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ON SOME NEW INEQUALITIES OF HERMITE – HADAMARD TYPE FOR FUNCTIONS WHOSE DERIVATIVES IN ABSOLUTE VALUE ARE s-CONVEX IN THE SECOND SENSE

ПРО ДЕЯКІ НОВІ НЕРІВНОСТІ ТИПУ ЕРМІТА – АДАМАРА ДЛЯ ФУНКЦІЙ З ПОХІДНИМИ, АБСОЛЮТНІ ЗНАЧЕННЯ ЯКИХ Є s-ОПУКЛИМИ В ДРУГОМУ СЕНСІ

Several new inequalities of Hermite – Hadamard type for functions whose derivatives in absolute value are s-convex in the second sense are established. Some applications to special means of positive real numbers are also presented.

Отримано кілька нових нерівностей типу Ерміта – Адамара для функцій з похідними, абсолютні значення яких є s-опуклими в другому сенсі. Наведено також деякі застосування до спеціальних середніх додатних дійсних чисел.

1. Introduction. Let $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$, $\varnothing \neq I \subseteq \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$, then the inequalities

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

hold and are known as Hermite – Hadamard inequalities. The inequalities in (1.1) hold in reversed order if f is a concave function.

In the paper [8], Hudzik and Maligranda considered, among others, the class of functions which are s-convex in the second sense. This class of functions is defined as follows:

A function $f\colon\ [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense if

$$f\left(\lambda x + (1-\lambda)\, y\right) \leq \lambda^s f(x) + (1-\lambda)^s f\left(y\right)$$

holds for all $x, y \in [0,\infty)$, $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$.

The class of s-convex functions in the second sense is usually denoted by K_s^2. It is to be noted that for $s = 1$, s-convexity is merely the usual convexity.

In [6], Dragomir and Fitzpatrick proved a variant of Hermite – Hadamard's inequality which holds for s-convex functions:

Theorem 1.1 [6]. *Suppose $f\colon\ [0,\infty) \to [0,\infty)$ is an s-convex function in the second sense, where $s \in (0,1)$ and let $a, b \in [0,\infty)$, $a < b$. If $f \in L^1[a,b]$, then the following inequalities hold*:

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1.2}$$

The constant $k = \dfrac{1}{s+1}$ is the best possible in the second inequality in (1.2). *The inequalities in* (1.2) *hold in reversed order if f is s-concave.*

In recent years, many authors have established several inequalities of Hermite – Hadamard type for convex functions and s-convex functions in the second sense see for instance the works in [1 – 20] and the references therein.

Most recently, Muddassar et al. [15] introduced a new class of functions which contains both s-convex functions in the first sense and (α, m)-convex functions. This class can be restated in the following definition.

Definition 1.1 [15]. *A function $f\colon [0,\infty) \to [0,\infty)$ is said to be s-(α, m)-convex function in the first sense, or f belongs to the class $K_{m,1}^{\alpha,s}$, if for all $x, y \in [0,\infty)$ and $\mu \in [0,1]$, the following inequality holds*:

$$f\left(\mu x + (1-\mu) y\right) \leq \left(\mu^{\alpha s}\right) f(x) + m\left(1-\mu^{\alpha s}\right) f\left(\frac{y}{m}\right),$$

where $(\alpha, m) \in [0,1]^2$ and for some fixed $s \in (0,1]$.

***Remark* 1.1.** It should be noted that in Definition 1.1, (α, m) must belong to $(0,1]^2$ instead of $[0,1]^2$. Moreover, in Definition 1.1, the range of the function f can be $\mathbb{R}$ the set of real numbers.

The following results from [15] were proved when $|f'|$ or $|f'|^q$, $q \geq 1$, belongs to the class $K_{m,1}^{\alpha,s}$.

Theorem 1.2 [15]. *Let $f\colon I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° (the interior of I), $a, b \in I$ with $a < b$, and $f' \in L^1[a,b]$. If $|f'|$ is s-(α, m)-convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int\limits_a^b f(u)\,du\right| \leq$$

$$\leq \frac{b-a}{2}\left(v_1\left|f'(a)\right| + v_2\left|f'\left(\frac{b}{m}\right)\right|\right), \tag{1.3}$$

where $v_1 = \left(1+2^{\alpha s}(\alpha s)\right)/2^{\alpha s}(\alpha s+1)(\alpha s+2)$ and $v_2 = m\left(\dfrac{1}{2} - v_1\right)$.

Under the assumptions of Theorem 1.2, some of the other results from [15] are the inequalities

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int\limits_a^b f(u)\,du\right| \leq$$

$$\leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\frac{\left|f'(a)\right|^q + m\alpha s\left|f'\left(\frac{b}{m}\right)\right|^q}{1+\alpha s}\right)^{\frac{1}{q}}, \tag{1.4}$$

where $p > 1$, such that $q = p/(p-1)$, and

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int\limits_a^b f(u)\,du\right| \leq$$

$$\leq \frac{b-a}{2^{\frac{(p+1)}{p}}}\left(v_1\left|f'(a)\right|^q + v_2\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}, \tag{1.5}$$

where $v_1 = (1+2^{\alpha s}(\alpha s))/2^{\alpha s}(\alpha s+1)(\alpha s+2)$, $v_2 = m\left(\frac{1}{2}-v_1\right)$ and $1/p+1/q=1$ with $q>1$.

The most representative work related to the Hermite – Hadamard type inequalities for convex mappings we refer the interested reader to author's work given in [14]. The interesting features of the established results from [14] are that they give, in fact, the estimates of difference between twice the middle and the sum of rightmost and leftmost terms connected with the Hermite – Hadamard's inequalities (1.1).

The main purpose of this paper is to establish some entirely new inequalities of Hermite – Hadamard type for functions whose derivatives in absolute value are s-convex in the second sense. The results proved in the present paper are more general and contain the results given in [14] which are connected with the Hermite – Hadamard inequalties given above in (1.1) as a special case. Some new estimates for the difference between the middle and the rightmost terms in (1.1) when $x=a$ or $x=b$ for s-convex functions are also given in Section 2. Applications of our results to special means are provided in Section 3.

2. Main results. In order to prove our results we need the following lemma:

Lemma 2.1 ([14, p. 2], Lemma 1)**.** *Let $f\colon I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, the interior of I, $a,b\in I^\circ$ with $a<b$. If $f'\in L[a,b]$, then the following equality holds*:

$$f(x)+\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{2}{b-a}\int_a^b f(u)\,du=$$

$$=\frac{(x-a)^2}{b-a}\int_0^1\frac{t}{2}f'\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)dt-\frac{(x-a)^2}{b-a}\int_0^1\frac{t}{2}f'\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)dt-$$

$$-\frac{(b-x)^2}{b-a}\int_0^1\frac{t}{2}f'\left(\frac{1+t}{2}x+\frac{1-t}{2}b\right)dt+\frac{(b-x)^2}{b-a}\int_0^1\frac{t}{2}f'\left(\frac{1-t}{2}x+\frac{1+t}{2}b\right)dt \qquad (2.1)$$

for all $x\in[a,b]$.

Theorem 2.1. *Let $f\colon I\to\mathbb{R}$, $I\subset[0,\infty)$, be a differentiable function on I° such that $f'\in$ $\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$. If $|f'|$ is s-convex on $[a,b]$, $s\in(0,1]$, then the following inequality holds*:

$$\left|f(x)+\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{2}{b-a}\int_a^b f(u)\,du\right|\le$$

$$\le\frac{s+2^{-s}}{(s+1)(s+2)}\left\{\frac{(x-a)^2}{b-a}\left[|f'(x)|+|f'(a)|\right]+\frac{(b-x)^2}{b-a}\left[|f'(x)|+|f'(b)|\right]\right\} \qquad (2.2)$$

for all $x\in[a,b]$.

Proof. From Lemma 2.1, we have

$$\left|f(x)+\frac{(b-x)\,f(b)+(x-a)\,f(a)}{b-a}-\frac{2}{b-a}\int\limits_a^b f\,(u)\,du\right|\le$$

$$\le\frac{(x-a)^2}{b-a}\left[\int\limits_0^1\frac{t}{2}\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\right|dt+\int\limits_0^1\frac{t}{2}\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)\right|dt\right]+$$

$$+\frac{(b-x)^2}{b-a}\left[\int\limits_0^1\frac{t}{2}\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}b\right)\right|dt+\int\limits_0^1\frac{t}{2}\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}b\right)\right|dt\right].\qquad(2.3)$$

Since $|f'|$ is s-convex on $[a,b]$, we obtain

$$\int\limits_0^1\frac{t}{2}f'\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)dt+\int\limits_0^1\frac{t}{2}\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)\right|dt\le$$

$$\le\left[\int\limits_0^1\frac{t}{2}\left(\frac{1-t}{2}\right)^s dt+\int\limits_0^1\frac{t}{2}\left(\frac{1+t}{2}\right)^s dt\right]\left[|f'(x)|+|f'(a)|\right]\qquad(2.4)$$

and

$$\int\limits_0^1\frac{t}{2}f'\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)dt+\int\limits_0^1\frac{t}{2}\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)\right|dt\le$$

$$\le\left[\int\limits_0^1\frac{t}{2}\left(\frac{1-t}{2}\right)^s dt+\int\limits_0^1\frac{t}{2}\left(\frac{1+t}{2}\right)^s dt\right]\left[|f'(x)|+|f'(b)|\right].\qquad(2.5)$$

By making use of the inequalties (2.4), (2.5) and the fact

$$\int\limits_0^1\frac{t}{2}\left(\frac{1-t}{2}\right)^s dt+\int\limits_0^1\frac{t}{2}\left(\frac{1+t}{2}\right)^s dt=\frac{s+2^{-s}}{(s+1)\,(s+2)}$$

in the inequality (2.3), we get the inequality (2.2).

Theorem 2.1 is proved.

***Corollary* 2.1.** *Under the assumptions of Theorem* 2.1, *if we take* $x=\dfrac{a+b}{2}$, *we get the following inequality*:

$$\left|f\left(\frac{a+b}{2}\right)+\frac{f(a)+f\,(b)}{2}-\frac{2}{b-a}\int\limits_a^b f\,(u)\,du\right|\le$$

$$\le\frac{(s+2^{-s})\left(2^{1-s}+1\right)}{(s+1)\,(s+2)}\left(\frac{b-a}{4}\right)\left[|f'(a)|+|f'(b)|\right].\qquad(2.6)$$

Proof. It follows from Theorem 2.1 and using the s-convexity of $|f'|$ on $[a,b]$.

***Corollary* 2.2.** *If the assumptions of Theorem* 2.1 *are fulfilled and if $x=a$ or $x=b$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \le \frac{(s+2^{-s})(b-a)}{2(s+1)(s+2)}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{2.7}$$

***Remark* 2.1.** If we take $s=1$ in Theorem 2.1 and Corollary 2.1 we get the inequalities given in [14, p. 86] (Theorem 1) and [14, p. 87] (Corollary 1) respectively.

***Remark* 2.2.** If we take $s=1$ in Corollary 2.2 we get Theorem 2.2 given in [5]. Also we get the same result [5] (Theorem 2.2) from (1.3) for $\alpha=m=s=1$.

The corresponding version of the inequality (2.2) for powers of the first derivative is incorporated as follows:

Theorem 2.2. *Let $f\colon I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$. If $|f'|^q$ is s-convex on $[a,b]$ for some fixed $q>1$, $s\in(0,1]$, then the following inequality holds*:

$$\left|f(x)+\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{2}{b-a}\int_a^b f(u)\,du\right|\le$$

$$\le\frac{1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left\{\frac{(x-a)^2}{b-a}\left[\left(\left(2-2^{-s}\right)\left|f'(x)\right|^q+2^{-s}\left|f'(a)\right|^q\right)^{\frac{1}{q}}+\right.\right.$$

$$\left.+\left(2^{-s}\left|f'(x)\right|^q+\left(2-2^{-s}\right)\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right]+$$

$$+\frac{(b-x)^2}{b-a}\left[\left(\left(2-2^{-s}\right)\left|f'(x)\right|^q+2^{-s}\left|f'(b)\right|^q\right)^{\frac{1}{q}}+\right.$$

$$\left.\left.+\left(2^{-s}\left|f'(x)\right|^q+\left(2-2^{-s}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]\right\} \tag{2.8}$$

for all $x\in[a,b]$ and $\dfrac{1}{p}+\dfrac{1}{q}=1$.

Proof. From Lemma 2.1 and using the well-known Hölder integral inequality, we have

$$\left|f(x)+\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{2}{b-a}\int_a^b f(u)\,du\right|\le$$

$$\le\frac{(x-a)^2}{b-a}\left(\int_0^1\left(\frac{t}{2}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}+$$

$$+\frac{(x-a)^2}{b-a}\left(\int_0^1\left(\frac{t}{2}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}+$$

$$+\frac{(b-x)^2}{b-a}\left(\int_0^1\left(\frac{t}{2}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}+$$

$$+\frac{(b-x)^2}{b-a}\left(\int_0^1\left(\frac{t}{2}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}} \tag{2.9}$$

for all $x \in [a,b]$.

Since $|f'|^q$ is s-convex on $[a,b]$, we obtain

$$\int_0^1\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\right|^q dt \le$$

$$\le \int_0^1\left[\left(\frac{1+t}{2}\right)^s\left|f'(x)\right|^q+\left(\frac{1-t}{2}\right)^s\left|f'(a)\right|^q\right]dt=$$

$$=\frac{2-2^{-s}}{s+1}\left|f'(x)\right|^q+\frac{2^{-s}}{s+1}\left|f'(a)\right|^q. \tag{2.10}$$

Similarly,

$$\int_0^1\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)\right|^q dt \le \frac{2^{-s}}{s+1}\left|f'(x)\right|^q+\frac{2-2^{-s}}{s+1}\left|f'(a)\right|^q, \tag{2.11}$$

$$\int_0^1\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}b\right)\right|^q dt \le \frac{2-2^{-s}}{s+1}\left|f'(x)\right|^q+\frac{2^{-s}}{s+1}\left|f'\left(b\right)\right|^q \tag{2.12}$$

and

$$\int_0^1\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}b\right)\right|^q dt \le \frac{2^{-s}}{s+1}\left|f'(x)\right|^q+\frac{2-2^{-s}}{s+1}\left|f'(b)\right|^q. \tag{2.13}$$

Using the inequalities (2.10) – (2.13) in (2.9) and the fact

$$\int_0^1\left(\frac{t}{2}\right)^p dt=\frac{1}{2^p}\frac{1}{p+1},$$

we get inequality (2.8).

Theorem 2.2 is proved.

***Remark* 2.3.** If in Theorem 2.2, we take $s = 1$, we get the inequality proved in [14, p. 87] (Theorem 2).

***Corollary* 2.3.** *Under the assumptions of Theorem* 2.2, *if we choose* $x = \dfrac{a+b}{2}$. *Then*

$$\left| f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} - \frac{2}{b-a}\int_a^b f(u)\,du \right| \le$$

$$\le \left(\frac{b-a}{8}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left\{\left[\left(2-2^{-s}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q + 2^{-s}\left|f'(a)\right|^q\right]^{\frac{1}{q}} + \right.$$

$$+\left[2^{-s}\left|f'\left(\frac{a+b}{2}\right)\right|^q + \left(2-2^{-s}\right)\left|f'(a)\right|^q\right]^{\frac{1}{q}} +$$

$$+\left[\left(2-2^{-s}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q + 2^{-s}\left|f'(b)\right|^q\right]^{\frac{1}{q}} +$$

$$\left. +\left[2^{-s}\left|f'\left(\frac{a+b}{2}\right)\right|^q + \left(2-2^{-s}\right)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\} \le$$

$$\le \left(\frac{b-a}{8}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[2^{-\frac{2s}{q}} + \left(2^{1-s}-2^{-2s}\right)^{\frac{1}{q}} + \right.$$

$$\left. + \left(2^{-2s}-2^{-s}+2\right)^{\frac{1}{q}} + \left(2^{1-s}-2^{-2s}+2^{-s}\right)^{\frac{1}{q}}\right]\left[\left|f'(a)\right| + \left|f'(b)\right|\right]. \qquad (2.14)$$

Proof. It follows from Theorem 2.2. The second inequality is obtained by using the s-convexity of $|f'|^q$ and the fact that

$$\sum_{k=1}^{n}(u_k+v_k)^r \le \sum_{k=1}^{n}(u_k)^r + \sum_{k=1}^{n}(v_k)^r$$

for all $u_k, v_k \ge 0, 1 \le k \le n$ and $0 \le r < 1$.

Corollary 2.3 is proved.

***Corollary* 2.4.** *Suppose the assumptions of Theorem* 2.2 *are satisfied and if* $x = a$ *or* $x = b$, *then the following inequality holds valid*:

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)\,du \right| \le$$

$$\le \frac{1}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}(b-a)\left\{\left(2^{-s}\left|f'(a)\right|^q + \left(2-2^{-s}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}} + \right.$$

$$\left. + \left(\left(2-2^{-s}\right)\left|f'(a)\right|^q + 2^{-s}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\}, \qquad (2.15)$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

***Remark* 2.4.** If in Corollary 2.3, we take $s = 1$, we get the inequality established in [14, p. 88] (Corollary 2). For $s = 1$, the inequality (2.15) becomes

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int\limits_a^b f(u)\,du\right| \le \frac{1}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}(b-a)\times$$

$$\times\left\{\left(\frac{|f'(a)|^q+3|f'(b)|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{3|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right\}, \qquad (2.16)$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1$.

***Remark* 2.5.** It is easy to observe that for $\alpha = m = 1$, the inequlity (2.16) provides a better estimate than given in (1.4).

***Remark* 2.6.** Since for $p,\ q > 1$ and $s \in (0,1]$, $\dfrac{1}{2} \le \left(\dfrac{1}{p+1}\right)^{\frac{1}{p}} \le 1$, $\left(\dfrac{1}{s+1}\right)^{\frac{1}{q}} \le 1$, we have from (2.14) the following inequality:

$$\left|f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}-\frac{2}{b-a}\int\limits_a^b f(u)\,du\right| \le$$

$$\le \left(\frac{b-a}{8}\right)\left\{\left[\left(2-2^{-s}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q+2^{-s}\left|f'(a)\right|^q\right]^{\frac{1}{q}}+\right.$$

$$+\left[2^{-s}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left(2-2^{-s}\right)\left|f'(a)\right|^q\right]^{\frac{1}{q}}+$$

$$+\left[\left(2-2^{-s}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q+2^{-s}\left|f'(b)\right|^q\right]^{\frac{1}{q}}+$$

$$\left.+\left[2^{-s}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left(2-2^{-s}\right)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\} \le$$

$$\le \left(\frac{b-a}{8}\right)\left[2^{-\frac{2s}{q}}+\left(2^{1-s}-2^{-2s}\right)^{\frac{1}{q}}+\right.$$

$$\left.+\left(2^{-2s}-2^{-s}+2\right)^{\frac{1}{q}}+\left(2^{1-s}-2^{-2s}+2^{-s}\right)^{\frac{1}{q}}\right]\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \qquad (2.17)$$

Theorem 2.3. *Let $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is s-convex on $[a,b]$ for some fixed $q > 1$, $s \in (0,1]$, then the following inequality holds*:

$$\left|f(x)+\frac{(b-x)\,f(b)+(x-a)\,f(a)}{b-a}-\frac{2}{b-a}\int\limits_a^b f(u)\,du\right| \le$$

$$\leq \frac{1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left\{\frac{(x-a)^2}{b-a}\left[\left|f'(x)\right|^q+\left|f'\left(\frac{x+a}{2}\right)\right|^q\right]^{\frac{1}{q}}+\right.$$

$$+\frac{(x-a)^2}{b-a}\left[\left|f'(a)\right|^q+\left|f'\left(\frac{x+a}{2}\right)\right|^q\right]^{\frac{1}{q}}+$$

$$+\frac{(b-x)^2}{b-a}\left[\left|f'(x)\right|^q+\left|f'\left(\frac{x+b}{2}\right)\right|^q\right]^{\frac{1}{q}}+$$

$$\left.+\frac{(b-x)^2}{b-a}\left[\left|f'(b)\right|^q+\left|f'\left(\frac{x+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right\} \tag{2.18}$$

for all $x\in[a,b]$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and using the well-known Hölder integral inequality, we have

$$\left|f(x)+\frac{(b-x)\,f(b)+(x-a)\,f(a)}{b-a}-\frac{2}{b-a}\int_a^b f(u)\,du\right|\leq$$

$$\leq \frac{(x-a)^2}{b-a}\left(\int_0^1\left(\frac{t}{2}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}+$$

$$+\frac{(x-a)^2}{b-a}\left(\int_0^1\left(\frac{t}{2}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}+$$

$$+\frac{(b-x)^2}{b-a}\left(\int_0^1\left(\frac{t}{2}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}+$$

$$+\frac{(b-x)^2}{b-a}\left(\int_0^1\left(\frac{t}{2}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}} \tag{2.19}$$

for all $x\in[a,b]$.

Since $|f'|^q$ is s-convex on $[a,b]$ so by using the inequality (1.2), we have

$$\int_0^1\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\right|^q dt\leq\frac{|f'(x)|^q+\left|f'\left(\frac{x+a}{2}\right)\right|^q}{s+1}. \tag{2.20}$$

Similarly,

$$\int_0^1 \left| f'\left(\frac{1-t}{2}x + \frac{1+t}{2}a\right)\right|^q dt \leq \frac{|f'(a)|^q + \left|f'\left(\frac{x+a}{2}\right)\right|^q}{s+1}, \tag{2.21}$$

$$\int_0^1 \left| f'\left(\frac{1+t}{2}x + \frac{1-t}{2}b\right)\right|^q dt \leq \frac{|f'(x)|^q + \left|f'\left(\frac{x+b}{2}\right)\right|^q}{s+1} \tag{2.22}$$

and

$$\int_0^1 \left| f'\left(\frac{1-t}{2}x + \frac{1+t}{2}b\right)\right|^q dt \leq \frac{|f'(b)|^q + \left|f'\left(\frac{x+b}{2}\right)\right|^q}{s+1}. \tag{2.23}$$

Using the inequalities (2.20) – (2.23) in (2.19) and the fact

$$\int_0^1 \left(\frac{t}{2}\right)^p dt = \frac{1}{2^p}\frac{1}{p+1},$$

we get inequality (2.18).

Theorem 2.3 is proved.

***Corollary* 2.5.** *Suppose all the conditions of Theorem* 2.3 *are satisfied and if* $x = \frac{a+b}{2}$, *then the inequality holds*:

$$\left| f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} - \frac{2}{b-a}\int_a^b f(u)\,du \right| \leq$$

$$\leq \left(\frac{b-a}{8}\right)\left\{\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q + \left|f'\left(\frac{3a+b}{4}\right)\right|^q\right]^{\frac{1}{q}} + \left[|f'(a)|^q + \left|f'\left(\frac{3a+b}{4}\right)\right|^q\right]^{\frac{1}{q}} + \right.$$

$$\left. + \left[\left|f'\left(\frac{a+b}{2}\right)\right|^q + \left|f'\left(\frac{a+3b}{4}\right)\right|^q\right]^{\frac{1}{q}} + \left[|f'(b)|^q + \left|f'\left(\frac{a+3b}{4}\right)\right|^q\right]^{\frac{1}{q}}\right\} \leq$$

$$\leq \left(\frac{b-a}{8}\right)\left\{\left[\left(\frac{1}{2}\right)^s + \left(\frac{3}{4}\right)^s\right]^{\frac{1}{q}} + \left[1 + \left(\frac{3}{4}\right)^s\right]^{\frac{1}{q}} + \right.$$

$$\left. + \left[\left(\frac{1}{2}\right)^s + \left(\frac{1}{4}\right)^s\right]^{\frac{1}{q}} + \left(\frac{1}{4}\right)^{\frac{s}{q}}\right\}\left[|f'(a)| + |f'(b)|\right]. \tag{2.24}$$

Proof. It follows directly from Theorem 2.3 and using similar arguments as that of Corollary 2.3 and Remark 2.6.

***Corollary* 2.6.** *Under the assumptions of Theorem* 2.3, *if* $x = a$ *or* $x = b$, *then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int\limits_a^b f(u)\,du\right| \le \frac{1}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}(b-a)\times$$

$$\times\left\{\left[\left|f'(a)\right|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}} + \left[\left|f'(b)\right|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right\}, \qquad (2.25)$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1.$

***Remark* 2.7.** For $s = 1$, the result (2.25) provides a better estimate than given in (1.3) – (1.5) for $\alpha = m = s = 1$.

Theorem 2.4. *Let* $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[a,b]$, *where* $a, b \in I^\circ$ *with* $a < b$. *If* $|f'|^q$ *is s-concave on* $[a,b]$ *for some fixed* $q > 1$, $s \in (0,1]$, *then the following inequality holds*:

$$\left|f(x) + \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{2}{b-a}\int\limits_a^b f(u)\,du\right| \le$$

$$\le \left(\frac{1}{p+1}\right)^{\frac{1}{p}} 2^{\frac{s-1}{q}-1}\left[\left|f'\left(\frac{3x+a}{4}\right)\right| + \left|f'\left(\frac{x+3a}{4}\right)\right| + \left|f'\left(\frac{3x+b}{4}\right)\right| + \left|f'\left(\frac{x+3b}{4}\right)\right|\right] \qquad (2.26)$$

for all $x \in [a,b]$ *and* $\dfrac{1}{p} + \dfrac{1}{q} = 1.$

Proof. From Lemma 2.1 and using the well-known Hölder integral inequality, we have

$$\left|f(x) + \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{2}{b-a}\int\limits_a^b f(u)\,du\right| \le$$

$$\le \frac{(x-a)^2}{b-a}\left(\int\limits_0^1 \left(\frac{t}{2}\right)^p dt\right)^{\frac{1}{p}}\left(\int\limits_0^1 \left|f'\left(\frac{1+t}{2}x + \frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}} +$$

$$+\frac{(x-a)^2}{b-a}\left(\int\limits_0^1 \left(\frac{t}{2}\right)^p dt\right)^{\frac{1}{p}}\left(\int\limits_0^1 \left|f'\left(\frac{1-t}{2}x + \frac{1+t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}} +$$

$$+\frac{(b-x)^2}{b-a}\left(\int\limits_0^1 \left(\frac{t}{2}\right)^p dt\right)^{\frac{1}{p}}\left(\int\limits_0^1 \left|f'\left(\frac{1+t}{2}x + \frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}} +$$

$$+\frac{(b-x)^2}{b-a}\left(\int\limits_0^1 \left(\frac{t}{2}\right)^p dt\right)^{\frac{1}{p}}\left(\int\limits_0^1 \left|f'\left(\frac{1-t}{2}x + \frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}} \qquad (2.27)$$

for all $x \in [a, b]$.

Since $|f'|^q$ is s-concave on $[a, b]$, by using the Hermite – Hadamard type inequality (1.2), we obtain

$$\int_0^1 \left| f'\left(\frac{1+t}{2}x + \frac{1-t}{2}a\right)\right|^q dt \le 2^{s-1}\left| f'\left(\frac{3x+a}{4}\right)\right|^q. \tag{2.28}$$

Similarly,

$$\int_0^1 \left| f'\left(\frac{1-t}{2}x + \frac{1+t}{2}a\right)\right|^q dt \le 2^{s-1}\left| f'\left(\frac{x+3a}{4}\right)\right|^q, \tag{2.29}$$

$$\int_0^1 \left| f'\left(\frac{1+t}{2}x + \frac{1-t}{2}b\right)\right|^q dt \le 2^{s-1}\left| f'\left(\frac{3x+b}{4}\right)\right|^q \tag{2.30}$$

and

$$\int_0^1 \left| f'\left(\frac{1-t}{2}x + \frac{1+t}{2}b\right)\right|^q dt \le 2^{s-1}\left| f'\left(\frac{x+3b}{4}\right)\right|^q. \tag{2.31}$$

By making use of the inequalities (2.28) – (2.31) in (2.27), we get (2.26).

Theorem 2.4 is proved.

***Corollary* 2.7.** *Suppose all the conditions of Theorem* 2.4 *are satisfied. If we choose* $x = \dfrac{a+b}{2}$ *and assume that* $|f'|$ *is a linear map, then we have the inequality*:

$$\left| f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} - \frac{2}{b-a}\int_a^b f(u)\,du\right| \le \left(\frac{1}{p+1}\right)^{\frac{1}{p}} 2^{\frac{s-1}{q}}\left[\left|f'(a+b)\right|\right]. \tag{2.32}$$

Proof. It is a direct consequence of Theorem 2.4.

***Corollary* 2.8.** *Under the assumptions of Theorem* 2.4 *and by choosing* $x = a$ *and* $x = b$, *we have*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)\,du\right| \le$$

$$\le \left(\frac{1}{p+1}\right)^{\frac{1}{p}} 2^{\frac{s-1}{q}-2}\left[\left|f'(a)\right| + \left|f'\left(\frac{3a+b}{4}\right)\right| + \left|f'\left(\frac{a+3b}{4}\right)\right| + \left|f'(b)\right|\right], \tag{2.33}$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Proof. By replacing $x = a$ and then $x = b$ in (2.26), and adding the resulting inequalities side by side, we get (2.33).

Theorem 2.5. *Let $f\colon I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$. If $|f'|^q$ is s-convex on $[a,b]$ for some fixed $q\geq 1,\ s\in(0,1]$, then the following inequality holds*:

$$\left|f(x)+\frac{(b-x)\,f(b)+(x-a)\,f(a)}{b-a}-\frac{2}{b-a}\int\limits_a^b f\,(u)\,du\right|\leq$$

$$\leq\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{1}{(s+1)\,(s+2)}\right)^{\frac{1}{q}}\left\{\frac{(x-a)^2}{b-a}\left[\left(\left(s+2^{-s-1}\right)\left|f'(x)\right|^q+2^{-s-1}\left|f'(a)\right|^q\right)^{\frac{1}{q}}+\right.\right.$$

$$\left.+\left(2^{-s-1}\left|f'(x)\right|^q+\left(s+2^{-s-1}\right)\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right]+$$

$$+\frac{(b-x)^2}{b-a}\left[\left(\left(s+2^{-s-1}\right)\left|f'(x)\right|^q+2^{-s-1}\left|f'(b)\right|^q\right)^{\frac{1}{q}}+\right.$$

$$\left.\left.+\left(2^{-s-1}\left|f'(x)\right|^q+\left(s+2^{-s-1}\right)\left|f'\,(b)\right|^q\right)^{\frac{1}{q}}\right]\right\}\tag{2.34}$$

for all $x\in[a,b]$.

Proof. From Lemma 2.1 and using the well-known power-mean integral inequality, we have

$$\left|f(x)+\frac{(b-x)\,f(b)+(x-a)\,f(a)}{b-a}-\frac{2}{b-a}\int\limits_a^b f\,(u)\,du\right|\leq$$

$$\leq\frac{(x-a)^2}{b-a}\left(\int\limits_0^1\frac{t}{2}dt\right)^{\frac{1}{q}-1}\left(\int\limits_0^1\frac{t}{2}\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}+$$

$$+\frac{(x-a)^2}{b-a}\left(\int\limits_0^1\frac{t}{2}dt\right)^{\frac{1}{q}-1}\left(\int\limits_0^1\frac{t}{2}\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}+$$

$$+\frac{(b-x)^2}{b-a}\left(\int\limits_0^1\frac{t}{2}dt\right)^{\frac{1}{q}-1}\left(\int\limits_0^1\frac{t}{2}\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}+$$

$$+\frac{(b-x)^2}{b-a}\left(\int\limits_0^1\frac{t}{2}dt\right)^{\frac{1}{q}-1}\left(\int\limits_0^1\frac{t}{2}\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\tag{2.35}$$

for all $x\in[a,b]$.

Since $|f'|^q$ is s-convex on $[a,b]$, we get

$$\int_0^1 \frac{t}{2}\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}a\right)\right|^q dt \leq$$

$$\leq \left|f'(x)\right|^q \int_0^1 \frac{t}{2}\left(\frac{1+t}{2}\right)^s dt + \left|f'(a)\right|^q \int_0^1 \frac{t}{2}\left(\frac{1-t}{2}\right)^s dt =$$

$$= \frac{s+2^{-s-1}}{(s+1)(s+2)}\left|f'(x)\right|^q + \frac{2^{-s-1}}{(s+1)(s+2)}\left|f'(a)\right|^q. \tag{2.36}$$

Similarly,

$$\int_0^1 \frac{t}{2}\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}a\right)\right|^q dt \leq$$

$$\leq \frac{2^{-s-1}}{(s+1)(s+2)}\left|f'(x)\right|^q + \frac{s+2^{-s-1}}{(s+1)(s+2)}\left|f'(a)\right|^q, \tag{2.37}$$

$$\int_0^1 \frac{t}{2}\left|f'\left(\frac{1+t}{2}x+\frac{1-t}{2}b\right)\right|^q dt \leq$$

$$\leq \frac{s+2^{-s-1}}{(s+1)(s+2)}\left|f'(x)\right|^q + \frac{2^{-s-1}}{(s+1)(s+2)}\left|f'(b)\right|^q \tag{2.38}$$

and

$$\int_0^1 \frac{t}{2}\left|f'\left(\frac{1-t}{2}x+\frac{1+t}{2}b\right)\right|^q dt \leq$$

$$\leq \frac{2^{-s-1}}{(s+1)(s+2)}\left|f'(x)\right|^q + \frac{s+2^{-s-1}}{(s+1)(s+2)}\left|f'(a)\right|^q. \tag{2.39}$$

Using the inequalities (2.36) – (2.39) in (2.35) and the fact

$$\int_0^1 \frac{t}{2}dt = \frac{1}{4},$$

we get inequality (2.34).

Theorem 2.5 is proved.

***Corollary* 2.9.** *Suppose all the conditions of Theorem* 2.3 *are satisfied. If we choose* $x = \dfrac{a+b}{2}$ *and using similar arguments as in Corollary* 2.3, *we have the inequalities*

$$\left|f(x) + \frac{(b-x)\,f(b)+(x-a)\,f(a)}{b-a} - \frac{2}{b-a}\int_a^b f(u)\,du\right| \leq$$

$$\leq \left(\frac{1}{4}\right)^{1-\frac{1}{q}} \left(\frac{1}{(s+1)(s+2)}\right)^{\frac{1}{q}} \left(\frac{b-a}{4}\right) \times$$

$$\times \left\{ \left[\left(s + 2^{-s-1}\right) \left| f'\left(\frac{a+b}{2}\right)\right|^q + 2^{-s-1} \left|f'(a)\right|^q \right]^{\frac{1}{q}} + \right.$$

$$+ \left[2^{-s-1} \left| f'\left(\frac{a+b}{2}\right)\right|^q + \left(s + 2^{-s-1}\right) \left|f'(a)\right|^q \right]^{\frac{1}{q}} +$$

$$+ \left[\left[\left(s + 2^{-s-1}\right) \left| f'\left(\frac{a+b}{2}\right)\right|^q + 2^{-s-1} \left|f'(b)\right|^q \right]^{\frac{1}{q}} + \right.$$

$$\left. + \left[2^{-s-1} \left| f'\left(\frac{a+b}{2}\right)\right|^q + \left(s + 2^{-s-1}\right) \left|f'(b)\right|^q \right]^{\frac{1}{q}} \right\} \leq$$

$$\leq \left(\frac{1}{4}\right)^{1-\frac{1}{q}} \left(\frac{1}{(s+1)(s+2)}\right)^{\frac{1}{q}} \left(\frac{b-a}{4}\right) \left[\left(2^{-s}s + 2^{-2s-1} + 2^{-s-1}\right)^{\frac{1}{q}} + \right.$$

$$\left. + \left(2^{-2s-1} + s + 2^{-s-1}\right)^{\frac{1}{q}} + \left(2^{-s}s + 2^{-2s-1}\right)^{\frac{1}{q}} + 2^{\frac{-2s-1}{q}} \right] \left[\left|f'(a)\right| + \left|f'(b)\right| \right]. \tag{2.40}$$

***Corollary* 2.10.** *By choosing $x = a$ or $x = b$ in Theorem* 2.5, *we obtain*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(u)\, du \right| \leq$$

$$\leq \left(\frac{1}{4}\right)^{1-\frac{1}{q}} \left(\frac{1}{(s+1)(s+2)}\right)^{\frac{1}{q}} (b-a) \times$$

$$\times \left\{ \left(\left(s + 2^{-s-1}\right) \left|f'(a)\right|^q + 2^{-s-1} \left|f'(b)\right|^q \right)^{\frac{1}{q}} + \right.$$

$$\left. + \left(2^{-s-1} \left|f'(a)\right|^q + \left(s + 2^{-s-1}\right) \left|f'(b)\right|^q \right)^{\frac{1}{q}} \right\}. \tag{2.41}$$

***Remark* 2.8.** If we take $s = 1$ in Theorem 2.5 and Corollary 2.9, we get the inequalities [14, p. 88] (Theorem 3) and [14, p. 89] (Corollary 3) respectively. The result given in (2.41) gives poor estimate than all the estimates established in this paper as well as given above by (1.3)–(1.5) for $\alpha = m = s = 1$.

3. Applications to special means. In [8], the following example is given:

Let $s \in (0, 1)$ and $a, b, c \in \mathbb{R}$. We define function $f : [0, \infty) \to \mathbb{R}$ as

$$f(t) = \begin{cases} a, & t = 0, \\ bt^s + c, & t > 0. \end{cases}$$

If $b \geq 0$ and $0\leq c \leq a$, then $f \in K_s^2$. Hence for $b = 1$ and $a = c = 0$, we have $f\colon [0,1] \to [0,1]$, $f(t) = t^s$, $f \in K_s^2$.

Now, using the results of Section 2, we give some applications to special means of real numbers. We shall consider the means for arbitrary real numbers a, b $(a \neq b)$. We take

(1) The arithmetic mean:

$$A(a,b) = \frac{a+b}{2}, \quad a,b \in \mathbb{R}.$$

(2) Generalized log-mean:

$$L_n(a,b) = \left[\frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}}, \quad a,b \in \mathbb{R}, \quad n \in \mathbb{Z}\setminus\{-1,0\}, \quad a \neq b.$$

Therefore, by considering the s-convex mapping $f\colon [0,1] \to [0,1]$, $f(x) = x^s$, $s \in (0,1)$, the following results hold:

Proposition 3.1. *Let a, $b \in (0,1)$ with $a < b$ and $0 < s < 1$. Then we have*

$$\left|A^s(a,b) + A(a^s,b^s) - 2L_s^s(a,b)\right| \leq \frac{s\left(s+2^{-s}\right)(b-a)}{(s+1)(s+2)} A\left(|a|^{s-1}, |b|^{s-1}\right).$$

Proof. It follows from Corollary 2.1 when applied to the s-convex function $f\colon [0,1] \to [0,1]$, $f(x) = x^s$, $s \in (0,1)$.

Proposition 3.2. *Let a, $b \in (0,1)$ with $a < b$ and $0 < s < 1$. Then for $q > 1$ we have*

$$\left|A^s(a,b) + A(a^s,b^s) - 2L_s^s(a,b)\right| \leq$$

$$\leq s\left(\frac{b-a}{8}\right)\left\{\left(\left(2-2^{-s}\right)\left|\frac{a+b}{2}\right|^{q(s-1)} + 2^{-s}|a|^{q(s-1)}\right)^{\frac{1}{q}} + \right.$$

$$+\left(2^{-s}\left|\frac{a+b}{2}\right|^{q(s-1)} + \left(2-2^{-s}\right)|a|^{q(s-1)}\right)^{\frac{1}{q}} +$$

$$+\left(\left(2-2^{-s}\right)\left|\frac{a+b}{2}\right|^{q(s-1)} + 2^{-s}|b|^{q(s-1)}\right)^{\frac{1}{q}} +$$

$$\left.+\left(2^{-s}\left|\frac{a+b}{2}\right|^{q(s-1)} + \left(2-2^{-s}\right)|b|^{q(s-1)}\right)^{\frac{1}{q}}\right\} \leq$$

$$\leq s\left(\frac{b-a}{4}\right)\left[2^{-\frac{2s}{q}} + \left(2^{1-s} - 2^{-2s}\right)^{\frac{1}{q}} + \left(2^{-2s} - 2^{-s} + 2\right)^{\frac{1}{q}} + \right.$$

$$\left. + \left(2^{1-s} - 2^{-2s} + 2^{-s}\right)^{\frac{1}{q}}\right] A\left(|a|^{s-1}, |b|^{s-1}\right).$$

Proof. The assertion follows from Remark 2.6 when applied to the s-convex function $f\colon [0,1] \to [0,1]$, $f(x) = x^s$, $s \in (0,1)$.

Proposition 3.3. *Let a, $b \in (0,1)$ with $a < b$ and $0 < s < 1$. Then for $q > 1$ we have*

$$|A^s(a,b) + A(a^s, b^s) - 2L_s^s(a,b)| \leq$$

$$\leq s\left(\frac{b-a}{8}\right)\left\{\left[\left|\frac{a+b}{2}\right|^{q(s-1)} + \left|\frac{3a+b}{4}\right|^{q(s-1)}\right]^{\frac{1}{q}} + \left[|a|^{q(s-1)} + \left|\frac{3a+b}{4}\right|^{q(s-1)}\right]^{\frac{1}{q}} + \right.$$

$$\left. + \left[\left|\frac{a+b}{2}\right|^{q(s-1)} + \left|\frac{a+3b}{4}\right|^{q(s-1)}\right]^{\frac{1}{q}} + \left[|b|^{q(s-1)} + \left|\frac{a+3b}{4}\right|^{q(s-1)}\right]^{\frac{1}{q}}\right\} \leq$$

$$\leq s\left(\frac{b-a}{4}\right)\left\{\left[\left(\frac{1}{2}\right)^s + \left(\frac{3}{4}\right)^s\right]^{\frac{1}{q}} + \left[1 + \left(\frac{3}{4}\right)^s\right]^{\frac{1}{q}} + \right.$$

$$\left. + \left[\left(\frac{1}{2}\right)^s + \left(\frac{1}{4}\right)^s\right]^{\frac{1}{q}} + \left(\frac{1}{4}\right)^{\frac{s}{q}}\right\} A\left(|a|^{s-1}, |b|^{s-1}\right).$$

Proof. The assertion follows from Corollary 2.5 when applied to the s-convex function $f\colon [0,1] \to [0,1]$, $f(x) = x^s$, $s \in (0,1)$.

Proposition 3.4. *Let $a, b \in (0,1)$ with $a < b$ and $0 < s < 1$. Then for $q \geq 1$ we have*

$$|A^s(a,b) + A(a^s, b^s) - 2L_s^s(a,b)| \leq$$

$$\leq s\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{1}{(s+1)(s+2)}\right)^{\frac{1}{q}}\left(\frac{b-a}{4}\right) \times$$

$$\times \left\{\left[\left(s + 2^{-s-1}\right)\left|\frac{a+b}{2}\right|^{q(s-1)} + 2^{-s-1}|a|^{q(s-1)}\right]^{\frac{1}{q}} + \right.$$

$$+ \left[2^{-s-1}\left|\frac{a+b}{2}\right|^{q(s-1)} + \left(s + 2^{-s-1}\right)|a|^{q(s-1)}\right]^{\frac{1}{q}} +$$

$$+ \left[\left[\left(s + 2^{-s-1}\right)\left|\frac{a+b}{2}\right|^{q(s-1)} + 2^{-s-1}|b|^{q(s-1)}\right]^{\frac{1}{q}} + \right.$$

$$\left. + \left[2^{-s-1}\left|\frac{a+b}{2}\right|^{q(s-1)} + \left(s + 2^{-s-1}\right)|b|^{q(s-1)}\right]^{\frac{1}{q}}\right\} \leq$$

$$\leq s\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{1}{(s+1)(s+2)}\right)^{\frac{1}{q}}\left(\frac{b-a}{2}\right)\left[\left(2^{-s}s + 2^{-2s-1} + 2^{-s-1}\right)^{\frac{1}{q}} + \right.$$

$$+\left(2^{-2s-1}+s+2^{-s-1}\right)^{\frac{1}{q}}+\left(2^{-s}s+2^{-2s-1}\right)^{\frac{1}{q}}+2^{\frac{-2s-1}{q}}\Big]A\left(|a|^{s-1},|b|^{s-1}\right).$$

Proof. The assertion follows from Corollary 2.9 when applied to the s-convex function $f\colon [0,1]\to[0,1]$, $f(x)=x^s$, $s\in(0,1)$.

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