right ideal of $\mathcal{O}$. Then $\tilde{L} = \underbrace{(L, \ldots, L)}_{n}$ is a finitely generated right ideal of $M_n(\mathcal{O})$. Therefore $\tilde{L}$ is a principal right ideal and $\tilde{L}/\tilde{L}R = (L/\mathcal{M}, \ldots, L/\mathcal{M})$. By Corollary 2.2, $L/\mathcal{M}$ is a simple module. Hence, L is a principal right ideal.

A converse statement follows from Lemma 1 [28].

The proposition is proved.

Let A be a finite ipri-ring. If A is an indecomposable ring then, by Theorem 3.2, $A = M_n(\mathcal{O})$, where $\mathcal{O}$ is a finite right uniserial ring.

Let $\mathcal{O}$ be a local ring with the unique (right, left and two-sided) ideal $\mathcal{M}$ and $\bar{\mathcal{O}} = \mathcal{O}/\mathcal{M}$. The module $\bar{\mathcal{O}}$ is the unique right (resp., left) simple $\mathcal{O}$-module U (resp., V).

Denote the number of elements in a finite set S by $|S|$.

Proposition 3.3. *Let $\mathcal{O}$ be a finite local ring with $\mathcal{M}^2 = 0$ ($\mathcal{M} \neq 0$), $\bar{\mathcal{O}} = \mathcal{O}/\mathcal{M}$ and $\mathcal{M}$ as a right $\mathcal{O}$-module is isomorphic to U^t. Then $|\mathcal{O}| = |U|^{t+1}$.*

Proof. Let $|U| = m$. Then, by the Lagrange theorem, $|\mathcal{O}| = [\mathcal{O} \colon \mathcal{M}] \cdot |\mathcal{M}| = m \cdot |U|^t = |U|^{t+1}$.

***Corollary* 3.4.** *Let $\mathcal{O}$ and $\mathcal{M}$ be as in Proposition* 3.3. *Then $\mathcal{M}$ as a left $\mathcal{O}$ is isomorphic to V^t.*

The proof follows from the previous proposition taking into account that $|U| = |V|$.

Proposition 3.4. *Right and left quivers of a finite local ring coincide.*

The proof follows from Corollary 3.4 and the definition of a quiver.

Proposition 3.5. *A right Artinian ring A with the radical R is right uniserial if and only if A/R^2 is a right uniserial ring.*

The proof follows immediately from [9] (Theorem 12.3.10).

Note, that a right uniserial ring $\mathcal{O}$ is local.

Proposition 3.6. *A right uniserial finite ring $\mathcal{O}$ is left uniserial.*

Proof. By Proposition 3.5 it is sufficient to prove that $\mathcal{O}/\mathcal{M}^2$ is left uniserial. For a right uniserial ring $\mathcal{O}$ with $\mathcal{M}^2 = 0$ $(\mathcal{M} \neq 0)$ we have $t = 1$. By Corollary 3.4 $\mathcal{M}$ is a simple left $\mathcal{O}$-module. Therefore, a ring $\mathcal{O}$ is uniserial, i.e., right and left uniserial.

As an immediate corollary of this proposition we obtain the following theorem proved by A. A. Nechaev in [23].

Theorem 3.7 (A. A. Nechaev [23]). *A finite ipri-ring is a principal ideal ring* (*not only pri-ring*).

For more details on finite uniserial (chain) rings see [5].

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