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# YETTER – DRINFEL'D HOPF ALGEBRAS ON BASIC CYCLE*

# ХОПФОВІ АЛГЕБРИ ЄТТЕРА – ДРІНФЕЛЬДА НА БАЗОВОМУ ЦИКЛІ

A class of Yetter – Drinfel'd Hopf algebras on basic cycle are constructed.

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**1. Introduction.** Let $H$ be a Hopf algebra. A Yetter – Drinfel'd module over $H$ is a $\mathbb{K}$-linear space $V$ such that $V$ is both an $H$-module and an $H$-comodule and satisfies a compatibility condition. Yetter – Drinfel'd Hopf algebras are Hopf algebras in Yetter – Drinfel'd module category. It is a class of braided Hopf algebras. Nichols algebras [11], $(G,\chi)$-Hopf algebras [12, p. 206] (10.5.11) and twisted Hopf algebras [10] are important examples of Yetter – Drinfel'd Hopf algebras.

Radford's projection theorem [13] leads to a decomposition of the given Hopf algebra into a Radford biproduct of two factors, one is no longer a Hopf algebra, but rather a Yetter – Drinfel'd Hopf algebra over the other factor. After Radford's work, some important advances are the followings. Doi considered Hopf modules in Yetter – Drinfel'd module category in [6]. Scharfschwerdt proved Nichols – Zoeller theorem for Yetter – Drinfel'd Hopf algebras, see [15]. Schauenburg proved that a Yetter – Drinfel'd module category is equivalent to a category of the left modules over the Drinfel'd double, and also to a Hopf bimodule category, see [16]. Sommerhäuser studied Yetter – Drinfel'd Hopf algebras over groups of prime order in [17]. Andruskiewitsch and Schneider studied Nichols algebras in [1]. Recently, Grana, Heckenberger and Vendramin classified Nichols algebras of irreducible Yetter – Drinfel'd module over nonabelian groups in [7].

The quiver methods in the representation theory of algebras were considered by Ringel in [14]. The coalgebra structure on quivers were considered by Chin and Montgomery in [4]. Quivers allow one to present algebras or coalgebras in a useful way. For example, Cibils and Rosso constructed Hopf quivers and quiver quantum groups in [3] and [5] respectively. Green and Solberg have investigated the structure of finite dimensional basic Hopf algebras in [8].

One can get a Hopf algebra or a quantum group via quivers. The constructions of braided Hopf algebras via quivers are not numerous. In this paper, we provide such an explicit construction via quivers. Let $C_d(n)$ be a subcoalgebra of the coalgebra $\mathbb{K}Z_n^c$ of paths in the oriented cycle quiver $Z_n^c$ of length n with basis the set of all paths of length strictly less than $d$. Assume that $G = \{1, g, \ldots, g^{n-1}\}$ is a group and $\mathbb{K}G$ a group Hopf algebra. In this paper, we prove that $C_d(n)$ is a Yetter – Drinfel'd module over $\mathbb{K}G$. Moreover, $C_d(n)$ is a Yetter – Drinfel'd Hopf algebra over $\mathbb{K}G$, see Theorem 5.

Throughout, $\mathbb{K}$ will denote a fixed field. All algebras, coalgebras, (co)modules, $\otimes$ and Hom are over $\mathbb{K}$. For basic definitions and facts about coalgebras, Hopf algebras and (co)modules we refer to Sweedler's book [18]. In particular, the comultiplication of a coalgebra C is denoted by

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$\Delta(c) = \sum c_1 \otimes c_2$ for all $c \in C$, and the structure map of a left $C$-comodule $V$ is denoted by $\rho(v) = \sum v^{-1} \otimes v^0$ for all $v \in V$. For quivers we refer to Auslander–Reiten–Smalϕ's book [2].

**2. Preliminaries.** Let $(H, m, u, \triangle, \epsilon, S)$ be a Hopf algebra with antipode $S$. A left *Yetter–Drinfel'd module* over $H$ is a $\mathbb{K}$-vector space $V$ such that $V$ is both a left $H$-module with action $\rightarrow$ and left $H$-comodule with coaction $\rho$, and satisfies the compatibility condition:

$$\sum (h \rightarrow v)^{-1} \otimes (h \rightarrow v)^0 = \sum h_1 v^{-1} S(h_3) \otimes h_2 \rightarrow v^0, \tag{1}$$

for all $h \in H$, $v \in V$. The category of left Yetter–Drinfel'd modules over $H$ is denoted by ${}^H_H\mathcal{YD}$. The category is a pre-braided category and the pre-braiding is given by

$$\tau_{V,W} : V \otimes W \longrightarrow W \otimes V, \qquad v \otimes w \longmapsto \sum (v^{-1} \rightarrow w) \otimes v^0.$$

The above map is a braiding when $H$ has a bijective antipode. Denote by $\bar{S}$ the inverse of $S$. The inverse of $\tau_{V,W}$ is

$$\tau^{-1}_{V,W} : W \otimes V \longrightarrow V \otimes W, \qquad w \otimes v \longmapsto \sum v^0 \otimes \bar{S}(v^{-1}) \rightarrow w.$$

Let $A$ be a Yetter–Drinfel'd module. We call the 6-tuple $(A, m, u, \triangle, \epsilon, S)$ *a Yetter–Drinfel'd Hopf algebra* (or *Hopf algebra in* ${}^H_H\mathcal{YD}$) if $A$ satisfies the following conditions:

($a_1$) $(A, m, u)$ is a left $H$-module algebra, i.e.,

$$h \rightarrow (ab) = \sum (h_1 \rightarrow a)(h_2 \rightarrow b), \qquad h \rightarrow 1_A = \epsilon(h) 1_A.$$

($a_2$) $(A, m, u)$ is a left $H$-comodule algebra, i.e.,

$$\rho(ab) = \sum (ab)^{-1} \otimes (ab)^0 = \sum a^{-1} b^{-1} \otimes a^0 b^0,$$

$$\rho(1_A) = 1_H \otimes 1_A.$$

($a_3$) $(A, \triangle, \epsilon)$ is a left $H$-module coalgebra, i.e.,

$$\triangle(h \rightarrow a) = \sum (h_1 \rightarrow a_1) \otimes (h_2 \rightarrow a_2), \qquad \epsilon_A(h \rightarrow a) = \epsilon_H(h)\epsilon_A(a).$$

($a_4$) $(A, \triangle, \epsilon)$ is a left $H$-comodule coalgebra, i.e.,

$$\sum a^{-1} \otimes (a^0)_1 \otimes (a^0)_2 = \sum {a_1}^{-1} {a_2}^{-1} \otimes {a_1}^0 \otimes {a_2}^0,$$

$$\sum a^{-1} \epsilon_A(a^0) = \epsilon_A(a) 1_H.$$

($a_5$) $\triangle$ and $\epsilon$ are algebra maps in ${}^H_H\mathcal{YD}$, i.e.,

$$\triangle(ab) = \sum a_1({a_2}^{-1} \rightarrow b_1) \otimes {a_2}^0 b_2,$$

$$\triangle(1) = 1 \otimes 1, \qquad \epsilon(ab) = \epsilon(a)\epsilon(b), \qquad \epsilon(1_A) = 1_k.$$

($a_6$) There exists a $\mathbb{K}$-linear map $S\colon A \longrightarrow A$ in ${}^H_H\mathcal{YD}$ such that it is a convolution inverse of identity, i.e., $S * \mathrm{Id} = u\epsilon = \mathrm{Id} * S$.

When the pre-braiding $\tau$ is trivial, Yetter – Drinfel'd Hopf algebras are ordinary Hopf algebras, see [18, p. 8] for details. However, generally, Yetter – Drinfel'd Hopf algebras are not ordinary Hopf algebras because the bialgebra axiom asserts that they obey ($a_5$).

Let $q \in \mathbb{K}$. For nonnegative integer $l$ and $0 \le m \le l$, the *Gaussian polynomials* is defined to be

$$\binom{l}{m}_q := \frac{(l)!_q}{m!_q(l-m)!_q}$$

where

$$l!_q := 1_q \ldots l_q, \qquad 0!_q := 1, \qquad l_q := 1 + q + \ldots + q^{l-1}.$$

Next, we will give several conclusions of Gaussian polynomials. They will be used in next section. Firstly, we recall the *q-Pascal identity*, it can be found in [9] (Proposition IV.2.1).

$$\binom{l}{m}_q = \binom{l-1}{m-1}_q + q^m\binom{l-1}{m}_q = \binom{l-1}{m}_q + q^{l-m}\binom{l-1}{m-1}_q. \tag{2}$$

For any scalar $a$ and a variable element $z$, for any positive integer $l$, Kassel proved that

$$(a-z)(a-qz)\ldots(a-q^{l-1}z) = \sum_{k=0}^{l}(-1)^k\binom{l}{k}_q q^{\frac{k(k-1)}{2}}a^{l-k}z^k$$

(see [9], IV.2.7). Especially, let $a = 1$ and $z = 1$, we have

$$\sum_{k=0}^{l}(-1)^k q^{\frac{k(k-1)}{2}}\binom{l}{k}_q = 0. \tag{3}$$

Moreover, the following equation also holds.

**Lemma 1.** *Let $l$ and $k$ be nonnegative integers. For any integer $s$, where $0 \le s \le l+k$, we have*

$$\sum_{\substack{m+p=s\\ 0\le m\le l,\, 0\le p\le k}} q^{m(k-p)}\binom{l+k-s}{l-m}_q\binom{s}{m}_q = \binom{l+k}{l}_q. \tag{4}$$

**3. Construction.** Let $Z_n^c$ denote the basic cycle of length $n$, i.e., an oriented graph with $n$ vertices $e_0, \ldots, e_{n-1}$, and a unique arrow $a_i$ from $e_i$ to $e_{i+1}$ for each $0 \le i \le n-1$. The indices are taken modulo $n$. Set $\gamma_i^m := a_{i+m-1}\ldots a_{i+1}a_i$ to be the path of length $m$ starting at the vertex $e_i$. Note that $\gamma_i^0 = e_i$ and $\gamma_i^1 = a_i$.

Let $C_d(n)$ be the subcoalgebra of $\mathbb{K}Z_n^c$ with basis the set of all paths of length strictly less than $d$. Observe that if the order of $q$ is $d$, then $\binom{d}{l}_q = 0$ for $1 \le l \le d-1$. Then $C_d(n)$ is a path coalgebra with comultiplication $\triangle(\gamma_i^l) = \sum_{k=0}^{l}\gamma_{i+k}^{l-k} \otimes \gamma_i^k$, and counit $\epsilon(\gamma_i^l) = \delta_{l,0}$. Here, $\delta_{l,0}$ is the Kronecker symbol.

Define a multiplication on $C_d(n)$ by

$$\gamma_i^l \gamma_j^s = \binom{l+s}{l}_q \gamma_{i+j}^{l+s}, \tag{5}$$

where $l+s<d$. Observe that if $l+s \geq d$, then $\gamma_i^l \gamma_j^s = 0$ since $q^d = 1$. It is easy to see that the unit element of $C_d(n)$ is $1 = \gamma_0^0$.

**Definition 1.** *Let $A$ be a vector space. We call $A$ a pre-bialgebra if $A$ is an algebra and a coalgebra.*

From Definition 1, we know that a pre-bialgebra is a bialgebra if and only if $\triangle$ and $\epsilon$ are algebra morphisms.

The following lemma is routine, we omit the proof.

**Lemma 2.** *Coalgebra $C_d(n)$ is a pre-bialgebra with multiplication* (5).

Let $G = \{1, g, g^2, \ldots, g^{n-1}\}$ be a group. Then $\mathbb{K}G$ is a Hopf algebra, see [12] (1.5.3). It is clear that $C_d(n)$ becomes a left $\mathbb{K}G$-module with the left module structure

$$g^s \rightharpoonup \gamma_i^l = q^{sl} \gamma_i^l \tag{6}$$

and $C_d(n)$ is also a left $\mathbb{K}G$-comodule with comodule structure

$$\rho(\gamma_i^l) = \sum g^l \otimes \gamma_i^l. \tag{7}$$

Then we have the following lemma.

**Lemma 3.** *Coalgebra $C_d(n)$ is a Yetter–Drinfel'd module over $\mathbb{K}G$ with module* (6) *and comodule* (7).

***Proof.*** Take $g^s \in \mathbb{K}G$ and $\gamma_i^l \in C_d(n)$. Recall that

$$\sum (g^s \rightharpoonup \gamma_i^l)^{-1} \otimes (g^s \rightharpoonup \gamma_i^l)^0 = q^{sl} g^l \otimes \gamma_i^l.$$

Moreover, we have

$$\sum (g^s)_1 (\gamma_i^l)^{-1} S((g^s)_3) \otimes (g^s)_2 \rightharpoonup \gamma_i^l = g^s g^l S(g^s) \otimes g^s \rightharpoonup \gamma_i^l = g^l \otimes q^{sl} \gamma_i^l.$$

This means that (1) holds. Thus $C_d(n)$ is a Yetter–Drinfel'd module over $\mathbb{K}G$.

Next, we will give the main theorem.

**Theorem 1.** *Coalgebra $C_d(n)$ is a Yetter–Drinfel'd Hopf algebra over $\mathbb{K}G$.*

***Proof.*** We divide the proof into six steps as the definition of Yetter–Drinfel'd Hopf algebras. In the following, we take $\gamma_i^l, \gamma_j^k \in C_d(n)$ and $g^s \in G$.

It is easy to check that ($a_1$)–($a_4$) hold. We only need to show ($a_5$) and ($a_6$).

($a_5$) It is obvious that $\triangle(1) = 1 \otimes 1$, $\epsilon(\gamma_i^l \gamma_j^k) = \binom{l+k}{l}_q \delta_{l+k,0} = \delta_{l,0}\delta_{k,0} = \epsilon(\gamma_i^l)\epsilon(\gamma_j^l)$ and $\epsilon(1) = 1$. Next, we will prove the comultiplication $\triangle$ is an algebra map in Yetter–Drinfel'd category. On one hand, we have

$$\triangle(\gamma_i^l \gamma_j^k) = \binom{l+k}{l}_q \triangle(\gamma_{i+j}^{l+k}) = \binom{l+k}{l}_q \sum_{s=0}^{l+k} \gamma_{i+j+s}^{l+k-s} \otimes \gamma_{i+j}^s. \tag{8}$$

On the other hand, we obtain

$$\sum (\gamma_i^l)_1 ((\gamma_i^l)_2{}^{-1} \rightharpoonup (\gamma_j^k)_1) \otimes (\gamma_i^l)_2{}^0 (\gamma_j^k)_2 =$$

$$= \sum_{m=0}^{l}\sum_{p=0}^{k} \gamma_{i+m}^{l-m}((\gamma_i^m)^{-1} \rightarrow \gamma_{j+p}^{k-p}) \otimes (\gamma_i^m)^0(\gamma_j^p) =$$

$$= \sum_{m=0}^{l}\sum_{p=0}^{k} \gamma_{i+m}^{l-m}(g^m \rightarrow \gamma_{j+p}^{k-p}) \otimes (\gamma_i^m \gamma_j^p) =$$

$$= \sum_{m=0}^{l}\sum_{p=0}^{k} q^{m(k-p)} \binom{l-m+k-p}{l-m}_q \binom{m+p}{m}_q \gamma_{i+j+m+p}^{l+k-m-p} \otimes \gamma_{i+j}^{m+p}. \qquad (9)$$

For $s = 0, 1, \ldots, l+k$, comparing the coefficient of $\gamma_{i+j+s}^{l+k-s} \otimes \gamma_{i+j}^{s}$ in equation (8) and equation (9), we get

$$\binom{l+k}{l}_q \gamma_{i+j+s}^{l+k-s} \otimes \gamma_{i+j}^{s} = \sum_{\substack{m+p=s \\ 0 \le m \le l, 0 \le p \le k}} q^{m(k-p)} \binom{l+k-s}{l-m}_q \binom{s}{m}_q \gamma_{i+j+s}^{l+k-s} \otimes \gamma_{i+j}^{s}$$

by (4). Thus

$$\binom{l+k}{l}_q \sum_{s=0}^{l+k} \gamma_{i+j+s}^{l+k-s} \otimes \gamma_{i+j}^{s} = \sum_{m=0}^{l}\sum_{p=0}^{k} q^{m(k-p)} \binom{l-m+k-p}{l-m}_q \binom{m+p}{m}_q \gamma_{i+j+m+p}^{l+k-m-p} \otimes \gamma_{i+j}^{m+p}.$$

That means

$$\triangle(\gamma_i^l \gamma_j^k) = \sum (\gamma_i^l)_1((\gamma_i^l)_2{}^{-1} \rightarrow (\gamma_j^k)_1) \otimes (\gamma_i^l)_2{}^0 (\gamma_j^k)_2.$$

Hence $\triangle$ is an algebra map in Yetter – Drinfel'd category.

($a_6$) Define $S\colon A \longrightarrow A$ by

$$S(\gamma_i^l) = (-1)^l q^{\frac{l(l-1)}{2}} \gamma_{-i-l}^l.$$

Then $S$ is a convolution inverse of identity, since

$$(S * Id)(\gamma_i^l) = \sum_{m=0}^{l} S(\gamma_{i+m}^{l-m})\gamma_i^m = \sum_{m=0}^{l} (-1)^{l-m} q^{\frac{(l-m)(l-m-1)}{2}} \gamma_{-i-l}^{l-m}\gamma_i^m =$$

$$= \sum_{m=0}^{l} (-1)^{l-m} q^{\frac{(l-m)(l-m-1)}{2}} \binom{l}{l-m}_q \gamma_{-l}^l.$$

If $l = 0$, we have $(S*Id)(\gamma_i^0) = \gamma_0^0$. If $l \neq 0$, we have $\sum_{m=0}^{l} (-1)^{l-m} q^{\frac{(l-m)(l-m-1)}{2}} \binom{l}{l-m}_q \gamma_{-l}^l =$ $= 0$ by (3). In a word, $(S * Id)(\gamma_i^l) = 0$. Similarly, $(Id * S)(\gamma_i^l) = 0$. So $S$ is the convolution inverse of identity.

Thus $C_d(n)$ is a Yetter – Drinfel'd Hopf algebra over the group algebra $\mathbb{K}G$.

Theorem 1 is proved.

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