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ON RINGS WITH WEAKLY PRIME CENTERS*

ПРО КІЛЬЦЯ З СЛАБКИМИ ПРОСТИМИ ЦЕНТРАМИ

We introduce a class of rings obtained as a generalization of rings with prime centers. A ring R is called *weakly prime center* (or, briefly, WPC) if $ab \in Z(R)$ implies that aRb is an ideal of R, where $Z(R)$ stands for the center of R. The structure and properties of these rings are studied, the relationships between prime center rings, strongly regular rings, and WPC rings are discussed, parallel with the relationship between WPC to commutativity.

Введено клас кілець, що є узагальненням кілець з простими центрами. Кільце R називається *слабко простим центром* (чи просто WPC), якщо з включення $ab \in Z(R)$ випливає, що aRb є ідеалом R, де $Z(R)$ — центр R. Вивчено структуру і властивості таких кілець та проаналізовано співвідношення між простими центральними кільцями, сильно регулярними кільцями та кільцями з слабко простим центром паралельно зі співвідношенням між слабко простим центром та комутативністю.

1. Introduction. Throughout this article, all rings considered are associative with identity, and all modules are unital, the symbols $J(R)$, $N(R)$, $U(R)$, $E(R)$, $Z(R)$ and $\mathrm{Max}_l(R)$ will stand respectively for the Jacobson radical, the set of all nilpotent elements, the set of all invertible elements, the set of all idempotent elements, the center and the set of all maximal left ideals of R. For any nonempty subset X of a ring R, $r(X) = r_R(X)$ and $l(X) = l_R(X)$ denote the right annihilator of X and the left annihilator of X, respectively.

A ring R is called

(1) *reduced* if $N(R) = 0$;

(2) *Abel* if $E(R) \subseteq Z(R)$;

(3) *left quasiduo* if every maximal left ideal of R is an ideal;

(4) *MELT* if every essential maximal left ideal of R is an ideal.

Recall that a ring R has prime (semiprime) center [8] if $ab \in Z(R)$ implies $a \in Z(R)$ or $b \in Z(R)$ $\big(a^n \in Z(R)$ implies $a \in Z(R)\big)$. Clearly, commutative rings have prime center. In [8], some basic properties of prime center rings are studied.

A ring R is called *periodic* [3] if for each $x \in R$, there exist distinct positive integers m and n for which $x^m = x^n$. In [8] (Theorem 1), it is shown that for a periodic ring R, R is commutative if and only if R has prime center.

In this paper, a new class of rings is introduced, which is a proper generalization of rings with prime centers. A ring R is called weakly prime center (or, briefly, WPC) if $ab \in Z(R)$ implies aRb is an ideal of R. Remark 2.1 points out that WPC rings are proper generalization of rings with prime centers. Proposition 2.6 shows that strongly regular rings are a class of WPC rings. Proposition 2.8 shows that a ring R is a division ring if and only if R is a WPC primitive ring.

Let R be a ring and $e \in E(R)$. e is called *left minimal idempotent* if Re is a minimal left ideal of R. We write $ME_l(R)$ for the set of all left minimal idempotents of R. A ring R is called *left min-Abel* if $(1-e)Re = 0$ for each $e \in ME_l(R)$. In [13] (Theorem 1.2), it is shown that a ring R

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is a left quasiduo ring if and only if R is a left min-Abel $MELT$ ring. The study of left min-Abel rings appears in [13, 15, 16]. Proposition 2.5 shows that WPC rings are left min-Abel.

Following [10], an element a of a ring R is called clean if a is a sum of a unit and an idempotent of R, and a is said to be exchange if there exists $e \in E(R)$ such that $e \in aR$ and $1 - e \in (1 - a)R$. A ring R is called clean if every element of R is clean, and R is said to be exchange if every element of R is exchange. According to [10], clean rings are always exchange, but the converse is not true, in general. In [18], it is shown that left quasiduo exchange rings are clean; in [19], it is shown that Abel exchange rings are clean; in [15], it is shown that quasinormal exchange rings are clean; in [16], it is shown that weakly normal exchange rings are clean. Theorem 3.1 shows that WPC exchange rings are clean and have stable range 1.

Following [4], a ring R is said to be *semiperiodic* if for each $x \in R\backslash(J(R) \cup Z(R))$, there exist $m, n \in \mathbb{Z}$, of opposite parity, such that $x^n - x^m \in N(R)$. Clearly, the class of semiperiodic rings contains all commutative rings, all Jacobson radical rings, and certain non-nil periodic rings. Theorem 4.2 shows that for a semiperiodic ring R with $J(R) \neq N(R)$, R is WPC if and only if R is commutative.

2. Some properties of WPC rings.

Definition 2.1. *A ring R is called weakly prime center* (WPC) *if for any $a, b \in R$, $ab \in Z(R)$ implies aRb is an ideal of R.*

Clearly commutative rings are WPC.

Proposition 2.1. *Prime center rings are WPC.*

Proof. Let R be a prime center ring and $a, b \in R$ with $ab \in Z(R)$. Since R is prime center, $a \in Z(R)$ or $b \in Z(R)$, one has $aRb = abR = Rab$. Hence aRb is an ideal of R and R is WPC.

Recall that a ring R is directly finite if $ab = 1$ implies $ba = 1$ for any $a, b \in R$. In [8], it is shown that prime center rings are directly finite.

Lemma 2.1. *WPC rings are directly finite.*

Proof. Let $a, b \in R$ with $ab = 1$. Let $e = ba$. Then $a = ae$ and $e \in E(R)$. Since $a(1 - e) = 0 \in Z(R)$, $aR(1 - e)$ is an ideal of R, that is, $R(1 - e)$ is an ideal of R because $aR = R$. Hence $(1 - e)a \in R(1 - e)$, which implies $(1 - e)a = (1 - e)ae = 0$. Then $a = ea$ and $1 = ab = eab = e = ba$, this shows that R is a directly finite ring.

Proposition 2.2. *Let R be a local ring. If $J(R)$ is commutative, then R is WPC.*

Proof. Assume that $ab \in Z(R)$. Then Rab is an ideal of R. If $Rab = R$, then $ab \in U(R)$, by Lemma 2.1, $a, b \in U(R)$, so $aRb = R$ is an ideal of R. If $Rab \subseteq J(R)$, then $ab \notin U(R)$. If $a \notin U(R)$ and $b \notin U(R)$, then $aR, Rb \subseteq J(R)$. Since $J(R)$ is commutative, $RaRbR = R\big(a(Rb)\big)R = RbaR = aRb$, this gives aRb is an ideal of R. If $a \in U(R)$ and $b \notin U(R)$, then $RaRbR = RbR = RabR = Rab = Rb = aRb$, so aRb is an ideal. Similarly, if $a \notin U(R)$ and $b \in U(R)$, we can show that aRb is an ideal. Hence R is WPC.

Proposition 2.3. *If R be a local prime center ring, then R is commutative.*

Proof. It is an immediate result of [8] (Basic Lemma 2(b)).

***Remark* 2.1.** By Proposition 2.2, one knows that division rings are WPC. By Proposition 2.3, noncommutative division rings need not be prime center. Thus there exists a WPC ring (noncommutative division rings) which is not prime center. Hence WPC rings are proper generalization of prime center rings.

Proposition 2.4. *If R is a semiprime WPC ring, then R is reduced.*

Proof. If $N(R) \neq 0$, then there exists $0 \neq a \in N(R)$ such that $a^2 = 0$. Since R is WPC, aRa is an ideal of R, this leads to $aRaR \subseteq aRa$, so $aRaRa \subseteq aRa^2 = 0$. Since R is semiprime, $a = 0$, which is a contradiction. Thus $N(R) = 0$.

Recall that a ring R is NCI [7] if either $N(R) = 0$ or $N(R)$ contains a nonzero ideal of R. By the proof of Proposition 2.4, we have the following corollary.

***Corollary* 2.1.** *WPC rings are NCI.*

***Remark* 2.2.** [7], Example 2.5, points out that NCI rings need not be directly finite, by Lemma 2.1, we know that the converse of Corollary 2.1 is not true.

***Remark* 2.3.** Simple rings need not be WPC. For example, let D be a division ring and $R = \begin{pmatrix} D & D \\ D & D \end{pmatrix}$. Then R be a simple ring. Clearly, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0 \in Z(R)$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}$. If R is WPC, then

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

which is a contradiction. Hence R is not a WPC ring.

Recall that a ring R is *left min-Abel* [13] if for every $e \in ME_l(R) = \big\{e \in E(R) |\ Re$ is a minimal left ideal of $R \big\}$, e is left semicentral in R. Clearly, R is a left min-Abel ring if and only if $(1-e)Re = 0$ for each $e \in ME_l(R)$.

Proposition 2.5. *If R is a WPC ring, then R is left min-Abel.*

Proof. Let $e \in ME_l(R)$. Since R is a WPC ring and $(1-e)e = 0 \in Z(R)$, $(1-e)Re$ is an ideal of R, this gives $R(1-e)Re \subseteq (1-e)Re$. If $(1-e)Re \neq 0$, then $R(1-e)Re = Re$, so $e \in eRe = eR(1-e)Re \subseteq e(1-e)Re = 0$, which is a contradiction. Therefore $(1-e)Re = 0$ and R is a left min-Abel ring.

***Remark* 2.4.** The converse of Proposition 2.5 is not true in general. For example, let $R = \begin{pmatrix} \mathbb{Z}_5 & \mathbb{Z}_5 \\ 0 & \mathbb{Z}_5 \end{pmatrix}$. Cleary, $\begin{pmatrix} 2 & 4 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \in Z(R)$ and $\begin{pmatrix} 2 & 4 \\ 0 & 4 \end{pmatrix} R \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} =$ $= \left\{\begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} | x, y \in \mathbb{Z}_5\right\}$. But $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix} \notin \begin{pmatrix} 2 & 4 \\ 0 & 5 \end{pmatrix} R \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}$, thus R is not WPC. Since R is a left quasiduo ring by [18], R is a left min-Abel ring by [13] (Theorem 1.2).

Recall that a ring R is *von Neumann regular* if $a \in aRa$ for any $a \in R$, and R is said to be *strongly regular* if $a \in a^2R$ for any $a \in R$. It is well known that a ring R is a strongly regular ring if and only if R is a reduced von Neumann regular ring.

Proposition 2.6. *The following conditions are equivalent for a ring R :*

(1) *R is a strongly regular ring*;

(2) *R is a WPC von Neumann regular ring.*

Proof. (1) $\Longrightarrow$ (2). Since R is a strongly regular ring, R is an Abel von Neumann regular ring. Hence, for any $a, b \in R$, $aR = eR = Re$ for some $e \in E(R)$, this gives $aRb = R(eb) = Rg = gR$ for some $g \in E(R)$. Thus aRb is an ideal of R and R is a WPC ring.

(2) $\Longrightarrow$ (1). Since von Neumann regular rings are semiprime, by Proposition 2.4, R is reduced. Hence R is a strongly regular ring.

Recall that a ring R is *left SF* if every simple left R-module is flat. It is well known that von Neumann regular rings are left SF. In [11] (Remark 3.13), it is shown that if R is a reduced left SF ring, then R is strongly regular. We can generalize this result as follows.

Proposition 2.7. *If R is a prime center left SF ring, then R is a commutative strongly regular ring.*

Proof. Let $a \in R$ with $a^2 = 0$. By [8] (Basic Lemma 2(a)), $a \in Z(R)$. If $a \neq 0$, then $l(a) \neq R$ and there exists a maximal left ideal M of R such that $l(a) \subseteq M$. Since R is a left SF ring, R/M is flat as left R-module. Since $a \in l(a) \subseteq M$, $a = am$ for some $m \in M$. Since $a \in Z(R)$, $a = ma$, one obtains $1 - m \in l(a) \subseteq M$, $1 \in M$, which is a contradiction. Hence $a = 0$, which implies R is reduced, by [11] (Remark 3.13), R is strongly regular. Now let $x \in R$. Then $x = xyx$ for some $y \in R$. Write $e = xy$ and $g = yx$. Then $e, g \in E(R)$ and $x = ex = xg$. Since R is Abel, $e, g \in Z(R)$. Since $yx = g \in Z(R)$, $y \in Z(R)$ or $x \in Z(R)$. If $x \in Z(R)$, we are done. If $y \in Z(R)$, then for any $r \in R$, we have $xr = xgr = xrg = xryx = xyrx = erx = rex = rx$, which implies $x \in Z(R)$. Hence R is commutative.

***Corollary* 2.2.** *If R is a prime center von Neumann regular ring, then R is a commutative strongly regular ring.*

***Remark* 2.5.** Since strongly regular rings need not be commutative, by Corollary 2.2, strongly regular rings need not be prime center.

***Corollary* 2.3.** *R is a field if and only if R is a prime center division ring.*

Proof. Fields are certainly prime center division rings. The converse is an immediate corollary of Corollary 2.2.

Proposition 2.8. *R is a division ring if and only if R is a WPC primitive ring.*

Proof. Division rings are certainly WPC primitive rings. Now let R be a WPC primitive ring. If R is not a division ring, then there exists a subring S of R such that $S \cong \begin{pmatrix} D & D \\ D & D \end{pmatrix}$, where D is a division ring. Clearly, $\begin{pmatrix} D & D \\ D & D \end{pmatrix}$ is not reduced, so S is not reduced, this implies R is not reduced. But by Proposition 2.4, R is reduced, which is a contradiction. Hence R is a division ring.

Corollary 2.3 and Proposition 2.8 give the following corollary.

***Corollary* 2.4.** *R is a field if and only if R is a prime center primitive ring.*

A ring R is called *weakly regular* if $a \in aRaR \cap RaRa$ for every $a \in R$. A left R-module M is called *YJ-injective* (*$Wnil$-injective* (see [14])) if for each $0 \neq a \in R$ $\big(0 \neq a \in N(R)\big)$, there exists a positive integer n such that $a^n \neq 0$ and each left R-homomorphism $Ra^n \longrightarrow M$ can be extended to $R \longrightarrow M$. It is easy to see that YJ-injective modules are $Wnil$-injective.

Proposition 2.9. *Let R be a WPC ring. If each singular simple left R-modules are $Wnil$-injective, then R is reduced.*

Proof. By Proposition 2.4, we only need to show that R is semiprime. Assume that $a \in R$ with $aRa = 0$. If $a \neq 0$, then there exists a maximal left ideal M of R such that $r(aR) \subseteq M$. We claim that M is an essential left ideal of R. If not, $M = l(e)$ for some $e \in ME_l(R)$. Since R is a WPC ring, R is left min-Abel by Proposition 2.5. Hence $aRe = aeRe = 0$ because $a \in r(aR) \subseteq M = l(e)$, this leads to $e \in r(aR) \subseteq l(e)$, which is a contradiction. Thus M is an essential left ideal of R and R/M is a singular simple left R-module, by hypothesis, R/M is $Wnil$-injective. Then the left R-homomorphism $f \colon Ra \longrightarrow R/M$ defined by $f(ra) = r + M$ can be extended into $R \longrightarrow R/M$, so there exists $d \in R$ such that $1 - ad \in M$. Since $ad(ad) = 0$, $1 - ad$ is a unit of R, so $M = R$, a contradiction. Hence $a = 0$.

***Corollary* 2.5.** *Let R be a WPC ring whose singular simple left R-modules are YJ-injective, then R is a reduced weakly regular ring.*

Proof. By Proposition 2.9, R is reduced. By [9] (Theorem 4), R is a reduced weakly regular ring.

3. Exchange $\boldsymbol{WPC}$ rings. Following [10], an element a of a ring R is called *clean* if a is a sum of a unit and an idempotent of R, and a is said to be *exchange* if there exists $e \in E(R)$ such that $e \in aR$ and $1 - e \in (1 - a)R$. A ring R is called *clean* if every element of R is clean, and R is said to be *exchange* if every element of R is exchange. According to [10], clean rings are always exchange, but the converse is not true unless R satisfies one of the following conditions: (1) R is a left quasiduo ring [18]; (2) R is an Abel ring [19]; (3) R is a quasinormal ring [15]; (4) R is a weakly normal ring [16].

Theorem 3.1. *Let R be a WPC ring and $a \in R$. Then*

(1) *If a is exchange, then a is clean.*

(2) *If R is an exchange ring, then R is a clean ring.*

(3) *If a^n is clean for some $n \geq 1$, then a is clean.*

(4) *If a^2 is clean, then a and $-a$ are clean.*

Proof. (1) Let $e \in E(R)$ such that $e \in aR$ and $1-e \in (1-a)R$. Write $e = ab$ and $1-e = (1-a)c$ for some $b = be, c = c(1 - e) \in R$. Then $(a - (1 - e))(b - c) = ab - ac - (1 - e)b + (1 - e)c = = ab + (1 - a)c - (1 - e)b - ec = 1 - (1 - e)b - ec$. Since R is a WPC ring and $b(1 - e) = = 0 \in Z(R)$, $bR(1 - e)$ is an ideal of R. Hence $bR(1 - e)R \subseteq bR(1 - e)$ and $bR(1 - e)Re = 0$, which implies $bR(1 - e)Rb = bR(1 - e)Rbe = 0$. Therefore $(R(1 - e)Rb)^2 = 0$, this leads to $(1 - e)b \in R(1 - e)Rb \subseteq J(R)$. Similarly, $ec \in J(R)$. Hence $1 - (1 - e)b - ec$ is a unit of R, by Lemma 2.1, one obtains $a - (1 - e)$ is an unit of R. Hence a is a clean element.

(2) It is an immediate result of (1).

(3) Since a^n is clean, there exist $u \in U(R)$ and $f \in E(R)$ such that $a^n = u + f$. Let $e = u(1 - -f)u^{-1}$. Then $(a^n-e)u = (u+f)u-u(1-f) = a^n(a^n-1) \in aR$, so $e = a^n+(a^n-a^{2n})u^{-1} \in aR$ and $1 - e \in (1 - a)R$, this implies a is exchange, by (1), a is clean.

(4) Since $a^2 = (-1a)^2$ is clean, by (3), a and $-a$ are clean.

***Corollary* 3.1.** *Let R be a WPC ring and idempotent can be lifted modulo $J(R)$. Let $a \in R$ be clean and $e \in E(R)$. Then*

(1) *ae is clean.*

(2) *If $-a$ is also clean, then $a + e$ is clean.*

Proof. (1) Since a is clean, $\bar{a}$ is clean in $\bar{R} = R/J(R)$. Since R is a WPC ring, $eR(1-e)$ is an ideal of R, which implies $((1-e)ReR)^2 = (eR(1-e)R)^2 = 0$. Hence $(\bar{1}-\bar{e})\bar{R}\bar{e} = \bar{e}\bar{R}(\bar{1}-\bar{e}) = \bar{0}$, that is, $\bar{e}$ is a central idempotent in $\bar{R}$. Since a is clean in R, there exist $u \in U(R)$ and $f \in E(R)$ such that $a = u+f$. Let $v \in R$ such that $uv = vu = 1$. Then, in $\bar{R}$, $\bar{a}\bar{e} = (\bar{u}\bar{e}+\bar{e}-\bar{1})+(\bar{f}\bar{e}+\bar{1}-\bar{e})$. Clearly, $(\bar{u}\bar{e}+\bar{e}-\bar{1})(\bar{v}\bar{e}+\bar{e}-\bar{1}) = (\bar{v}\bar{e}+\bar{e}-\bar{1})(\bar{u}\bar{e}+\bar{e}-\bar{1}) = \bar{1}$ and $(\bar{f}\bar{e}+\bar{1}-\bar{e})^2 = \bar{f}\bar{e}+\bar{1}-\bar{e}$, so $\bar{a}\bar{e}$ is clean in $\bar{R}$. Since idempotent can be lifted modulo $J(R)$, there exists $g \in E(R)$ such that $\bar{g} = \bar{f}\bar{e}+\bar{1}-\bar{e}$. Let $w \in R$ such that $\bar{w} = \bar{u}\bar{e}+\bar{e}-\bar{1}$. Then $w \in U(R)$ and $ae - w - g \in J(R)$. Let $ae - w - g = x \in J(R)$. Then $ae = g + w(1 + w^{-1}x)$. Since $w(1 + w^{-1}x) \in U(R)$, ae is clean in R.

(2) Since $-a$ is clean in R, $1+a$ is clean in R. Hence $\bar{a}$ and $\bar{1}+\bar{a}$ are all clean in $\bar{R} = R/J(R)$. Let $\bar{a} = \bar{u}+\bar{f}$ and $\bar{1}+\bar{a} = \bar{v}+\bar{g}$ where $u, v \in U(R)$ and $f, g \in E(R)$. Clearly, $\bar{a}+\bar{e} = \bar{a}(\bar{1}-\bar{e})+(\bar{1}+\bar{a})\bar{e}$, so $\bar{a}+\bar{e} = \bar{v}\bar{e}+\bar{u}(\bar{1}-\bar{e})+\bar{g}\bar{e}+\bar{f}(\bar{1}-\bar{e})$. Clearly, $(\bar{v}\bar{e}+\bar{u}(\bar{1}-\bar{e}))(\bar{v}^{-1}\bar{e}+\bar{u}^{-1}(\bar{1}-\bar{e})) = \bar{1}$ and $\bar{g}\bar{e}+\bar{f}(\bar{1}-\bar{e}) \in E(\bar{R})$. Therefore, $\bar{a}+\bar{e}$ is clean in $\bar{R}$, similar to (1), we obtain $a + e$ is clean in R.

In [5], it is showed that if R is a unit regular ring, then every element of R is a sum of two units. A ring R is called an $(S, 2)$-*ring* [5], if every element of R is a sum of two units of R. In [2], it is proved that if R is an Abel π-regular ring, then R is an $(S,2)$-ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of R.

Theorem 3.2. *Let R be a WPC π-regular ring. Then R is an $(S,2)$-ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of R.*

Proof. Since R is a WPC π-regular ring, $R/J(R)$ is π-regular ring. Since R is an exchange ring, idempotent can be lifted modulo $J(R)$. By the proof of Corollary 3.1(1), $R/J(R)$ is an Abel ring. By [2], $R/J(R)$ is an $(S,2)$-ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of $R/J(R)$. By [15] (Lemma 4.3), we are done.

In light of Theorem 3.2, we have the following corollaries:

***Corollary* 3.2.** *Let R be a WPC π-regular ring such that $2 = 1 + 1 \in U(R)$. Then R is an $(S,2)$-ring.*

***Corollary* 3.3.** *Let R be a WPC π-regular ring. Then R is an $(S,2)$-ring if and only if for some $d \in U(R)$, $1 + d \in U(R)$.*

Recall that a ring R is said to *have stable range* 1 [12] if for any $a, b \in R$ satisfying $aR+bR = R$, there exists $y \in R$ such that $a + by$ is right invertible. Clearly, R has stable range 1 if and only if $R/J(R)$ has stable range 1. In [19] (Theorem 6), it is showed that exchange rings with all idempotents central have stable range 1.

Theorem 3.3. *WPC exchange rings have stable range* 1.

Proof. Let R be a WPC exchange ring. Then $R/J(R)$ is exchange with all idempotents central, so, by [19] (Theorem 6), $R/J(R)$ has stable range 1. Therefore R has stable range 1.

In [17], A ring R is said to satisfy the *unit* 1*-stable condition* if for any $a, b, c \in R$ with $ab+c = 1$, there exists $u \in U(R)$ such that $au + c \in U(R)$. It is easy to prove that R satisfies the unit 1-stable condition if and only if $R/J(R)$ satisfies the unit 1-stable condition.

Theorem 3.4. *Let R be a WPC exchange ring, then the following conditions are equivalent*:

(1) *R is an $(S,2)$-ring.*

(2) *R satisfies the unit $1-$stable condition.*

(3) *Every factor ring of R is an $(S,2)$-ring.*

(4) *$\mathbb{Z}_2$ is not a homomorphic image of R.*

A ring R is called *left topologically boolean*, or a *tb-ring* [1] for short, if for every pair of distinct maximal left ideals of R there is an idempotent in exactly one of them.

Theorem 3.5. *Let R be a WPC exchange ring. Then R is a left tb-ring.*

Proof. Suppose that M and N are distinct maximal left ideals of R. Let $a \in M\backslash N$. Then $Ra + N = R$ and $1 - xa \in N$ for some $x \in R$. Clearly, $xa \in M\backslash N$. Since R is a WPC exchange ring, R is clean by Theorem 3.1, there exist an idempotent $e \in E(R)$ and a unit u in R such that $xa = e + u$. If $e \in M$, then $u = xa - e \in M$ from which it follows that $R = M$, a contradiction. Thus $e \notin M$. If $e \notin N$, then $Re + N = R$. Since R is a WPC ring, by the proof of Corollary 3.1(1), $(1-e)ReR \subseteq J(R) \subseteq N$, $1-e \in (1-e)R = (1-e)Re + (1-e)N \subseteq N$. Hence $u = (1-e) + (xa-1) \in N$. It follows that $N = R$ which is also impossible. We thus have that e is an idempotent belonging to N only.

4. $\boldsymbol{WPC}$ semiperiodic rings. Following [4], a ring R is said to be *semiperiodic* if for each $x \in R\backslash(J(R) \cup Z(R))$, there exist $m, n \in \mathbb{Z}$, of opposite parity, such that $x^n - x^m \in N(R)$. Clearly, the class of semiperiodic rings contains all commutative rings, all Jacobson radical rings, and certain nonnil periodic rings.

Lemma 4.1. *If R is a WPC semiperiodic ring, then $N(R) \subseteq J(R)$.*

Proof. Let $a \in N(R)$ with $a^k = 0$, and let $x \in R$. If $ax \in J(R)$, then ax is right quasiregular; and if $ax \in Z(R)$, then ax is nilpotent and again ax is right quasiregular. Suppose, then, that $ax \notin J \cup Z$, in which case [4] (Lemma 2.3 (iii)) gives $q \in \mathbb{Z}^+$ and an idempotent e of form ay such that $(ax)^q = (ax)^q e$. Since

$$e = ay = eay = ea(1-e)y + eaey = ea(1-e)y + ea^2y^2 =$$

$$= ea(1-e)y + ea^2(1-e)y^2 + ea^2ey^2 = ea(1-e)y + ea^2(1-e)y^2 + ea^3y^3 = \ldots$$

$$\ldots = \sum_{i=1}^{k-1} ea^i(1-e)y^i + ea^ky^k = \sum_{i=1}^{k-1} ea^i(1-e)y^i.$$

Since R is a WPC ring, $eR(1-e) \in J(R)$ by the proof of Corollary 3.1(1), which implies $e \in J(R)$, so $e = 0$ and $(ax)^q = 0$, which shows that ax is right quasiregular. Thus $a \in J(R)$.

Theorem 4.1. *If R is a WPC semiperiodic ring, then $R/J(R)$ is commutative.*

Proof. Let $\bar{R} = R/J(R)$. Clearly, $N(R) \subseteq J(R)$ by Lemma 4.1. Now let $\bar{a} \in \bar{R}$ with $\bar{a}^2 = 0$. Then $a^2 \in J(R) \subseteq N(R) \cup Z(R)$ by [4] (Lemma 2.6). If $a^2 \in N(R)$, then $a \in N(R)$. Hence $a \in J(R)$ by Lemma 4.1 and $\bar{a} = \bar{0}$. If $a^2 \in Z(R)$, then $\bar{a}^2 \in Z(\bar{R})$. If $\bar{a} \in Z(\bar{R})$, then $\bar{a}\bar{R}\bar{a} = \bar{0}$. Since $\bar{R}$ is semiprime, $\bar{a} = \bar{0}$. If $\bar{a} \notin Z(\bar{R})$, then $a \notin J(R) \cup Z(R)$. By [4] (Lemma 2.3(iii)), $a^q = a^q e$ for some $q \geq 1$ and $e \in E(R)$ with the form ay. Hence $e = eay = ea(1-e)y + eaey = = ea(1-e)y + ea^2y^2 \in J(R)$. Thus $e = 0$ and $a^q = 0$. This implies $a \in N(R) \subseteq J(R)$ by Lemma 4.1, which is a contradiction. Hence $\bar{a} \in Z(\bar{R})$ and so $\bar{a} = \bar{0}$. Therefore $\bar{R}$ is a reduced ring. Since $\bar{R}$ is also semiperiodic, by [4] (Lemma 4.4), $\bar{R}$ is commutative.

Theorem 4.2. *Let R be a WPC semiperiodic ring. Then*

(1) *$N(R)$ is an ideal of R.*

(2) *If $J(R) \neq N(R)$, then R is commutative.*

Proof. (1) Let $a, b \in N(R)$ and $x \in R$. Then $a - b, ax \in J(R)$ by Lemma 4.1. By [4] (Lemma 2.6), $a - b, ax \in N(R) \cup Z(R)$. If $a - b, ax \in N(R)$, we are done. If $a - b, ax \in Z(R)$. Then $(a-b)a = a(a-b)$ and $(ax)^n = a^n x^n$ for any $n \geq 1$, this gives $ab = ba$, thus $a-b, ax \in N(R)$. Similarly, $xa \in N(R)$. Therefore $N(R)$ is an ideal of R.

(2) By [4] (Lemma 2.6), it follows that

$$J(R) = (J(R) \cap N(R)) \cup (J(R) \cap Z(R). \tag{4.1}$$

By (1), viewing (4.1) as a relation holding on additive subgroup, we conclude that

$$J(R) = J(R) \cap N(R) \qquad \text{or} \qquad J(R) = J(R) \cap Z(R).$$

This implies that

$$J(R) \subseteq N(R) \qquad \text{or} \qquad J(R) \subseteq Z(R).$$

Since $J(R) \neq N(R)$, by Lemma 4.1, $J(R) \subseteq Z(R)$.

Now let $x \in R$. If $x \notin Z(R)$, then $x \notin J(R) \cup Z(R)$, so there exists positive integers n, m ($n \geq m$) of opposite parity such that $x^n - x^m \in N(R)$. Let $k \geq 1$ such that $(x^n - x^m)^k = 0$. Then $((x - x^{n-m+1})^m)^k = 0$, this gives $x - x^{n-m+1} \in N(R) \subseteq J(R) \subseteq Z(R)$. By Herstein's theorem [6], R is commutative.

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