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FREDHOLM QUASI-LINEAR MANIFOLDS AND DEGREE OF FREDHOLM QUASI-LINEAR MAPPING BETWEEN THEM

КВАЗІЛІНІЙНІ МНОГОВИДИ ФРЕДГОЛЬМА ТА СТЕПІНЬ КВАЗІЛІНІЙНИХ ВІДОБРАЖЕНЬ ФРЕДГОЛЬМА МІЖ НИМИ

In this article a new class of Banach manifolds and a new class of mappings between them are presented and also the theory of degree of such mappings is given.

Представлено новий клас многовидів Банаха та новий клас відображень між ними, а також наведено теорію степеня таких відображень.

0. Introduction. As it is known, the degree theory for infinite-dimensional mappings (of the kind “identical+compact”) for the first time was given by Leray and Schauder. Afterwards, this theory was expanded up to various classes of mappings (for example, up to class Fredholm proper mappings)[1]. However, these theories were not appropriate for solution of non-linear Hilbert problem. For solution of this problem the class of Fredholm Quasi-Linear (FQL) mappings, determined on Banach space, was introduced by A. I. Shnirelman, and was determined the degree of such a mapping, which has all the main properties of classical (finite-dimensional) degree (see [8]). Later, M. A. Efendiyev expanded this theory up to FQL-mappings, determined on quasicylindrical domains (see [6]). In the given article, this theory is expanded up to FQL-mappings, determined between FQL-manifolds. In more details:

In first part of this article an example of FQL-manifold, given in [2], is extended up to example of Banach manifold from a wide class, namely up to space $H_s(\mathrm{M},\mathrm{N})$, where $\mathbf{M}$ and $\mathbf{N}$ are compact smooth manifolds of dimensions m, respectively n and $\mathbf{N}$ doesn’t have boundary. First such structure is given in $H_s(\mathbf{M},\mathbf{N})$ at $m<n$, and later, at $m \geq n$. In the last case $(m \geq n)$ the FQL-manifold $H_s(\mathbf{M},\mathbf{N})$ is appeared as a submanifold of the FQL-manifold $H_s(\mathbf{M},\mathbf{N}^k)$, where $k\cdot(n-1) \leq m < n\cdot k$.

In second part of this article the degree of FQL-mapping is expanded up to FQL-mappings between FQL-manifolds and its basic properties are proved. However, in this part another form of FQL-mapping is used, as it is better adapted for definition of degree. We named it as Fredholm Special Quasi-Linear (FSQL) mapping. The proof of identity of FQL and FSQL-mappings is given in [1].

[1] See review article [4] and later works on this subject, for example [7].

As an example of an FQL-mapping, this mapping is given: $F_f: \ H_s(\mathbf{M},\mathbf{N}) \to \to H_s(\mathbf{M},\mathbf{N})$, $F_f: u \mapsto f(u)$, where $f: \mathbf{N} \to \mathbf{N}$ is a smooth mapping with a gradient distinct from zero in all points[2].

Various types of Nonlinear Hilbert Problem have been solved by means of the theory of degree of FQL-mapping (see [6, 8]).

The purpose of the given article is preparation of theoretical base for solution of practical problems.

In the end we noted that some definitions and theorems from [8], which will be used later, are given in Section 1.

1. Let X, Y be the real Banach spaces, Ω be a bounded domain in X and X_n be an n-dimensional Euclidian space. In addition, let $\pi_n: X \to X_n$ be a linear mapping and $X_\alpha^n = \pi_n^{-1}(\alpha)$, $\alpha \in X_n$.

Definition 1.1. *A continuous mapping* $f^n: \Omega \to Y$ *is called a Fredholm Linear* (*FL*), *if*

a) *on each plane* X_α^n, $\alpha \in X_n$, *which crosses with* Ω, $f_\alpha^n \equiv f^n \big|_{X_\alpha^n}$ *is an affine invertible mapping between* X_α^n *and its image* $Y_\alpha^n = f(X_\alpha^n)$, *which is closed in* Y *and its co-dimension in* Y *is equal to* n;

b) f_α^n continuously depends on α.

Definition 1.2. *Let a sequence of FL-mappings* $\left\{ f^{n_k} \,\middle|\, f^{n_k} : \Omega \to Y \right\}$ *uniformly approximate to the mapping* f *on* Ω *and*

$$\left\| f_\alpha^{n_k} \right\| < C(\Omega), \qquad \left\| (f_\alpha^{n_k})^{-1} \right\| < C(\Omega), \qquad \alpha \in \pi_{n_k}(\Omega) \quad at \quad k > k_0(\Omega), \tag{1.1}$$

where $C(\Omega)$ *does not depend on* k. *Then continuous mapping* $f: \Omega \to Y$ *is called a Fredholm Quasi-Linear* (*FQL*).

Theorem 1.3. *Any finite combination of linear* (*pseudo*) *differential operators and operators of superposition with smooth function of finite number of arguments with a gradient which is distinct from zero in all points, defines an FQL-mapping between* H_s *and* $H_{s-\alpha}$ *at some* α *and all sufficiently greater* s.

2. Quasi-Linear manifolds. Let X be a real infinite-dimensional Banach manifold, $\{X_j\}$, $X_{j-1} \subset X_j$, $j = 1, 2, \ldots$, be a system of open sets covering to X (i.e., $X = \bigcup X_j$), $\xi_j = (Y_j, P_j, B_j)$ be an affine bundle with the total space Y_j, with the base space B_j, which is a finite-dimensional manifold and with the continuous epimorphism $P_j: Y_j \to B_j$. Let D_j be a bounded domain in Y_j and $\varphi_j: X_j \to D_j$ be a homeomorphism. In this case we shall call (φ_j, X_j) an L-chart on X_j and we shall say that, on X_j an L-structure is introduced. If an L-structure is determined on X_{j+1}, then obviously, it is determined also on X_j (as an induced structure). Let

[2] Proof of quasilinearity of similar mapping is given in [2].

$\varphi_{j'}: X_{j'} \to D_{j'}$, $\varphi_{j''}: X_{j''} \to D_{j''}$, j', $j'' \geq j$, be an L-structures on X_j. Then the transition functions $\varphi_{j''} \circ \varphi_{j'}^{-1}: D_{j'} \to D_{j''}$ and $\varphi_{j'} \circ \varphi_{j''}^{-1}: D_{j''} \to D_{j'}$ will arise. Let's suppose that each of them is an FQL-mapping between affine bundles $\xi_{j'} = (Y_{j'}, P_{j'}, B_{j'})$ and $\xi_{j''} = (Y_{j''}, P_{j''}, B_{j''})$, i.e., an FQL-mapping in charts of $\xi_{j'}$ and $\xi_{j''}$ in sense of Definition 1.2. In this case we shall say that two L-structures on X_j are equal.

Definition 2.1. *A class of equivalent L-structures on X_j is called an FQL-structure on X_j.*

Obviously, the FQL-structure on X_{j+1} induces an FQL-structure on X_j, too. The FQL-structures on X_j and X_{j+1} are called coordinated, if the FQL-structure on X_j coincides with the induced structure.

Definition 2.2. *A collection of FQL-structures on X_j, $j = 1, 2, 3, \ldots$, which are coordinated between each other is called an FQL-structure on X.*

The Banach manifold X with FQL-structure is called FQL-manifold.

Now, let us define an FQL and FSQL-mappings between FQL-manifolds.

Let X, X' be FQL-manifolds,

$$\forall j: \ X_j \subset X_{j+1}, \quad X = \bigcup X_j \quad \text{and} \quad \forall i: \ X'_i \subset X'_{i+1}, \quad X' = \bigcup X'_i.$$

In addition, let (φ_j, X_j), (φ'_i, X'_i) be L-charts on X, X' and $\varphi_j(X_j) = D_j$, $\varphi'_i(X'_i) = D'_i$ be the bounded domains of $\xi_j = (Y_j, P_j, B_j)$ and $\xi'_i = (Y'_i, P'_i, B'_i)$, respectively.

Definition 2.3. *A continuous mapping $f: X \to X'$ between FQL-manifolds X and X' is called a Fredholm Quasi-Linear (FQL), if*

a) $\forall j \ \ \exists i: \ f(X_j) \subset X'_i$;

b) $\varphi'_i \circ f \circ \varphi_j^{-1}: D_j \to D'_i$ *are FQL-mappings in charts of affine bundles ξ_j and ξ'_i (in sense of Definition 1.2).*

Definition 2.4. *A continuous mapping $f_{ji} = Y_j \to Y'_i$ is called a Fredholm Special Linear (FSL) mapping between affine bundles ξ_j and ξ'_i, if there exist subbundles $\xi_{j,r} = (Y_j, P_{j,r}, B_{j,r})$ of ξ_j and $\xi'_{i,r} = (Y'_i, P'_{i,r}, B'_{i,r})$ of ξ'_i (respectively), with identical dimension r of base spaces, such that f_{ji} is a <u>bimorphism</u> between $\xi_{j,r}$ and $\xi'_{i,r}$.*

In this case we will denote f_{ji} by $f_{ji,r}$. The restriction of FSL-mapping onto any domain D_j, $\bar{D}_j \subset Y_j$ shall be named an FSL-mapping, too.

Definition 2.5. *A continuous mapping $f_{ji} = Y_j \to Y'_i$ is called a Fredholm Special Quasi-Linear (FSQL) mapping between affine bundles ξ_j and ξ'_i, if there exists* *a sequence of FSL-mappings* $f_{ji,r} = Y_j \to Y_i'$, $r = 1, 2, 3, \ldots$, *which uniformly converges to* f_{ji} *in each bounded domain* $D_j \subset Y_j$ *and estimates* (1.1) *are satisfied.*

Definition 2.6. *A continuous mapping* $f: X \to X'$ *between FQL-manifolds* X *and* X' *is called a Fredholm Special Quasi-Linear* (*FSQL*), *if*

a) $\forall j \;\; \exists i, \;\; f(X_j) \subset X_i'$;

b) $\varphi_i' \circ f \circ \varphi_j^{-1}: D_j \to D_i'$ *is an FSQL-mapping between* ξ_j *and* ξ_i'.

As it was mentioned in introduction, the proof of identity of FQL and FSQL-mappings is given in [1].

3. Example of FQL-manifold (in case of $m < n$). Let be $X = H_s(\mathbf{M}, \mathbf{N})$, where $\mathbf{M}$, $\mathbf{N}$ are the compact smooth manifolds of the dimensions m, n $(m < n)$ respectively and $\mathbf{N}$ has no boundary. Besides, let $\mathbf{N}$ be embedded in R^{2n+1} [3]. Obviously, on X one can introduce the smooth structure [5]; the Hilbert real space $H_s(\mathbf{M}, R^n)$ will be its tangential space.

Now let's introduce an FQL-structure on X. Suppose that X is naturally embedded in $H_s(\mathbf{M}, R^{2n+1})$, $X_j = \left\{ u \in X \,\middle|\, \|u\|_s < j \right\}$, where j and s are the natural numbers, and $\| \;\|_s$ is a norm in $H_s(\mathbf{M}, R^{2n+1})$). In order to solve this problem we shall construct an affine bundle $\left(Y_j, P_j, B_j\right)$ with finite-dimensional base space B_j, shall pick out a bounded domain D_j in Y_j, shall construct homeomorphisms Φ_j: $D_j \to X_j$ (L-charts), $j = 1, 2, 3, \ldots$, and also shall prove that homeomorphisms $\Phi_i^{-1} \circ \Phi_j: D_j \to D_i$ are FQL-mappings.

Lemma 3.1. *If* $m < n$, *then*

$$\exists \gamma(j, s) > 0 \quad \forall u \in X_j \quad \exists y(u) \in \mathbf{N}: \quad \rho(y, u) \geq \gamma,$$

where $\rho(y, u) = \min_x \left\{\rho(y, u(x))\right\}$, *and* $\rho(y, u(x))$ *is a distance between* y *and* $u(x)$, $x \in \mathbf{M}$, *on* $\mathbf{N}$.

Proof. Let's suppose the contrary:

$$\forall \gamma > 0 \quad \exists u_\gamma \in X_j \quad \forall y \in \mathbf{N}: \quad \rho\left(y, u_\gamma\right) < \gamma,$$

so, $u_\gamma(\mathbf{M})$ is a γ-network of $\mathbf{N}$. For simplicity, let's suppose that $n = m + 1$. Let K be an $(m+1)$-dimensional unit cube, homeomorphic to a (closed) domain of $\mathbf{N}$. Besides, let k be a cube, belonging to K with the same dimension, its sides are parallel to the relevant sides of K and the distance between them is γ.

***Remark* 3.1.** On the contrary assumption, a part of surface $u_\gamma(\mathbf{M})$, which is the γ-network of k, will belong to K.

Let's take m-dimensional sections of k in form of m-dimensional planes, which are parallel to a m-dimensional side of k and are on a distance of 2γ from

[3] For simplicity, the embedding mappings are not written in the text.

each other. On the opposite assumption, between two (such) next planes has to be part of surface $u_\gamma(\mathbf{M})$. The m-dimensional volume of each similar part will be more or equal to $(1-2\gamma)^m$. A number of such parts is not less than $\left[\frac{1}{2\gamma}\right]$ ($[\cdot]$ shows the whole part of the number). Therefore, the total volume of all similar parts will be more or equal to $\left(\left[\frac{1}{2\gamma}\right]\right)\cdot(1-2\gamma)^m$. Obviously, $\left(\left[\frac{1}{2\gamma}\right]\right)\cdot(1-2\gamma)^m \to \infty$ at $\gamma \to 0$, so, the volume of surface $u_\gamma(\mathbf{M})$, $u_\gamma \in X_j$, will increase infinitely at $\gamma \to 0$. On the other hand, as

$$\forall u \in X_j: \quad \|u\|_{C'} \le c \cdot \|u\|_s < c \cdot j.$$

Then

$$\forall u \in X_j: \quad V_m(u) \le c \cdot j \cdot V_m(\mathbf{M}),$$

where $V_m(u)$, $V_m(\mathbf{M})$ are m-dimensional volumes of $u(\mathbf{M})$ and $\mathbf{M}$ respectively, and c there is a constant which is not dependent from u $(u \in X_j)$. In other words, all the numbers $V_m(u)$, $u \in X_j$, are bounded from above (by $c \cdot j \cdot V_m(\mathbf{M})$). This paradox proves the contention of lemma.

Now we shall start construction of FQL-structure on $H_s(\mathbf{M},\mathbf{N})$. Let $\{x_1,\ldots,x_N\}$ be a δ-network of $\mathbf{M}$. Let's assign $p_N(u) = (u(x_1),\ldots,u(x_N)) \in [\mathbf{N}]^N$ to each mapping $u \in X_j$ [4]. Let

$$B_j = \Big\{\bar{y} = (y_1,\ldots,y_N) \in [\mathbf{N}]^N \,\Big|\, \exists u \in X_j: \\ u(x_1) = y_1,\, u(x_2) = y_2,\ldots,u(x_N) = y_N\Big\}.$$

Obviously, B_j is a domain in $[\mathbf{N}]^N$, therefore it will also be a manifold of dimension $n \cdot N$.

Now for every point $\bar{y} \in B_j$ we shall construct mapping $H_s(\mathbf{M},\mathbf{N})$, $U_{\bar{y}}(x_i) = y_i$, $i = \overline{1,N}$, as follows: Let $\bar{U}_{\bar{y}}: \mathbf{M} \to R^{2n+1}$ be such a mapping that, $U_{\bar{y}}(x_i) = y_i$, $i = \overline{1,N}$, and in addition, $\|\bar{U}_{\bar{y}}\|_s$ has a minimum among all such mappings. Such a mapping $\bar{U}_{\bar{y}}(x)$ exists, is unique and continuously depends on $\bar{y}$; it results from convexity of function $u \mapsto \|u\|_s^2$. In this case, $\|\bar{U}_{\bar{y}}\|_s < j$, because according to the construction, there exists such a mapping $u \in X_j$ that $p_N(u) = \bar{y}$, and $\|\bar{U}_{\bar{y}}\|_s \le \|u\|_s$ for each similar $u(x)$.

[4] In the given article (because of technical problem) the same number is designated by symbols $\mathbf{N}$ and N, namely, number of elements in δ-network of $\mathbf{M}$.

As known, $\mathbf{N}$ has a tubular neighborhood in R^{2n+1}. Let's denote its radius by ε $(\varepsilon > 0)$. There exists a nearest point $\pi(y) \in \mathbf{N}$ for each point y from this neighborhood. Moreover, the mapping $y \mapsto \pi(y)$ is smooth, surjective and non-degenerative. Let

$$u \in H_s(\mathbf{M}, R^{2n+1}), \quad \|u\|_s < j.$$

As $\|u\|_{C^1} \le K \cdot \|u\|_s$ at sufficiently greater s, then $\|u\|_{C^1} \le K \cdot j$. Therefore,

$$\forall x \in \mathbf{M}: \ \|u'(x)\|_{R^{2n+1}} < K \cdot j.$$

Then

$$\forall x', \ x'' \in \mathbf{M}: \ \forall u \in H_s(\mathbf{M}, R^{2n+1}):$$

$$\|u\|_s < j \, \|u(x') - u(x'')\|_{R^{2n+1}} < K \cdot j \cdot d(x', x''),$$

where d is the distance on $\mathbf{M}$. Therefore,

$$\forall x', \ x'' \in \mathbf{M}: \ \forall u \in H_s(\mathbf{M}, R^{2n+1}):$$

$$\|u\|_s < j \, \|u(x') - u(x'')\|_{R^{2n+1}} < \varepsilon$$

when $d(x', x'') < \delta$ $\left(\delta = \varepsilon / (K \cdot j)\right)$. Let $x \in \mathbf{M}$. Obviously,

$$\exists i: \quad d(x, x_i) < \delta.$$

Therefore,

$$\forall u \in H_s(\mathbf{M}, R^{2n+1}): \quad \|u\|_s < j \, \|u(x) - u(x_i)\|_{R^{2n+1}} < \varepsilon.$$

As a result of that $u(x)$, $u \in H_s(\mathbf{M}, R^{2n+1})$, belongs to the ε-tubular neighborhood of $\mathbf{N}$ (in R^{2n+1}) when $\|u\|_s < j$ and $u(x_i) \in \mathbf{N}$, $i = \overline{1, N}$. Therefore it is possible to project it smoothly on $\mathbf{N}$ (by help of π). As $\|\bar{U}_{\bar{y}}\| < j$, then all of this is true also for $\bar{U}_{\bar{y}}$. Let $U_{\bar{y}}(x) = \pi \circ \bar{U}_{\bar{y}}(x)$. According to the construction, this mapping belongs to $p_N^{-1}(\bar{y})$, so, $U_{\bar{y}}(x_i) = y_i$, $i = \overline{1, N}$.

***Remark* 3.2.** Due to the smoothness of π, $\|U_{\bar{y}}\|_s \le C \cdot \|\bar{U}_{\bar{y}}\|_s < C \cdot j$. Thus, $U_{\bar{y}} \notin X_j$, but $U_{\bar{y}} \in X_{C \cdot j}$.

Let $\exp_y : T_y \mathbf{N} \to \mathbf{N}$ be the exponential mapping. Obviously, $\exp_y$ is diffeomorphism between some neighborhoods of zero (in $T_y\mathbf{N}$) and of point $y \in \mathbf{N}$. Let's denote these neighborhoods by $\delta_1(y)$ and $\varepsilon_1(y)$, relatively. We can suppose that $\varepsilon_1(y)$ and $\delta_1(y)$ are independent from $y \in \mathbf{N}$, as $\exp_y$ is smooth and $\mathbf{N}$ is compact.

Analogously to proved above, one can show that the ε_1-neighborhood of $U_{\bar{y}}(x)$, $\bar{y} \in B_j$ includes all $u(x)$ from $p_N^{-1}\left(p_N(U_{\bar{y}})\right) \cap X_{C \cdot j}$ when δ is small enough.

Let $\bar{y}_0 \in B_j$. Let's take n of vector fields in neighborhood of $U_{\bar{y}_0}(x)$, which are tangential to $\mathbf{N}$, orthogonal to each other and have the unit length. Let's denote them by $\vec{g}_1(y),\ldots,\vec{g}_n(y)$. According to the Lemma 3.1 this vector fields will be defined lengthways of each $U_{\bar{y}}(x)$, where $\bar{y} \in \theta_{\bar{y}_0}$, and $\theta_{\bar{y}_0}$ is γ-neighborhood of point $\bar{y}_0$ in B_j. B_j can be covered by help of finite-number of similar γ-neighborhoods, because it is relatively compact and finite-dimensional. Let's denote them by $\theta_{\bar{y}_1},\ldots,\theta_{\bar{y}_l}$, where $\bar{y}_1,\ldots,\bar{y}_l$ are some points from B_j. Let

$$F^N = \left\{ \vec{v} \in \mathbf{M} \to R^n \,|\, \vec{v} \in H_s, \vec{v}(x_1) = \ldots = \vec{v}(x_N) = 0 \right\};$$

it is a linear subspace of $H_s(\mathbf{M}, R^n)$ of finite co-dimension nN, where $\{\vec{e}_1,\ldots,\vec{e}_n\}$ is an orthonormal basis in R^n. Obviously, any function $\vec{v} \in F^N$ has the following form in this basis:

$$\vec{v}(x) = v_1(x)\cdot\vec{e}_1 + \ldots + v_n(x)\cdot\vec{e}_n,$$

where $v_k(k)$, $k = \overline{1,n}$, is a scalar function, $v_k \in H_s(\mathbf{M}, R^1)$, $v_k(x_i) = 0$, $k = \overline{1,n}$, $i = \overline{1,N}$.

Let's consider the mapping

$$\Phi_p : \theta_{\bar{y}_p} \times F^N \to p_N^{-1}\left(\theta_{\bar{y}_p}\right), \quad \Phi_p\left(\bar{y}, \vec{v}\right)(x) = \exp_{U_{\bar{y}(x)}} \vec{g}(x), \quad p = \overline{1,l},$$

where

$$\vec{g}(x) = v_1(x)\cdot\vec{g}_1\left(U_{\bar{y}}(x)\right) + \ldots + v_n(x)\cdot\vec{g}_n\left(U_{\bar{y}}(x)\right).$$

Obviously,

1) $\Phi_p\left(\bar{y}', \vec{v}\right) \neq \Phi_p\left(\bar{y}'', \vec{w}\right)$ $\forall \vec{v}, \vec{w} \in F^N$ at $\bar{y}' \neq \bar{y}''$, $\bar{y}', \bar{y}'' \in \theta_{\bar{y}_p}$, as (according to construction) $\Phi_p\left(\bar{y}', \vec{v}\right) \in p_N^{-1}\left(\bar{y}'\right)$, and $\Phi_p\left(\bar{y}'', \vec{w}\right) \in p_N^{-1}\left(\bar{y}''\right)$;

2) $\Phi_p\left(\bar{y}, \vec{v}\right) \neq \Phi_p\left(\bar{y}, \vec{w}\right)$ $\forall \bar{y} \in \theta_{\bar{y}_p}$ $\forall p = \overline{1,l}$ at $\|\vec{v}\|_C < \delta_1$, $\|\vec{w}\|_C < \delta_1$ and $\vec{v} \neq \vec{w}$, as $\exp_y$ is diffeomorphism in δ_1-neighborhood of $0_y \in T_y\mathbf{N}$.

It follows from here that Φ_p, $p = \overline{1,l}$, is a diffeomorphism between $\theta_{\bar{y}_p} \times \times \left\{ \vec{v} \in F^N \,|\, \|\vec{v}\|_C < \delta_1 \right\}$ and neighborhood $\left\{ u(x) \,\middle|\, \left\|U_{\bar{y}}(x) - u(x)\right\|_C < \varepsilon_1 \right\}$, where $\bar{y} \in \theta_{\bar{y}_p}$, $p_N\left(u(x_i)\right) = p_N\left(U_{\bar{y}}(x_i)\right)$, $i = \overline{1,N}$. According to the construction, this neighborhood contains the set $p_N^{-1}\left(\theta_{\bar{y}_p}\right) \cap X_j$. Obviously, $D_p = = \Phi_p^{-1}\left(p_N^{-1}(\theta_{\bar{y}_p}) \cap X_j\right)$ is a bounded domain in $\theta_{\bar{y}_p} \times F^N$. Let's paste D_p, $D_{p'}$, $p, p' = \overline{1,l}$, by the help of $\Phi_{p'}^{-1} \circ \Phi_p$; as a result we shall receive some set D_j.

Let's construct an affine bundle, in which D_j will be a bounded domain. Let $\vec{g}_{1,p}(y),\ldots,\vec{g}_{n,p}(y)$ and $\vec{g}_{1,p'}(y),\ldots,\vec{g}_{n,p'}(y)$ be the two vector fields, defined (as above) in neighborhoods of $U_{\overline{y}_p}(x)$ and $U_{\overline{y}_{p'}}(x)$ respectively and $\overline{y} \in \theta_{\overline{y}_p} \bigcap \theta_{\overline{y}_{p'}}$. Besides, let $\lambda_{p,p',\overline{y}}(x)$ be an orthogonal matrix, which transforms the first basis into the second in the point $y = U_{\overline{y}}(x)$. The diffeomorphism $\Phi_{p'}^{-1} \circ \Phi_p$ will transform $(\overline{y}, \vec{v}) \in \theta_{\overline{y}_p} \times F^N$ into $(\overline{y}, \vec{w}) \in \theta_{\overline{y}_{p'}} \times F^N$, where

$$\vec{w}(x) = \lambda_{p,p',\overline{y}}(x) \cdot \vec{v}(x). \tag{3.1}$$

The function (3.1) is a linear isomorphism, which smoothly depends on $\overline{y} \in \theta_{\overline{y}_p} \bigcap \theta_{\overline{y}_{p'}}$. Pasting all $\theta_{\overline{y}_p} \times F^N$, $p = \overline{1,l}$, by the help of $\Phi_{p'}^{-1} \circ \Phi_p$, we shall receive an affine bundle, which we will denote by (Y_j, P_j, B_j). According to the construction, D_j will be the bounded domain in Y_j. Now let's paste $\Phi_1, \ldots, \Phi_l$ by the help of transition functions; as a consequence we shall receive one diffeomorphism between D_j and X_j, which we shall denote by Φ_j. Thus, construction of the L-chart (Φ_j^{-1}, X_j) on X_j is finished.

Now we shall show that L-structures on X_j and X_i are coordinated for different j and i. For this purpose it is enough to prove that transition function $\Phi_i^{-1} \circ \Phi_j$ is a FQL-mapping between affine bundles (Y_j, P_j, B_j) and (Y_i, P_i, B_i). Let $(x_1, \ldots, x_N)$, $(x_1', \ldots, x_L')$ be points from $\mathbf{M}$, which have been used at definition of L-structures on X_j, X_i and $\overline{y} = (y_1, \ldots, y_N)$, $\overline{y}' = (y_1', \ldots, y_L')$ be points from B_j, B_i respectively. Moreover, let $U_{\overline{y}}(x)$, $U_{\overline{y}'}(x)$ be mappings, constructed by the help of above mentioned method, $\vec{g}_1(y), \ldots, \vec{g}_n(y)$ and $\vec{g}_1'(y), \ldots, \vec{g}_n'(y)$ be the vector fields, defined (as above) in the neighborhoods of $U_{\overline{y}}(x)$, $U_{\overline{y}'}(x)$, respectively.

Let

$$F^N = \left\{ \vec{v} \in H_s(\mathbf{M}, R^n) \,\|\, \vec{v}(x_1) = \ldots = \vec{v}(x_N) = 0 \right\},$$

$$F^L = \left\{ \vec{v} \in H_s(\mathbf{M}, R^n) \,\|\, \vec{v}(x_1') = \ldots = \vec{v}(x_L') = 0 \right\},$$

be vector subspaces of $H_s(\mathbf{M}, R^n$, which are isomorphic to layers of affine bundles (Y_j, P_j, B_j), (Y_i, P_i, B_i) respectively. Without loss of generality, we can suppose that $x_m \neq x_r'$, $m = \overline{1,N}$, $r = \overline{1,L}$. Let

$$[\mathbf{N}]^N F^{N+L} = \left\{ \vec{v} \in H_s(\mathbf{M}, R^n) \,|\, \vec{v}(x_m) = \vec{v}(x_r') = 0, \quad m = \overline{1,N}, \quad r = \overline{1,L} \right\}.$$

Obviously, $F^N = F^{N+L} + F_L$, where F_L is orthogonal complement to F^{N+L} in F^N and $\theta_{\overline{y}_p} \times F^N = \left(\theta_{\overline{y}_p} \times F_L\right) \times F^{N+L}$. Pasting $\left(\theta_{\overline{y}_p} \times F_L\right) \times F^{N+L}$, $p = \overline{1,l}$, by the help of diffeomorphisms (3.1), we shall get a new affine bundle. Let's denote it by $\left(Y_j, P_{ji}, B_{ji}\right)$.

Let $(\bar{y}, \vec{z}) \in \theta_{\bar{y}_p} \times F_L$. Let's look at the function

$$u(x) = \exp_{U_{\bar{y}}(x)}\left(\sum_{k=1}^{n}(z_k(x)+v_k(x))\cdot \vec{g}_k\left(U_{\bar{y}}(x)\right)\right),$$

where $v_k(x_m) = v_k(x'_r) = 0$, that is $\vec{v} = (v_1, \ldots, v_n) \in F^{N+L}$. For each such $u(x)$, $u(x_m) = y_m$, $u(x'_r) = y'_r$, $m = \overline{1, N}$, $r = \overline{1, L}$. Therefore,

$$\exp^{-1}_{U_{\bar{y}'}(x)} u(x) = (\bar{y}', \vec{w}(x)),$$

where $\bar{y}' = (y'_1, \ldots, y'_L)$, $\vec{w}(x) = (w_1(x), \ldots, w_n(x))$. Thus $\Phi_i^{-1} \circ \Phi_j$ will transform the layer $P_{ji}^{-1}(\bar{y}, \vec{z})$ above $(\bar{y}, \vec{z})$ into the layer $P_i^{-1}(\bar{y}')$ above $\bar{y}'$, where $\bar{y}' = (u(x'_1), \ldots, u(x'_L))$. Then it will transform $P_{ji}^{-1}\left(\theta_{\bar{y},\vec{z}}\right)$ in $P_i^{-1}\left(\theta_{\bar{y}'_q}\right)$, where $\bar{y}' \in \theta_{\bar{y}'_q}$, $\theta_{\bar{y}'_q}$ is a chart of a fixed atlas on B_i, and $\theta_{\bar{y},\vec{z}}$ is a neighborhood of $(\bar{y}, \vec{z})$ in B_{ji}. This transition function has the following form:

$$(\bar{y}, \vec{z}, \vec{v}) \mapsto (\bar{y}', \vec{w}) = (\bar{y}', (w_1, \ldots, w_n)),$$

where

$$\bar{y}' = (u(x'_1), \ldots, u(x'_L)), \qquad u = \Phi_j(\bar{y}, \vec{z} + \vec{v}),$$

and

$$w_k(x) = \left(\vec{g}'_k\left(U_{\bar{y}'}(x)\right), \vec{h}(x)\right), \quad k = \overline{1, n},$$

is the scalar product of vectors, tangential to $\mathbf{N}$ in point $U_{\bar{y}'}(x)$,

$$\vec{h}(x) = \exp^{-1}_{U_{\bar{y}'}(x)} u(x) \quad \left(\vec{h}(x) \in T_{U_{\bar{y}'}(x)}\mathbf{N}\right).$$

It is obvious from the foresaid formulas that in charts of the mentioned affine bundles, the transition function $\Phi_i^{-1} \circ \Phi_j$ is given by an operator of superposition with smooth functions. As all used functions have gradients different from zero in all points, then according to the Theorem 6.3, such a function is an FQL-mapping. Therefore, it is an FQL-mapping between L-charts on X_j and X_i. It follows from here that the structure included in X is Fredholm Quasi-Linear.

4. Example of FQL-manifold (the case $m \geq n$). For simplicity, let's suppose that $m < 2n$. Let $R^{4n+2} = R^{2n+1} \times R^{2n+1}$, and $\mathbf{N}^2 = \mathbf{N} \times \mathbf{N}$ is embedded in R^{4n+2} such that $\mathbf{N}$ is embedded in. Let $X = H_s(\mathbf{M}, \mathbf{N})$. Obviously, $X^2 = H_s(\mathbf{M}, \mathbf{N}^2)$. Let $X_j^2 = X_j \times X_j$, $X_{j,0} = X_j \times O \subset X_j^2$, where $X_j = \{u \in X \mid \|u\|_s < j\}$, $O : \mathbf{M} \to 0$ and $\mathbf{N}_0 = \mathbf{N} \times 0$, where 0 is the origin of R^{4n+2}. In addition, let $\{x_1,\ldots,x_N\}$ be a δ-network of $\mathbf{M}$. Without loss of generality, also let's suppose that the origin of R^{4n+2} coincides with a point of $\mathbf{N}_0$ and consequently, of $\mathbf{N}^2$. To each mapping $u=(u_1,u_2)\in X_j^2$ we shall assign the point

$$p_{N^2}(u_1,u_2) \;=\; \big((u_1(x_1),u_2(x_1)),\ldots,(u_1(x_N),u_2(x_N))\big)\in[\mathbf{N}^2]^{\mathrm{N}},$$

and to each mapping $(u_1,O)\in X_{j,0}$, the point

$$\tilde{p}_N(u_1,O) \;=\; \big((u_1(x_1),O),\ldots,(u_1(x_N),O)\big)\in[\mathbf{N}_0]^{\mathrm{N}}.$$

Let

$$\tilde{B}_j \;=\; \Big\{\tilde{y}=(y_1,y_2)=((y_{11},y_{21}),\ldots,(y_{1N},y_{2N}))\in[\mathbf{N}^2]^{\mathrm{N}}\,\Big|\,\exists(u_1,u_2)\in X_j^2:$$
$$(u_1(x_1),u_2(x_1))=(y_{11},y_{21}),\ldots,(u_1(x_N),u_2(x_N))=(y_{1N},y_{2N})\Big\},$$

$$B_{j,0} \;=\; \Big\{(y_1,0)=((y_{11},0),\ldots,(y_{1N},0))\in[\mathbf{N}_0]^{\mathrm{N}}\,\Big|\,\exists(u_1,O)\in X_{j,0}:$$
$$(u_1(x_1),0)=(y_{11},0),\ldots,(u_1(x_N),0)=(y_{1N},0)\Big\}.$$

Obviously, $\tilde{B}_j$ $(B_{j,0})$ is a domain in $[\mathbf{N}^2]^{\mathrm{N}}$ $\big([\mathbf{N}_0]^{\mathrm{N}}\big)$, therefore it is a $2nN$ (nN)-dimensional manifold. Moreover, $B_{j,0}$ will be submanifold of $\tilde{B}_j$ and

$$\forall(y_1,0)\in B_{j,0}:\ \tilde{p}_N^{-1}(y_1,0)\subset p_{N^2}^{-1}(y_1,0).$$

Let's denote a mapping by Π, which transforms each point (y_1,y_2) of the ε-tubular neighborhood of $\mathbf{N}^2$ (in R^{4n+2}) into the nearest point of $\mathbf{N}^2$. By the help of the aforecited method (see, the case $m<n$) to each point $\tilde{y}\in\tilde{B}_j$ at first, we shall assign such a mapping $\bar{U}_{\tilde{y}}=\big(\bar{U}_{1,\tilde{y}},\bar{U}_{2,\tilde{y}}\big)\in H_s\big(\mathbf{M},R^{2n+1}\times R^{2n+1}\big)$ that $\bar{U}_{\tilde{y}}(x_i)=$ $=(y_{1i},y_{2i})$, $i=\overline{1,N}$, and later, we shall assign a mapping $U_{\tilde{y}}=\big(U_{1,\tilde{y}},U_{2,\tilde{y}}\big)=$ $=\Pi\big(\bar{U}_{1,\tilde{y}},\bar{U}_{2,\tilde{y}}\big)\in H_s(\mathbf{M},\mathbf{N}^2)$, which also will satisfy the condition $U_{\tilde{y}}(x_i)=$ $=(y_{1i},y_{2i})$, $i=\overline{1,N}$.

We need the following in advance.

Lemma 4.1. *Let* $\bar{U}_{(y_1,0)}=\big(\bar{U}_{1,(y_1,0)},\bar{U}_{2,(y_1,0)}\big)\in H_s\big(\mathbf{M},R^{2n+1}\times R^{2n+1}\big)$ *be a mapping such that* $\bar{U}_{(y_1,0)}(x_i)=\big(\bar{U}_{1,(y_1,0)}(x_i),\bar{U}_{2,(y_1,0)}(x_i)\big)=(y_{1i},0)$, $i=\overline{1,N}$, *and in addition,* $\big\|\bar{U}_{(y_1,0)}\big\|_s'$ *has a minimum among all such mappings. Then* $\bar{U}_{2,(y_1,0)}\equiv O$, *in other words, such* $\bar{U}_{(y_1,0)}$ *will belong to* $H_s(\mathbf{M},R^{2n+1}\times R^{2n+1})$.

Here $\|(u_1,u_2)\|_s'=\|u_1\|_s+\|u_2\|_s$ *and* $\|\ \|_s$ *are the norms in* $H_s(\mathbf{M},R^{4n+1})$ *and* $H_s(\mathbf{M},R^{2n+1})$ *respectively.*

Proof. As $(y_{1i}, 0) \in R^{2n+1}$, $i = \overline{1, N}$, then obviously, such $\bar{U}_{(y_1, 0)}$ will belong to $H_s(\mathbf{M}, R^{2n+1} \times 0)$.

According to the Lemma 4.1, the mapping $U_{(y_1, 0)} = \Pi \circ \bar{U}_{(y_1, 0)}$ will belong to $H_s(\mathbf{M}, \mathbf{N}_0)$.

Let $\tilde{y}_0 \in \tilde{B}_j$. Let's take $2n$ vector fields in neighborhood of $U_{\tilde{y}_0}(x)$, which are tangential to $\mathbf{N}^2$, orthogonal to each other and have the unit length. Let's denote them by $\vec{g}_1(y), \ldots, \vec{g}_{2n}(y)$. According to the Lemma 3.1, these vector fields will be defined lengthways of each $U_{\tilde{y}}(x)$, where $\tilde{y} \in \tilde{\theta}_{\tilde{y}_0}$ and $\tilde{\theta}_{\tilde{y}_0}$ is γ-neighborhood of point $\tilde{y}_0$ in $\tilde{B}_j$. One can cover $\tilde{B}_j$ by the help of finite number of similar γ-neighborhoods $\tilde{\theta}_{\tilde{y}_1}, \ldots, \tilde{\theta}_{\tilde{y}_{l'}}$ ($\tilde{y}_1, \ldots,$ are points from $\tilde{B}_j$), because $\tilde{B}_j$ is relatively compact and finite-dimensional. Let $\theta_{p,0} = \tilde{\theta}_{\tilde{y}_p} \cap B_{j,0}$, $p = \overline{1, l'}$. Obviously, the collection $\{\{\theta_{1,0}, \ldots, \theta_{l',0}\}$ will cover $B_{j,0}$. Let

$$F^{2N} = \left\{ \vec{\nu} \in H_s(\mathbf{M}, R^{2n}) \,|\, \vec{\nu}(x_1) = \ldots = \vec{\nu}(x_N) = 0 \right\},$$

it is a linear subspace of $H_s(\mathbf{M}, R^{2n})$ of finite co-dimension $2nN$. Let $\{\vec{e}_1, \ldots, \vec{e}_{2n}\}$ be an orthonormal basis in R^{2n}. Obviously, each function $\vec{v} \in F^{2N}$ will have in this basis the following form:

$$\vec{v}(x) = v_1(x) \cdot \vec{e}_1 + \ldots + v_{2n}(x) \cdot \vec{e}_{2n}.$$

Here $v_k(x)$, $k = \overline{1, 2n}$, is a scalar function, $v_k \in H_s(\mathbf{M}, R^1)$, $v_k(x_i) = 0$, $k = \overline{1, 2n}$, $i = \overline{1, N}$. Let's consider a mapping

$$\tilde{\mathbf{\Phi}}_p : \tilde{\theta}_{\tilde{y}_p} \times F^{2N} \to p_{N^2}^{-1}\left(\tilde{\theta}_{\tilde{y}_p}\right), \quad \tilde{\mathbf{\Phi}}_p(\tilde{y}, \vec{v})(x) = \exp_{U_{\tilde{y}}(x)} \vec{g}(x), \quad p = \overline{1, l'},$$

where

$$\vec{g}(x) = v_1(x) \cdot \vec{g}_1\left(U_{\tilde{y}}(x)\right) + \ldots + v_{2n}(x) \cdot \vec{g}_{2n}\left(U_{\tilde{y}}(x)\right), \tag{4.1}$$

$$\tilde{y} \in \tilde{\theta}_{\tilde{y}_p}, \quad p = \overline{1, l'}.$$

As in the case $m < n$, one can show that the $\tilde{\mathbf{\Phi}}_p$, $p = \overline{1, l'}$, is a diffeomorphism between $\tilde{\theta}_{\tilde{y}_p} \times \left\{\left\{ \vec{v} \in F^{2N} \,|\, \|\vec{v}\|_C < \delta_1 \right\}\right.$ and neighborhood $\left\{ u(x) \,\left|\, \left\| U_{\tilde{y}}(x) - u(x) \right\|_C < \right.\right.$ $\left. < \varepsilon_1 \right\}$, where $\tilde{y} \in \tilde{\theta}_{\tilde{y}_p}$, $p_{N^2}(u(x_i)) = p_{N^2}\left(U_{\tilde{y}}(x_i)\right)$, $i = \overline{1, N}$. According to the construction, this neighborhood contains the set $p_{N^2}^{-1}\left(\tilde{\theta}_{\tilde{y}_p}\right) \cap X_j^2$.

Now let's construct such subbundle of $\tilde{\theta}_{\tilde{y}_p} \times F^{2N}$, which $\tilde{\mathbf{\Phi}}_p$ would transform onto $\tilde{p}_N^{-1}(\theta_{p,0})$. Let $\tilde{T}_{\tilde{y}}$, $\tilde{y} \in \tilde{\theta}_{\tilde{y}_p}$, be a space of mappings $\vec{g} : \mathbf{M} \to T \to T(\mathbf{N}^2)$ of form (4.1). Obviously, it is linear and isomorphic to F^{2N}. In addition, $\tilde{T}_{\tilde{y}'} \cap \cap \tilde{T}_{\tilde{y}''} = \varnothing$ at $\tilde{y}' \neq \tilde{y}''$ and $\tilde{T}_{\tilde{y}}$ continuously depends on $\tilde{y} \in \tilde{\theta}_{\tilde{y}_p}$. Therefore, the family $\left\{ \tilde{T}_{\tilde{y}} \,\middle|\, \tilde{y} \in \tilde{\theta}_{\tilde{y}_p} \right\}$, $p = \overline{1, l'}$ will induce an affine bundle with the total space $\tilde{T}_{\tilde{y}_p} = \bigcup \tilde{T}_{\tilde{y}}$, $\tilde{y} \in \tilde{\theta}_{\tilde{y}_p}$, with the layer F^{2N}, with the projection $\tilde{P}_{\tilde{y}_p}(\vec{g}) = \tilde{y}$ and the base space $\tilde{\theta}_{\tilde{y}_p}$. Let's denote it by $\left(\tilde{T}_{\tilde{y}_p}, \tilde{P}_{\tilde{y}_p}, \tilde{\theta}_{\tilde{y}_p} \right)$. Obviously, the mapping

$$\tilde{G}_p : \tilde{\theta}_{\tilde{y}_p} \times F^{2N} \to \tilde{T}_{\tilde{y}_p}, \quad \tilde{G}_p(\tilde{y}, \vec{v}) = \vec{g},$$

$$\tilde{y} \in \tilde{\theta}_{\tilde{y}_p}, \quad \vec{v} \in F^{2N}, \quad p = \overline{1, l'},$$

will be an isomorphism between Cartesian product $\tilde{\theta}_{\tilde{y}_p} \times F^{2N}$ and $\left(\tilde{T}_{\tilde{y}_p}, \tilde{P}_{\tilde{y}_p}, \tilde{\theta}_{\tilde{y}_p} \right)$.

Let $T_{(y_1, 0)}$, $(y_1, 0) \in \theta_{p,0}$ be a space of mappings $\vec{g} : \mathbf{M} \to T \to T(\mathbf{N}_0)$, where

$$\vec{g}(x) = \sum_{k=1}^{2n} v_k(x) \cdot \vec{g}_k \left(U_{(y_1,0)}(x) \right), \quad (y_1, 0) \in \theta_{p,0}, \quad \vec{v} \in F^{2N}.$$

For each $(y_1, 0) \in \theta_{p,0}$, $T_{(y_1,0)}$ will be linear subspace of $\tilde{T}_{\tilde{y}}$, $\tilde{y} \in \tilde{\theta}_{\tilde{y}_p}$, where $\tilde{y} = = \tilde{y} = (y_1, 0)$. In addition, $T_{(y_1,0)}$ continuously depends on $(y_1, 0) \in \theta_{p,0}$ and $T_{(y_1',0)} \cap T_{(y_1'',0)} = \varnothing$, when $(y_1', 0) \neq (y_1'', 0)$. Therefore, the family $\left\{ T_{(y_1,0)} \,\middle|\, (y_1, 0) \in \theta_{p,0} \right\} = \varnothing$, $p = \overline{1, l'}$, will induce an affine bundle with the total space $T_{p,0} = \bigcup T_{(y_1,0)}$, $(y_1, 0) \in \theta_{p,0}$, with the projection $P_{p,0}(\vec{g}) = (y_1, 0)$ and the base space $\theta_{p,0}$. Let's denote it by $(T_{p,0}, P_{p,0}, \theta_{p,0})$. According to the construction, it will be a subbundle of $\left(\tilde{T}_{\tilde{y}_p}, \tilde{P}_{\tilde{y}_p}, \tilde{\theta}_{\tilde{y}_p} \right)$. As $\tilde{G}_p$ is the isomorphism, then $\tilde{G}_p^{-1}(T_{p,0})$ will be an affine subbundle of $\theta_{p,0} \times F^{2N}$. According to the construction, the mapping $\tilde{\Phi}_p$ will transform $\tilde{G}_p^{-1}(T_{p,0})$ onto $\tilde{p}_N^{-1}(\theta_{p,0})$. Obviously, $\tilde{D}_p = = \tilde{\Phi}_p^{-1}\left(p_{N^2}^{-1}\left(\tilde{\theta}_{\tilde{y}_p} \right) \cap X_j^2 \right)$ $\Big($and also $D_{p,0} = \tilde{\Phi}_p^{-1}\left(\tilde{p}_N^{-1}(\theta_{p,0}) \cap (X_{j,0}) \right)$ is a bounded domain in $\tilde{\theta}_{\tilde{y}_p} \times F^{2N}$ (accordingly, in $\tilde{G}_p^{-1}(T_{p,0})$). Let's paste $\tilde{D}_p$ and $\tilde{D}_{p'}$ (and also $D_{p,0}$ and $D_{p',0}$), $p, p' = \overline{1, l'}$, by the help of mappings $\tilde{\Phi}_{p'}^{-1} \circ \tilde{\Phi}_p$; as a result we shall receive a set $\tilde{D}_j$ (accordingly, $D_{j,0}$).

Now let's construct such two affine bundles that in one of them each $\tilde{D}_j$ and in the other each $D_{j,0}$ will be a bounded domain. Let $\vec{g}_{1,p}(y), \ldots, \vec{g}_{2n,p}(y)$ and $\vec{g}_{1,p'}(y), \ldots, \vec{g}_{2n,p'}$ are vector fields, defined (as above) in neighborhoods of $U_{\tilde{y}_p}(x)$ and $U_{\tilde{y}_{p'}}(x)$, respectively. Let $\tilde{y} \in \tilde{\theta}_{\tilde{y}_p} \cap \tilde{\theta}_{\tilde{y}_{p'}}$ and $\mu_{p,p',\tilde{y}}(x)$ is an orthogonal matrix, which transforms the first basis onto second basis in point $y = U_{\tilde{y}}(x)$. The diffeomorphism $\tilde{\Phi}_{p'}^{-1} \circ \tilde{\Phi}_p$ will transform the element $(\tilde{y}, \vec{v}) \in \tilde{\theta}_{\tilde{y}_p} \times F^{2N}$ into $(\tilde{y}, \vec{w}) \in \tilde{\theta}_{\tilde{y}_{p'}} \times F^{2N}$, where

$$\vec{w}(x) = \mu_{p,p',\tilde{y}}(x) \cdot \vec{v}(x). \tag{4.2}$$

The function (4.2) is the linear isomorphism, smoothly depending on $\tilde{y} \in \tilde{\theta}_{\tilde{y}_p} \cap \tilde{\theta}_{\tilde{y}_{p'}}$. Pasting together all $\tilde{\theta}_{\tilde{y}_p} \times F^{2N}$ (and also all $\tilde{G}_p^{-1}(T_{p,0})$), $p = \overline{1, l'}$, by the help of mappings $\tilde{\Phi}_{p'}^{-1} \circ \tilde{\Phi}_p$, we shall receive an affine bundle (accordingly, a subbundle). Let's denote it by $(\tilde{Y}_j, \tilde{P}_j, \tilde{B}_j)$ (accordingly, by $(Y_{j,0}, P_{j,0}, B_{j,0})$). According to the construction, $\tilde{D}_j$ (and also $D_{j,0}$) will be the bounded domain in Y_j (accordingly, in $Y_{j,0}$).

Now we shall paste together $\tilde{\Phi}_1, \ldots, \tilde{\Phi}_l$ by the help of transition functions; as a consequence we shall receive one diffeomorphism between $\tilde{D}_j$ (and also $D_{j,0}$) and X_j^2 (accordingly, $X_{j,0}$), which we will denote by $\tilde{\Phi}_j$. Thus, the construction of the L-chart (Φ_j^{-1}, X_j^2) (and also $(\tilde{\Phi}_j^{-1}, X_{j,0})$) on X_j^2 (accordingly, on $X_{j,0}$), is completed.

Similarly to the case $m < n$, the transition function between affine bundles $(\tilde{Y}_j, \tilde{P}_j, \tilde{B}_j)$ and $(\tilde{Y}_i, \tilde{P}_i, \tilde{B}_i)$ will be an FQL-mapping. Therefore, the L-structures on X_j^2 and X_i^2, $j \neq i$, will be coordinated. It follows from here that $\tilde{\Phi}_i^{-1} \circ \tilde{\Phi}_j$ will be an FQL-mapping between subbundles $(Y_{j,0}, P_{j,0}, B_{j,0})$ and $(Y_{i,0}, P_{i,0}, B_{i,0})$, too. In other words, the L-structures on $X_{j,0}$ and $X_{i,0}$, $j \neq i$, also will be coordinated. Thus, the structure, introduced in X, will also be Fredholm Quasi-Linear.

Remark. It is obvious from all of the above-established facts that at $(n-1) \cdot k \leq m < n \cdot k$, $k \geq 3$, all constructions and proofs will be similar to the case $m < 2n$.

5. A degree of FSQL-mapping. At the definition of the degree of FSQL-mapping between FQL-manifolds we shall consider a more simple case, namely when the following conditions are satisfied:

(1) FQL-manifolds X and X' are embedded in Banach spaces E_1 and E_2, respectively.

(2) The open sets X_j and X_i' (see, the definition of FQL-manifold) have forms $X_j = X \cap B_1(R_j)$ and $X_i' = X' \cap B_2(r_i)$, where $B_1(R_j)$ and $B_2(r_i)$ are the open balls in E_1 and E_2 with centers at zero and of the radiuses R_j and r_i, respectively, R_j, $r_i \to \infty$, when $j, i \to \infty$.

(3) For each j and i, the L-charts φ_j, φ_j^{-1}, φ_i', $(\varphi_i')^{-1}$ are uniformly continuous.

(4) FSQL-mapping $f : X \to X'$ satisfies the following a priori estimate

$$\|x\|_1 \leq \Phi\left(\|f(x)\|_2\right), \tag{5.1}$$

where Φ is a positive monotone function, and $\|\cdot\|_1$, $\|\cdot\|_2$ are norms in E_1 and E_2, respectively.

Now we shall start defining the degree of FSQL-mapping $f : X \to X'$. For simplicity let's suppose that Φ is the identical mapping. Let's consider the equation

$$f(x) = x_0', \quad x_0' \in X'. \tag{5.2}$$

When condition (5.1) is satisfied, all the solutions of the equation (5.2) will belong to $X_{R_0} = X \bigcap B_1(R_0)$, $R_0 = \|x_0'\|_2$. According to the assumption,

$$\exists j_0 \quad \forall j \geq j_0 : \ X_j \supset X_{R_0},$$

and according to the Definition 2.6,

$$\exists i_0 \quad \forall i \geq i_0 : \ f(X_j) \subset X_i'.$$

Let j and i be numbers, for which all above mentioned conditions are satisfied. Then, while defining deg (f) in the point $x_0' \in X'$ we may consider the restriction of the mapping f onto X_j. As φ_j and φ_i' are the homeomorphisms, then to solve equation (5.2) in X_{R_0} will be equivalent to solve equation

$$f_{ji}(y) = y_0', \qquad y_0' = \varphi_i'(x_0')$$

in $\varphi_j(X_{R_0})$, where $f_{ji} \equiv \varphi_i' \circ f \circ \varphi_j^{-1}$. According to Definition 2.6, f_{ji} is an FSQL-mapping between affine bundles ξ_j and ξ_i'. Let $\left\{ f_{ji,r} \right\}$ be a sequence of FSL-mappings, which uniformly converges to f_{ji} on D_j. Let's consider the equation

$$f_{ji,r}(y) = y_0', \quad y_0' = \varphi_i'(x_0'); \tag{5.3}$$

we will search its solutions in $\varphi_j(X_{R_0'})$, where $X_{R_0'} = X \bigcap B_1\left(R_0'\right)$, $R_0' = \|x_0'\|_2 + 2\delta$, $\delta > 0$.

***Remark* 5.1.** Obviously, $X_{R_0'} \subset X_j$ when j is big enough, therefore $\varphi_j(X_{R_0'}) \subset D_j$.

This problem can be transformed to finite-dimension problem. Indeed, as $f_{ji,r}$ is a bimorphism, then it will induce the finite-dimensional continuous mapping

$$g_{ji,r} : \ B_{j,r} \to B_{i,r}'$$

between base spaces of affine bundles ξ_j and ξ_i'. Let's consider this finite-dimensional equation

$$g_{ji,r}(\beta) = \beta_0', \qquad \beta_0' = P_{i,r}'(y_0'), \tag{5.4}$$

where $P'_{i,r}$ is the projection of subbundle $\xi'_{i,r} = \left(Y'_i, P'_{i,r}, B'_{i,r}\right)$. Let's prove that when r is big enough; finding the solutions of equation (5.3) is equivalent to finding the solutions of equation (5.4).

Indeed, let $y \in \varphi_j\left(X_{R'_0}\right)$ and it is a solution of equation (5.3). Obviously, there exists unique $\beta \in P_{j,r}\left(\varphi_j\left(X_{R'_0}\right)\right)$, such that $y \in P_{j,r}^{-1}(\beta)$ and $f_{ji,r,\beta}(y) = y'_0$, where $P_{j,r}$ is the projection of subbundle $\xi_{j,r} = \left(Y_j, P_{j,r}, B_{j,r}\right)$, and $f_{ji,r,\beta}$ is the restriction of $f_{ji,r}$ onto layer $Y_{j,\beta} = P_{j,r}^{-1}(\beta)$. Therefore, $f_{ji,r,\beta}\left(Y_{j,\beta}\right) = Y'_{i,\beta'_0}$, where Y'_{i,β'_0} is the layer of $\xi'_{i,r} = \left(Y'_i, P'_{i,r}, B'_{i,r}\right)$, which contains point y'_0. Therefore, β will be solution of the equation (5.4).

Conversely, let β be a solution of (5.4). This means that

$$f_{ji,r,\beta}\left(Y_{j,\beta}\right) = Y'_{i,\beta'_0}.$$

As $f_{ji,r,\beta}$ is an isomorphism, then there exists unique point $y \in P_{j,r}^{-1}(\beta)$ such that

$$f_{ji,r,\beta}(y) = y'_0, \tag{5.5}$$

i.e., the equation (5.3) is solved. Let's show that $y \notin \varphi_j\left(X_{R'_0}\right)$ is not possible. Obviously,

$$\left\| f(x) - (\varphi'_i)^{-1} \circ f_{ji,r} \circ \varphi_j(x) \right\|_2 < \delta, \quad x \in D_j,$$

when r is big enough. If $y \notin \varphi_j(X_{R'_0})$, then $x = \varphi_j^{-1}(y) \notin X_{R'_0}$, i.e., $\|x\|_1 > R'_0$. Then, it follows from estimate (5.1) that

$$\left\| (\varphi'_i)^{-1} \circ f_{ji,r} \circ \varphi_j(x) \right\|_2 \geq \|f(x)\|_2 - \left\| f(x) - (\varphi'_i)^{-1} \circ f_{ji,r} \circ \varphi_j(x) \right\|_2 \geq$$

$$\geq \left(\|x'_0\|_2 + 2\delta \right) - \delta > \|x'_0\|_2,$$

i.e., $(\varphi'_i)^{-1} \circ f_{ji,r} \circ \varphi_j(x) \neq x'_0$, hence $f_{ji,r}(y) \neq y'_0$. This contradicts to equality (5.5). So, $y \in \varphi_j(X_{R'_0})$.

Thus, the equation (5.3) is transformed to finite-dimension equation (5.4). Now we can define the degree of FSL-mapping $f_{ji,r}$.

Definition 5.1. $\deg\left(f_{ji,r}\right) = \deg\left(g_{ji,r}\right)$.

The sign of this degree depends on orientations in $B_{j,r}$ and $B'_{i,r}$, but its absolute value is invariable. The last circumstance is not important for proof of the existence of a solution of (5.2) (see Theorem 5.1 and Definition 5.2).

Theorem 5.1. *Let $f_{ji,r_1}, f_{ji,r_2} \colon Y_j \to Y'_i$ be FSL-mappings, which are close enough to FQL-mapping $f_{i,j} \colon Y_j \to Y'_i$ in D_j. Then*

$$\left|\deg\left(f_{ji,r_1}\right)\right| = \left|\deg\left(f_{ji,r_2}\right)\right|.$$

The proof of this theorem is similar to proof of the Theorem 2.3 from [1].

By the Theorem 5.1 the sequence $\left\{\left|\deg\left(f_{ji,r_1}\right)\right|\right\}$ will be stable when r is big enough. Therefore, we can give the next definition.

Definition 5.2. $\deg\left(f_{ji}\right) = \lim\limits_{r\to\infty}\left|\deg\left(f_{ji,r}\right)\right|$.

As φ_j and φ_i' are homeomorphisms, then we can give the next definition.

Definition 5.3. $\deg(f) = \deg\left(f_{ji}\right)$.

Obviously, $\deg(f)$ does not depend on L-charts on X and Y.

Theorem 5.2. *Let* $\{f_t\}$ *be a family of FSQL-mappings, continuously* (*uniformly in each* X_j) *depending on parameter* $t \in [0,1]$. *Let's suppose also that the conditions* (1)–(4) *are satisfied for each* t. *Then*

$$\deg(f_0, x') = \deg(f_1, x'), \quad x' \in X'.$$

Here the function Φ *does not depend on* t.

Proof. Using compactness of $[0,1]$, uniform continuity (according to t) of the family $\{f_t\}$ and also of the mappings φ_j and $(\varphi_i')^{-1}$, we can approximate the family of FSQL-mappings $f_{t,ji}\colon Y_j \to Y_i'$ by the help of the family $\left\{f_{t,i,j,r}\right\}$ of FSL-mappings. According to the Theorem 5.1, the absolute value of degree of FSL-mapping will be locally stable. Therefore,

$$\left|\deg\left(f_{0,ji,r}, y'\right)\right| = \left|\deg\left(f_{1,ji,r}, y'\right)\right|, \quad y' = \varphi_i'(x'),$$

when r is big enough. Hence

$$\deg\left(f_{0,ji}, y'\right) = \deg\left(f_{1,ji}, y'\right).$$

From here,

$$\deg(f_0, x') = \deg(f_1, x').$$

Theorem 5.2 is proved.

Theorem 5.3. *At the conditions* (1)–(4)

$$\deg(f, x_1') = \deg(f, x_2'), \quad x_1', x_2' \in X'.$$

Proof. Let $X_j \supset X \cap B_1(R)$, $R \geq \Phi\left(\max\left\{\|x_1'\|_2, \|x_2'\|_2 + 2\delta\right\}\right)$ and X_i' is such that $f(X_j) \subset X_i'$. Let $\left\{f_{ji,r}\right\}$ be a sequence of FSL-mappings, which converges to the f_{ji} in D_j. As

$$\left|\deg\left(f_{ji,r}, y_l'\right)\right| = \left|\deg\left(f_{ji}, y_l'\right)\right|, \quad y_l' = \varphi_i'(x_l'), \quad l = 1, 2,$$

when r is big enough, then it is enough to prove that

$$\deg\left(f_{ji,r}, y_1'\right) = \deg\left(f_{ji,r}, y_2'\right).$$

For this purpose it is enough to prove that

$$\deg\left(g_{ji,r}, \beta'_1 \right) = \deg\left(g_{ji,r}, \beta'_2 \right), \quad \beta'_l = P'_{i,r}(y'_l), \quad l = 1, 2 .$$

The last equality is known from the classical (finite-dimensional) analysis.

Theorem 5.3 is proved.

Theorem 5.4. *Let the conditions* (1)–(4) be satisfied and $\deg(f) \neq 0$. *Then the equation* (5.2) *has a solution for each* $x'_0 \in X'$.

The similar theorem has been proved in [3] (see also [8]).

Remark **5.2.** Specific examples of FQL-mappings with calculated degrees (in relatively simple cases) are given in [6] and [8].

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