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NEW RESULTS IN G-BEST APPROXIMATION IN G-METRIC SPACES

НОВІ РЕЗУЛЬТАТИ ЩОДО G-НАЙКРАЩОГО НАБЛИЖЕННЯ В G-МЕТРИЧНИХ ПРОСТОРАХ

The purpose of this paper is to introduce and discuss the concepts of G-best approximation and a_0-orthogonality in G-metric spaces theory. We consider the relation between these concepts and the dual X, and obtain some results on the subsets of G-metric spaces similar to normed spaces.

Мета роботи — ввести та обговорити поняття G-найкращого наближення та a_0-ортогональності в теорії G-метричних просторів. Розглянуто співвідношення між цими поняттями та дуальним X, отримано деякі результати щодо підмножин G-метричних просторів, що подібні до нормованих просторів.

1. Introduction. In 2005, Zead Mustafa and Brailey Sims introduced a new structure of generalized metric spaces (see [1]), which are called G-metric spaces as generalization of metric space (X, d) to develop and introduce a new fixed point theory for a various mappings in this new structure. In this section, we give a brief introduction of G-metric, G-continuous and G-contraction. First, we recall some basic notations of G-metric theory.

Definition 1 [1]. *Let X be a nonempty set and $G: X \times X \times X \to [0, \infty)$ a function satisfying the following properties*:

(G_1) $G(x, y, z) = 0$ *if* $x = y = z$,

(G_2) $0 < G(x, x, y)$ *for all* $x, y \in X$ *with* $x \neq y$,

(G_3) $G(x, x, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$ *with* $z \neq y$,

(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (*symmetry in all three variables*) *and*

(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for all* $x, y, z, a \in X$ (*rectangle inequality*).

Then the function G is called a generalized metric or more specifically, a G-metric on X and the pair (X, G) is called a G-metric space.

Clearly these properties are satisfied when $G(x, y, z)$ is the perimeter of the triangle with vertices at x, y and z in $\mathbf{R}^2$; moreover, taking a in the interior of the triangle shows that (G_5) is the best possible.

If (X, d) is an ordinary metric space, then (X, d) can define G-metrics on X by

(G_s) $\quad G_s(d)(x,y,z) = d(x,y) + d(y,z) + d(x,z)$ *and*

(G_m) $\quad G_m(d)(x,y,z) = \max\{d(x,y),\, d(y,z),\, d(x,z)\}$.

Proposition 1 [1]. *Let (X,G) be a G-metric space. Then for any x, y, z and $a \in X$ it follows that*

(1) *if* $G(x,y,z) = 0$, *then* $x = y = z$,

(2) $G(x,y,z) \le G(x,x,y) + G(x,x,z)$,

(3) $G(x,y,y) \le 2G(y,x,x)$,

(4) $G(x,y,z) \le G(x,a,z) + G(a,y,z)$,

(5) $G(x,y,z) \le (2/3)\big(G(x,y,a) + G(x,a,z) + G(a,y,z)\big)$,

(6) $G(x,y,z) \le \big(G(x,a,a) + G(y,a,a) + G(z,a,a)\big)$.

Proposition 2 [1]. *Every G-metric space (X,G) will define a metric space (X,d_G) by*

$$d_G(x,y) = G(x,y,y) + G(y,x,x) \quad \textit{for all} \quad x,\ y \in X.$$

Definition 2 [1]. *Let (X,G) be a G-metric space. Then for $x_0 \in X$ and $r > 0$ the G-ball with center x_0 and radius r is $B_G(x_0,r) = \{y \in X : G(x_0,y,y) < r\}$.*

Proposition 3 [1]. *Let (X,G) be a G-metric space. Then for any $x_0 \in X$ and $r > 0$ we have*

(i) *if $G(x_0,x,y) < r$, then $x,\ y \in B_G(x_0,r)$,*

(ii) *if $y \in B_G(x_0,r)$, then there exists a $\delta > 0$ such that $B_G(y,\delta) \subseteq$ $\subseteq B_G(x_0,r)$.*

Definition 3 [1]. *Let (X_1,G_1) and (X_2,G_2) be two G-metric spaces and $f : (X_1,G_1) \to (X_2,G_2)$ a function, then f is said to be G-continuous at a point $a \in X_1$ if and only if for given $\varepsilon > 0$, there exists $\delta > 0$ such that $x,\ y \in X_1$ and $G_1(a,x,y) < \delta$ implies $G_2(f(a), f(x), f(y)) < \varepsilon$.*

A function f is G-continuous at X_1 if and only if it is G-continuous at all $a \in X_1$.

Definition 4 [1]. *Let (X,G) be a G-metric space, (x_n) a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence (x_n) if $\lim\limits_{n,m\to\infty} G(x, x_n, x_m) = 0$. In this case we say that the sequence (x_n) is G-convergent to x.*

Thus if $x_n \to x$ in a G-metric space (X,G), then for any $\varepsilon > 0$, there exists $l \in N$ such that $G(x, x_n, x_m) < \varepsilon$ for all $n,\ m > l$.

Definition 5 [1]. *A G-metric space (X,G) is called symmetric G-metric space if*

$$G(x,y,y) = G(x,x,y) \quad \textit{for every} \ x,\ y \in X.$$

Definition 6. *Let (X_1,G_1) and (X_2,G_2) be two G-metric spaces. A function $f : X_1 \to X_2$ is called G-isometric when f preserves distances, i.e.,*

$$G_1(x, y, z) = G_2(f(x), f(y), f(z)) \quad \text{for every } x, y, z \in X.$$

We can show that onto G-isometric maps are G-homeomorphisms (i.e., G-continuous inverse).

Definition 7. *Let (X_1, G_1) and (X_2, G_2) be two G-metric spaces. A function $f: X_1 \to X_2$ is called G-contraction when there is a constant $0 \le k < 1$ such that*

$$G_2(f(x), f(y), f(z)) = kG_1(x, y, z) \quad \text{for every } x, y, z \in X.$$

It follows that f is G-continuous, because

$$G_1(x, y, z) < \delta := \varepsilon / k \quad \Rightarrow \quad G_2(f(x), f(y), f(z)) \le \varepsilon.$$

2. New results. The field of approximation theory has become so vast that it intersects with every other branch of analysis and plays an important role in applications in the applied sciences and engineering. Fixed point theorems have been used in many instances in approximation theory. In the subject of approximation theory one often wishes to know whether some useful properties of the function being approximated are inherited by the approximating function. In this section we will prove several theorems.

Theorem 1. *Let (X, G) be a G-complete space and $f: X \to X$ a G-contraction map. Then f has a unique fixed point $x = f(x)$.*

Proof. We consider the interaction $x_{n+1} = f(x_n)$ with $x_0 = a$ any point in X. Note that

$$G(x_{n+1}, x_n, x_{n-1}) = G(f(x_{n+1}), f(x_n), f(x_{n-1})) \le kG(x_{n+1}, x_n, x_{n-1}).$$

Hence by induction

$$G(x_{n+1}, x_n, x_{n-1}) \le kG(x_2, x_1, x_0).$$

For all $n, m \in N$, $n < m$, by rectangle inequality that

$$G(x_n, x_m, x_m) \le G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) + \ldots$$

$$\ldots + G(x_{m-1}, x_m, x_m) \le (k^n + k^{n+1} + \ldots + k^{m-1})G(x_2, x_1, x_0) \le$$

$$\le \frac{k^n}{1-k} G(x_2, x_1, x_0). \tag{1}$$

Then $G(x_n, x_m, x_m) \to 0$ as $m, n \to \infty$ and thus (x_n) is a G-Cauchy sequence. Due to the completeness of (X, G), there exists a $x \in X$ such that $x_n \to x_0$ and by G-continuity of f

$$x = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} f(x_{n+1}) = f\left(\lim_{n \to \infty} x_n \right) = f(x).$$

Moreover, the rate of convergence is given by (1).

Suppose there are two fixed point $x = f(x)$ and $y = f(y)$, then

$$G(x, y, y) = G(f(x), f(y), f(y)) \le kG(x, y, y)$$

so that $G(x, y, y) = 0$ since $k < 1$, therefore, $x = y$.

The theorem is proved.

3. G-best approximations.

Definition 8. *Let Y be a subspace of a metric space (X,d) and $x \in X$. The point x_0 is a best approximation to x from X if*

(i) $x_0 \in Y$ *and*

(ii) $d(x_0, x) \le d(x, y)$ *for every* $y \in Y$.

For a normed linear space X and $x, y \in X$ a point x is said to be Birkhoff-orthogonal to y and it denoted by $x \perp y$ if and only if $\|x\| \le \|x + \alpha y\|$ for all scalar α (see [2 – 4]).

In general if X is a symmetric G-metric space and $x, y, a_0 \in X$ we call x a_0-orthogonal to y and denoted by $x \perp_G^{a_0} y$ if and only if $d_G(x, a_0) \le d_G(x, y)$ or $G(x, a_0, a_0) \le G(x, y, y)$.

Let Y_1 and Y_2 be subsets of X. Then $Y_1 \perp_G^{a_0} Y_2$ if and only if for all $y_1 \in Y_1$, $y_2 \in Y_2$, $y_1 \perp_G^{a_0} y_2$. If $x \perp_G^{a_0} y$ then it is not necessary $y \perp_g^{a_0} x$.

Let X be a symmetric G-metric space and Y a subset of X. A point $y_0 \in Y$ call a G-best approximation for $x \in X$ if $x \perp_G^{y_0} Y$ or $d_G(x, y_0) \le d_G(x, y)$ for every $y \in Y$. The set of all G-best approximations of x in Y is shown by $P_G^Y(x)$.

If Y is a subset of X it is clearly that

$$P_G^Y(x) = \left\{ y_0 \in Y : x \perp_G^{y_0} y \quad \forall y \in Y \right\}.$$

In continue we obtain some results on G-best approximation in symmetric G-metric spaces.

Theorem 2. *Let (X,G) be a G-metric symmetric space. For $x, y, z, a_0 \in X$*

(i) $x \perp_G^{a_0} x$ *if and only if* $x = a_0$,

(ii) *if* $x \perp_G^{a_0} y$ *and* $y \perp_G^{x} z$, *then* $x \perp_G^{a_0} z$,

(iii) $x \perp_G^{a_0} a_0$ *and* $a_0 \perp_G^{a_0} x$.

Proof. (i) If $x \perp_G^{a_0} x$ then $G(x, a_0, a_0) \le G(x, x, x) = 0$. Therefore, $G(x, a_0, a_0) = 0$, i.e., $x = a_0$.

(ii) Since $x \perp_G^{a_0} y$ then $G(x, a_0, a_0) \le G(x, y, y)$ also $y \perp_G^{x} z$ then $G(y, x, x) \le G(y, z, z)$ and G is symmetric. Hence $G(x, a_0, a_0) \le G(y, z, z)$ that is $x \perp_G^{a_0} z$.

(iii) It is clear.

Theorem 3. *Let (X,G) be a symmetric G-metric space and A a subset of X.*

(i) *If* $x \in X$, *then* $x \in P_G^A(x)$.

(ii) *If* $x \in A$, *then* $P_G^A(x) = \{x\}$.

Proof. (i) Since $G(x,x,x) = 0$, therefore, $x \in P_G^A(x)$.

(ii) If $x \in A$ and $y_0 \in P_G^A(x)$, then $G(x,y_0,y_0) \le 0 = G(y,x,x)$. Therefore, $x = y_0$.

Theorem 4. *Let (X,G) be a symmetric G-metric space x, $a_0 \in X$ and A a subset of X. Then the following statements are equivalent*:

(i) $x \perp_G^{a_0} A$,

(ii) *there is a function $f : X \to [0,\infty)$ such that $d_G(x,a_0) \le f(y) \le d_G(x,y)$ for all $y \in A$.*

Proof. (i) $\Rightarrow$ (ii) we define $f : X \to [0,\infty)$ by $f(z) = d_G(x,z)$. Suppose $y \in A$ since $x \perp_G^{a_0} y$

$$d_G(x,a_0) \le d_G(x,y) = f(y).$$

(ii) $\Rightarrow$ (i) By definition $x \perp_G^{a_0} y$ for every $y \in A$.

***Corollary* 1.** *Let (X,G) be a symmetric G-metric space x, $a_0 \in X$, Y a subset of X and $y_0 \in Y$. Then the following statements are equivalent*:

(i) $y_0 \in P_G^Y(x)$,

(ii) *there is a function $f : X \to [0,\infty)$ such that $d_G(x,y_0) \le f(y) \le d_G(x,y)$ for all $y \in Y$.*

***Corollary* 2.** *Let (X,G) be a symmetric G-metric space, x, $a_0 \in X$, Y a subset of X and $E \subseteq Y$. If there is a function $f : X \to [0,\infty)$ such that $d_G(x,y_0) \le f(y) \le d_G(x,y)$ for all $y_0 \in E$ and for all $y \in Y$, then $E \subseteq P_G^Y(x)$.*

Example. We can define a symmetric G-metric on $X = \{a,b,c\}$ such that

$$G(a,b,b) = G(a,a,b) = G(a,c,c) = G(c,a,a) = G(b,c,c) = G(c,b,b) = 1,$$

$$G(a,a,a) = G(b,b,b) = G(c,c,c) = 0$$

and

$$G(a,b,c) = G(b,c,a) = G(c,a,b) = 2.$$

It is clear that G is a symmetric G-metric. Then $a \perp_G^b c$, $b \perp_G^a c$ and $c \perp_G^a b$.

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