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ON A REGULARITY OF DISTRIBUTION FOR SOLUTION OF SDE OF A JUMP TYPE WITH ARBITRARY LEVY MEASURE OF THE NOISE*

ПРО РЕГУЛЯРНІСТЬ РОЗПОДІЛУ РОЗВ'ЯЗКУ СДР ЗІ СТРИБКАМИ З ДОВІЛЬНОЮ МІРОЮ ЛЕВІ

In the paper the local properties of distributions of solutions of SDE's with jumps are studied. Using the method, based on the "time-wise" differentiation on the space of functionals from Poisson point measure, we give a full analogue of Hörmander condition, sufficient for the solution to have a regular distribution. This condition is formulated only in terms of coefficients of the equation and does not require any regularity properties of the Levy measure of the noise.

Вивчаються локальні властивості розв'язків СДР зі стрибками. При застосуванні методу, який базується на „диференційованні за часом" на просторі функціоналів від пуассонової точкової міри, наведено умову, яка аналогічна умові Хьормандера та достатня для того, щоб розв'язок мав регулярний розподіл. Ця умова формулюється тільки у термінах коефіцієнтів рівняння та не вимагає від міри Леві виконання будь-яких властивостей регулярності.

Introduction. In this article we deal with the following general problem. Let $X(x,t,r)$ be the solution of the SDE

$$X(x,t,r) = x + \int_r^t a(s, X(x,s,r))\, ds +$$

$$+ \int_r^t \int_{\mathbb{R}^d} c(s, X(x,s-,r),u)\tilde{\nu}(ds,du), \quad t \in [r,+\infty), \tag{0.1}$$

where ν is a random Poisson point measure on $\mathbb{R}^d \times \mathbb{R}^+$ with the Levy measure Π, $\tilde{\nu}$ is corresponding compensated measure (we are not going into details with introducing this standard objects, referring the reader, if necessary, to [1]), and coefficients a, c satisfy standard conditions, sufficient for equation (0.1) to have unique strong solution. Denote by $P(x,t,r,dy)$ the distribution of this solution, $P(x,t,r,dy) \equiv P(X(x,t,r) \in dy)$. It is natural both from probabilistic and analytical points of view to consider the following family of questions: does measure $P(x,t,r,dy)$ have a density $p(x,t,r,y)$ w.r.t. Lebesgue measure? Does this density, considered either as a function from y under fixed $t,x,$ or as a function from (x,t,r,y), possess some regularity property, for instance belongs to some $L_{p,\mathrm{loc}}$, is locally bounded, belongs to classes C^k or C^∞, etc? These questions were studied by numerous authors, let us emphasize two big groups of results in this direction, which are based on different ideas and impose essentially different conditions on the Levy measure Π of the noise.

The first group is based on the approach proposed by J. Bismut [2], in which some Malliavin-type calculus on a space of trajectories of Levy processes is introduced via transformations of trajectories, which change values of its jumps (see [2 – 5] and references there). In this approach Levy measure is supposed to have some (regular) density

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w.r.t. Lebesgue measure, which is a natural condition, sufficient for such transformations to be admissible.

The second group is based on the method by J. Picard [6], in which some version of stochastic calculus of variations for Poisson point measure is proposed. This method uses perturbations of the point measure by adding point into it and requires method some limitations on the asymptotic behavior of the Levy measure at the origin.

It is natural to try to give some sufficient conditions for the regular density to exist, which would not involve any specific conditions on the Levy measure. As a first possible answer on this question, let us mention recent results by V. N. Kolokol'tsov and A. D. Tyukov [7], who developed an analytical approach for SDE's of some special form, which, we believe, is not crucial and is caused by the framework of characteristics method for stochastic heat equation with a jump noise. This approach allows to prove regularity results for *small time* part of the initial distribution, this means that instead of $P(x,t,r,dy)$ the measures $E 1\!\!1_{X(x,t,r)\in dy}\cdot 1\!\!1_{t\leq\tau},$ where τ is some specific stopping time, are considered.

Another point of view on this problem was given in the recent work by the author [8]. It was motivated by a natural idea, that without any conditions on the Levy measure there always exist admissible transformations of the Poisson point measure $\nu,$ which change the moments of jumps, and one can construct some kind of stochastic calculus of variations based on these transformations. This idea is not very new, it was mentioned in the introduction to [6]. However, the rigorous development of this idea is nontrivial, it appears that the corresponding calculus have some new properties, which does not exist in Malliavin calculus for diffusions or Bismut calculus for jump processes with regular Levy measures (see discussions in [8] and Example 1.4 below).

In the work [8] the following two problems remained unsolved. First, sufficient condition for $P(x,t,r,dy)$ to have a density was given in the following form: some combination of differential and difference operators, defined by the coefficients of initial equation, has to be nondegenerated. This can be interpreted as a partial analogue of Hörmander condition, as soon as Hörmander condition is formulated in the terms not of one, but of a sequence of vector fields. Thus, it is natural to try to give a regularity result under a full analogue of Hörmander condition. Another problem is regularity properties of the density. It was shown in [8] (see also Example 1.4 below), that the density, considered as a function of $y,$ can be extremely nonregular, for instance, there exist situations in which it does not belong to $L_{1+\varepsilon,\mathrm{loc}}$ for $\varepsilon>0.$ At the same time, the properties of the density as a function of (t,x) were not studied. In this paper we solve the first problem and prove the regularity of the density under a full analogue of Hörmander condition.

1. Main result. We suppose that coefficients $a:\ \mathbb{R}^+\times\mathbb{R}^m\to\mathbb{R}^m,\ c:\mathbb{R}^+\times\mathbb{R}^m\times\times\mathbb{R}^d\to\mathbb{R}^m$ of equation (0.1) are measurable functions which are infinitely differentiable in (s,x) and locally bounded together with their derivatives. We also assume that

$$\exists K\quad \forall x,y\in\mathbb{R}^m,\quad s\in\mathbb{R}^+:$$

$$\int\limits_{\mathbb{R}^d}\|c(s,x,u)-c(s,y,u)\|^2\Pi(du)\leq K\|x-y\|^2,$$

so that (0.1) has a unique strong solution which is a process with cádlág trajectories. Also we suppose the following condition to hold true,

$$\int\limits_{\mathbb{R}^d} \sup_{s\in[0,T],\|x\|\le R} \Big[\|c(s,x,u)\|_{\mathbb{R}^m} + \|\nabla_x c(s,x,u)\|_{\mathbb{R}^m\times\mathbb{R}^m}\Big]\Pi(du) < +\infty \tag{1.1}$$

$$\text{for any positive} \quad T,\ R.$$

Under this condition (0.1) can be rewritten in the equivalent form

$$X(x,t,r) = x + \int\limits_r^t \tilde a(s,X(x,s,r))\,ds + \int\limits_r^t\int\limits_{\mathbb{R}^d} c(s,X(x,s-,r),u)\nu(ds,du), \quad t\in\mathbb{R}^+, \tag{1.2}$$

with $\tilde a(s,x) = a(s,x) - \displaystyle\int_{\mathbb{R}^d} c(s,x,u)\Pi(du)$.

Let us introduce some notations. For every function $\Upsilon(s,x,u),\ s\in\mathbb{R}^+,\ x\in\mathbb{R}^m$, $u\in\mathbb{R}^d$, which takes values in $\mathbb{R}^m$ and is smooth w.r.t. (s,x), we define $\Lambda\Upsilon\equiv\Lambda_{\tilde a}\Upsilon$ by

$$(\Lambda\Upsilon)(s,x,u) = \nabla_x\Upsilon(s,x,u)\tilde a(s,x) + \Upsilon'_s(s,x,u) - \nabla_x\tilde a(s,x)\Upsilon(s,x,u),$$

here ∇_x denotes vector derivative w.r.t. variable x. We also define $\Xi_u\Upsilon\equiv\Xi_{c,u}\Upsilon$ by

$$(\Xi_u\Upsilon)(s,x,u) = [I_{\mathbb{R}^m} + \nabla_x c(s,x,u))]^{-1}\Upsilon(s,x+c(x,u),u).$$

Note that the function $\Xi_u\Upsilon$ is well defined only for s,x,u satisfying assumption

$$-1\notin\sigma(\nabla_x c(s,x,u)), \tag{1.3}$$

we denote the set of such (s,x,u) by Θ and put $\Theta_{s,x} = \{u|(s,x,u)\in\Theta\}$.

For $(s,x,u)\in\Theta$ we put

$$\Delta(s,x,u) = [I_{\mathbb{R}^m} + \nabla_x c(s,x,u))]^{-1}\times$$
$$\times\Big[\{\tilde a(s,x+c(s,x,u)) - \tilde a(s,x)\} - \nabla_x c(s,x,u)\tilde a(s,x) - c'_s(s,x,u)\Big],$$

and introduce the family of $\mathbb{R}^m$-valued functions $\{\Delta_k^{i_0,\ldots,i_k},\ k\ge 0,\ i_r\in\mathbb{Z}_+,\ r = 0,\ldots,k\}$ by

$$\Delta_k^{i_0,\ldots,i_k}(s,x,u_0,\ldots,u_k) = \Lambda^{i_k}\Xi_{u_k}\Lambda^{i_{k-1}}\ldots\Lambda^{i_1}\Xi_{u_1}\Lambda^{i_0}\Delta(s,x,u_0),$$
$$s\in\mathbb{R}^+,\quad x\in\mathbb{R}^m,\quad u_0,\ldots,u_k\in\Theta_{s,x}.$$

Next, we denote by $\mathfrak{L}_k(s,x,u_0,\ldots,u_k),\ k\ge 0$ the linear span (in $\mathbb{R}^m$) of the vectors

$$\Big\{\Delta_j^{i_0,\ldots,i_j}(s,x,u_{0+r},\ldots,u_{j+r}),\quad i_0,\ldots,i_j\ge 0,\quad r=0,\ldots,k-j,\quad j=0,\ldots,k\Big\}.$$

One can see that the family $\{\mathfrak{L}_k\}$ is monotonous in a sense that $\mathfrak{L}_k(s,x,u_0,\ldots,u_k)\subset\mathfrak{L}_{k+1}(s,x,u_0,\ldots,u_{k+1})$.

At last, let us denote

$$\Pi^*_{k+1}(A) = \sup_{n\ge 1}\frac{\Pi^{\otimes(k+1)}\left(A\cap\left\{(u_0,\ldots,u_k)\in(\mathbb{R}^d)^{k+1}\,\middle|\,\|u_i\|>\dfrac{1}{n},\ i=0,\ldots,k\right\}\right)}{\left\{\Pi\left(\left\{u\in\mathbb{R}^d|\ \|u\|>\dfrac{1}{n}\right\}\right)\right\}^k},$$
$$A\in\mathcal{B}\Big(\big(\mathbb{R}^d\big)^{k+1}\Big).$$

For every k function Π^*_{k+1} posses the following properties:

1) $\Pi^*_{k+1}(A) \le \Pi^*_{k+1}(B)$ for $A \subset B$;

2) $\Pi^*_{k+1}(A) = \lim_{n\to\infty} \Pi^*_{k+1}(A_n)$ for $A_n \uparrow A$, $n \to +\infty$;

3) $\Pi^*_{k+1}(A) \le \sum_{n=1}^{\infty} \Pi^*_{k+1}(A_n)$ for $A \subset \bigcup_{n=1}^{\infty} A_n$.

Informally one can interpret the space $(\mathbb{R}^d)^{k+1}$ as the space of $(k+1)$-point configurations and Π^*_{k+1} as the outer measure, generated on this space by initial Levy measure.

Now we can formulate our main result.

Theorem 1.1. A. *Suppose that*

$$\forall x \in \mathbb{R}^m, \quad s \in \mathbb{R}^+, \quad \bar{l} \in \mathbb{R}^m \backslash \{0\}:$$

$$\Pi\Big\{u \in \Theta_{s,x}: \ \bar{l} \text{ is not orthogonal to } \mathfrak{L}_0(s,x,u)\Big\} = +\infty. \tag{1.4}$$

Then for every $x \in \mathbb{R}^m$, $0 \le r < t$

$$\mathrm{P} \circ [X(x,t,r)]^{-1} \ll \lambda^m.$$

B. *Suppose that there exists* $k > 0$ *such that for every* $x \in \mathbb{R}^m$, $s \in \mathbb{R}^+$, $\bar{l} \in \mathbb{R}^m \backslash \{0\}$

$$\Pi^*_{k+1}\Big\{(u_0,\ldots,u_k) \in [\Theta_{s,x}]^{k+1}:$$

$$\bar{l} \text{ is not orthogonal to } \mathfrak{L}_k(s,x,u_0,\ldots,u_k)\Big\} = +\infty. \tag{1.5}$$

Suppose also two following additional conditions to hold true:

$\mathcal{A}$) *there exists* $C > 0$ *such that functions* $a(\cdot,\cdot)$ *and* $c(\cdot,\cdot,u)$ *for* Π-*almost all* $u \in \mathbb{R}^d$ *are analytical functions in every point* $(s,x) \in \mathbb{R}^+ \times \mathbb{R}^m$ *with the radius of analyticity not less than* C;

$\mathcal{B}$) $\displaystyle\int_{\mathbb{R}^d} \sup_{s\in[0,T],\|x\|\le R} \Big[\|(\nabla_x)^j c(s,x,u)\|_{(\mathbb{R}^m)^{\times j}}\Big]\Pi(du) < +\infty$ *for any* $j \in \mathbb{N}$, $T > 0$, $R > 0$.

Then for every $x \in \mathbb{R}^m$, $0 \le r < t$

$$\mathrm{P} \circ [X(x,t,r)]^{-1} \ll \lambda^m.$$

The proof of Theorem 1.1 will be given in Section 2, some improvements will be given in Section 3. Here let us make some discussion.

It was shown in [8] (see also Example 1.4 below) that the density $p(x,t,r,y)$ of distribution of $X(x,t,r)$, considered as a function of y, can be extremely nonregular, for instance, there exist situations in which it does not belong to $L_{p,\mathrm{loc}}$ for every $p > 1$. At the same time, the properties of this density, considered as a function of (t,x), were not studied.

Proposition 1.1. *Under conditions of Theorem* 1.1 *the function*

$$\mathbb{R}^m \times \{(t,r) \in (\mathbb{R}^+)^2 | t > r\} \ni (x,t,r) \mapsto p(x,t,r,\cdot) \in L_1(\mathbb{R}^m)$$

is continuous.

The proof of this statement is a subject of a separate paper [9] and is based in the methods, developed recently in [10].

Next, the statement of Theorem 1.1 can be rewritten in a form, which is natural from the point of view of theory of pseudo-differential operators, let us do this in time-homogeneous case.

For an $\mathbb{R}^m$-valued function $\Upsilon(x)$ denote by the same letter Υ differential operator on $\mathbb{R}^m$, defined by

$$(\Upsilon f) = (\nabla f, \Upsilon)_{\mathbb{R}^m}, \quad f \in C_b^1(\mathbb{R}^m),$$

then it is clear that

$$\Lambda\Upsilon = [\tilde{a}, \Upsilon] \equiv \tilde{a} \cdot \Upsilon - \Upsilon \cdot \tilde{a}.$$

Also denote by C_u (for every $u \in \mathbb{R}^d$) difference operator defined by

$$C_u f(x) = f(x + c(x,u)) - f(x), \quad f \in C_b(\mathbb{R}^m).$$

Under more strong version of condition (1.3),

$$\sup_{u\in\mathbb{R}^d, x\in\mathbb{R}^m} \|\nabla_x c(x,u)\| < 1, \tag{1.3$'$}$$

operator $I + C_u$ is invertible (here I is identity operator), and one can see that

$$\Xi_u\Upsilon = (I + C_u)\Upsilon(I + C_u)^{-1}, \quad \Delta(\cdot, u) = \tilde{a} - \Xi_u\tilde{a}.$$

Statements of Theorem 1.1.A and Proposition 1.1 now can be reformulated in the following form.

***Corollary* 1.1.** *Let us say that the family of operators* $\{\Psi_{u,k}, k \geq 0\}$, *indexed by* $u \in \mathbb{R}^d$, *is nondegenerated w.r.t. measure* Π *if for every* $x \in \mathbb{R}^m$ *and every* $f \in C^1(\mathbb{R}^m)$ *such that* $\nabla f(x) \neq 0$

$$\Pi\Big(\{u | \exists k \geq 0 : \Psi_{u,k} f(x) \neq 0\}\Big) = +\infty.$$

Consider the family

$$\Psi_{0,u} = [\tilde{a}, C_u](I+C_u)^{-1} = \tilde{a} - (I+C_u)\tilde{a}(I+C_u)^{-1}, \quad \Psi_{k,u} = [\tilde{a}, \Psi_{k-1,u}], \quad k > 0,$$

and suppose that it is nondegenerated w.r.t. measure Π. *Then for the PDO* L, *given by the formula*

$$L = \tilde{a} + \int_{\mathbb{R}^d} C_u\,\Pi(du),$$

the fundamental solution of equation

$$u_t' = Lu$$

is usual (*not generalized*) *function, which is continuous while considered as a function from* $\mathbb{R}^m \times \mathbb{R}^+$ *to* $L_1(\mathbb{R}^m)$.

Statement B of Theorem 1.1 also can be reformulated in the same way, we omit this in order to shorten exposition.

At last, let us give several examples, illustrating different features of the regularity result, given by Theorem 1.1.

The first example shows that Theorem 1.1 is a crucial improvement of Theorem 4.2 [8]. It is motivated by the classical Kolmogorov's example of a diffusion, which hypoellipticity can not be provided only by condition on a diffusion part, see [11] or [12], Chapter 5, Example 8.1.

***Example* 1.1.** Consider SDE

$$X_1(t,\bar{x}) = x_1 + \int_0^t X_2(s,\bar{x})\,ds + \eta_t,$$
$$X_2(t,\bar{x}) = x_2 + \int_0^t X_1(s,\bar{x})\,ds, \tag{1.6}$$

where $\bar{x} = (x_1, x_2) \in \mathbb{R}^2$ and η_t is some compensated Levy process with infinite Levy measure. Then this equation is of the type (0.1) with $m = 2,\ d = 1$ and

$$a(s,\bar{x}) = (x_2, x_1)^T, \qquad c(s,\bar{x},u) = (u,0)^T.$$

One has

$$\Delta(\bar{x},u) = (0,u)^T, \qquad \Delta_0^1(\bar{x},u) = -(u,0)^T,$$

which means that for every $x \in \mathbb{R}^2,\ u \neq 0\ \ \mathfrak{L}_0(x,u) = \mathbb{R}^2$ and therefore condition of Theorem 1.1 holds true. Note that for $\bar{l} = (1,0)^T\ \ \Pi(u|(\bar{l},\Delta(\bar{x},u)) \neq 0) = 0$ for every $x \in \mathbb{R}^2,$ which means that condition of Theorem 4.2 [8] fails.

It is worth to mention that equation (1.6) is not the full analogue of Example 8.1 [12]. The corresponding analogue should be written as follows:

$$X_1(t,\bar{x}) = x_1 + \int_0^t X_1(s,\bar{x})\,ds + \eta_t,$$
$$X_2(t,\bar{x}) = x_2 + \int_0^t X_1(s,\bar{x})\,ds. \tag{1.6$'$}$$

Equation $(1.6')$ gives the simple counterexample, which shows that conditions of the Theorem 1.1 are close to necessary ones. Namely, in this case $\Delta_k^{i_0,\ldots,i_k}(\bar{x},u_0,\ldots,u_k) = (u_0,u_0)^T$ for every $k \geq 1,\ i_0,\ldots,i_k \geq 1,\ \bar{x} \in \mathbb{R}^2, u_0,\ldots,\ u_k \in \mathbb{R},$ and conditions of the Theorem fail. On the other hand, one can choose Π in such a way that the distribution of $X_1(t) - X_2(t) = \eta_t$ is not absolutely continuous (see Example 1.4 below), and for Π the joint distribution $(X_1(t), X_2(t))$ definitely is not absolutely continuous.

Next, let $\psi \in C^\infty(\mathbb{R})$ be globally Lipschitz. Let us consider equation

$$X_1(t,\bar{x}) = x_1 + \int_0^t \psi(X_2(s,x))\,ds + \eta_t,$$
$$X_2(t,\bar{x}) = x_2 + \int_0^t X_1(s,x)\,ds. \tag{1.7}$$

One has that

$$\Delta(\bar{x},u) = (0,u)^T, \qquad \Delta_0^r(\bar{x},u) = (-u)^r(\psi^{(r)}(x_2),0)^T.$$

Now let us take $\psi(x)$ which is equal in some neighborhood of 0 to $x^r,\ r \in \mathbb{N},$ then equation (1.7) shows that, in general, we can not replace in statement A the family $\mathfrak{L}_0(x,u)$ by $\mathfrak{L}_0^r(x,u) \equiv \langle \Delta_0^i(x,u),\ i \leq r \rangle.$

In the second example conditions of statement B of the Theorem hold true, but conditions of statement A fail.

Example **1.2.** Consider SDE

$$X_1(t,\bar{x}) = x_1 + \int\limits_0^t X_1(s,\bar{x})\,ds + \eta_t^1,$$
$$X_2(t,\bar{x}) = x_2 + \int\limits_0^t X_1(s,\bar{x})X_2(s,\bar{x})\,ds + \eta_t^2. \tag{1.8}$$

We suppose that the Levy measure Π of the Levy process $\eta_t = (\eta_t^1, \eta_t^2)$ is concentrated on the set $\{(u_1,u_2)\in\mathbb{R}^2 : u_1\cdot u_2 = 0\}$ and denote by Π^1, Π^2 restrictions of Π on the axis $\{(u_1,u_2) : u_2 = 0\}$ and $\{(u_1,u_2) : u_1 = 0\}$ correspondingly. Straightforward computations give that

$$\Delta(\bar{x},\bar{u}) = (u_1, u_1x_2 + u_2x_1)^T, \qquad \Delta_0^r(\bar{x},\bar{u}) = (-1)^r(u_1, P_{r,u_1,u_2}(x_1,x_2))^T,$$

where P_{r,u_1,u_2} is some polynom with the free term equal to zero. This means that for $\bar{x} = (0,0)^T$, $\bar{l} = (0,1)^T$ condition (1.4) fails.

On the other hand, consider vectors

$$\Delta(\bar{x},\bar{u}^0), \qquad \Delta_1^{0,0}(\bar{x},\bar{u}^0,\bar{u}^1) = \left(u_1^0, u_1^0(x_2+u_2^1) + u_2^0(x_1+u_1^1)\right)^T,$$

one can see that these vectors generate $\mathbb{R}^2$ for every $\bar{u}^0,\bar{u}^1\in$ supp Π such that $u_1^0\neq 0$, $u_2^1\neq 0$. For the set A of such pairs $(\bar{u}^0,\bar{u}^1)$ we have that

$$\Pi_2^*(A) = \sup_n \frac{\Pi^1\left(\|\bar{u}\| > \dfrac{1}{n}\right)\Pi^2\left(\|\bar{u}\| > \dfrac{1}{n}\right)}{\Pi\left(\|\bar{u}\| > \dfrac{1}{n}\right)},$$

and $\Pi_2^*(A) = +\infty$ under condition

$$\Pi^1(\mathbb{R}^2) = +\infty, \qquad \Pi^2(\mathbb{R}^2) = +\infty. \tag{1.9}$$

It is easy to see that condition (1.9) is a nessesary one: if it fails, then for the solution of (1.8) with $x_1 = x_2 = 0$ the distribution either of $X_1(t)$ or $X_2(t)$ has an atom.

The third example shows the following interesting feature. In the usual Hörmander condition the linear subspace, generated by the corresponding family of vector fields, is supposed to have maximal possible dimension. This example shows, that condition of Theorem 1.1 can hold true even if dim $\mathfrak{L}_k(s,x,u) < m$ for every s,x,u,k.

Example **1.3.** Consider SDE

$$X_1(t,\bar{x}) = x_1 + \int\limits_0^t X_1(s,\bar{x})\,ds + \eta_t^1,$$
$$X_2(t,\bar{x}) = x_2 + \int\limits_0^t X_2(s,\bar{x})\,ds + \eta_t^2,$$

where $\eta_t = \left(\eta_t^1, \eta_t^2\right)$ is a compensated two-dimensional Levy process. One has that for every $k,\, i_0,\ldots,i_k\in\mathbb{Z}_+$

$$\Delta_k^{i_0,\dots,i_k}(\bar{x}, u_0, u_1, \dots, u_k) = \Delta(\bar{x}, u_0) = u_0, \qquad u_0, \dots, u_k, \ \bar{x} \in \mathbb{R}^2,$$

and therefore dim $\mathfrak{L}_k(\bar{x}, u_0, \dots, u_k) = 1$ for $u_0 \neq 0$ and dim $\mathfrak{L}_k(\bar{x}, u_0, \dots, u_k) = 0$ otherwise. On the other hand, if for every $\bar{l} \in \mathbb{R}^2$

$$\Pi(\bar{u} \notin \langle \bar{l} \rangle) = +\infty,$$

condition of Theorem 1.1 hold true.

One can say, that in the considered examples solution of equation

$$X(t, x) = x + \int_0^t X(s, x)\, ds + \eta_t \tag{1.10}$$

plays the role of an analogue of one-dimensional diffusion with constant coefficients. Indeed, in condition of Theorem 1.1, considered as an analogue of Hörmander condition, function Δ corresponds to the vector field generated by diffusion coefficient, and for equation (1.10) Δ does not depend on x. The following example (last in this section) shows, that even in this simplest case one can hardly expect to obtain for the density, given by Theorem 1.1, any regularity properties better than given in Proposition 1.1.

***Example* 1.4.** Consider (1.10) with $\eta_t = \sum_{k=1}^{\infty} \frac{1}{\alpha^k} \eta_t^k$, where $\alpha > 1$, $\{\eta^k\}$ are independent Poisson processes with the same intensity λ. Let for simplicity $x = 0$, then one can show (see [8], Chapter 5) that there exists $\beta = \beta(\alpha) > 0$ such that

$$\limsup_{\varepsilon \to 0+} \varepsilon^{\frac{-\lambda t}{\beta}} \mathrm{P}(X(t, 0) > \varepsilon) > 0. \tag{1.11}$$

We have $\Delta(x, u) = u$, i.e., conditions of Theorem 1.1 hold true and there exists a density $p(t, \cdot)$ of distribution of $X(t, 0)$. We have $\mathrm{P}(X(t, 0) < 0) = 0$, therefore from (1.11) we obtain that

$$\begin{gathered} p(t, \cdot) \notin C^k(\mathbb{R}), \quad \text{if} \quad \lambda t < \beta(k+1), \quad k \geq 1, \\ p(t, \cdot) \notin L_{\infty,\mathrm{loc}}(\mathbb{R}), \quad \text{if} \quad \lambda t < \beta, \\ p(t, \cdot) \notin L_{p,\mathrm{loc}}(\mathbb{R}), \quad \text{if} \quad \lambda t < \beta\left(1 - \frac{1}{p}\right), \quad p \in (1, +\infty). \end{gathered} \tag{1.12}$$

It appears, that statements (1.12) are rather precise, namely, as soon as (1.10) is a linear equation, one can calculate the characteristic function $\varphi_t(\cdot)$ of $X(t, 0)$ explicitly and then obtain the estimate $\varphi_t(z) = o\left(|z|^{-\frac{\lambda t}{\gamma}}\right)$, $|z| \to \infty$, with some $\gamma = \gamma(\alpha) > 0$, which means that

$$p(t, \cdot) \in C^k(\mathbb{R}), \qquad \lambda t > \gamma(k+1), \quad k \geq 0.$$

This forms the phenomenon, which can be call “gradual hypoellipticity”: fundamental solution of the corresponding equation with PDO becomes smooth not instantly, but after some period of time, and this period depends linearly on the rate of smoothness which is to be achieved.

The feature of “gradual hypoellipticity” is interesting, but not very common, this can be illustrated by the following modification of the example. Using the same arguments to those made in [8] (Chapter 5), one can construct for every given function φ with $\varphi(0+) = +\infty$ a sequence $\{\alpha_k\} \in \mathbb{R}^+$, such that for the solution of (1.10) with $\eta_t = \sum_k \alpha_k \eta_t^k$ the following analogue of (1.11) holds true:

$$\forall t > 0: \quad \limsup_{\varepsilon \to 0+} \varphi(\varepsilon) \mathrm{P}(X(t,0) > \varepsilon) > 0. \tag{1.13}$$

This can be reformulated in the following form. Let Φ be positive convex function on $\mathbb{R}$, denote by L^{Φ}_{loc} the space of the functions f on $\mathbb{R}$ such that

$$\int\limits_I \Phi(f(x))\, dx < +\infty$$

for every finite interval I. The following statement is due to (1.13) and Jensen's inequality.

Proposition 1.2. *For any fixed* Φ *with* $\dfrac{\Phi(x)}{|x|} \to \infty,\ |x| \to \infty,$ *there exists an equation of the type* (1.10) *such that*

$$p(t,\cdot) \notin L^{\Phi}_{\mathrm{loc}}, \quad t > 0.$$

It is worth to be mentioned that these interesting features do not occur in classes of equations, which can be treated by methods of J. Bismut or J. Picard. Another new feature is that, at least for some values of α (say, integers or so called P.V. numbers), distribution of $\eta_t = \sum_{k=1}^{\infty} \dfrac{1}{\alpha^k} \eta_t^k$ is singular for every t. The situation, when the solution of SDE is regular while the noise, driving this equation, is not, seems not to be studied systematically yet.

As a conclusive remark let us say that our method of proof (some modification of stratification method) appears to be well suited for a "boundary region" of equations of the type (0.1), in which such phenomenons, as "gradual hypoellipticity" or singularity of initial noise, hold, and which can not be treated by other known methods. However, the price is that this method does not allow one to obtain general results on regularity, more strong than given in Proposition 1.1.

2. Proof. First we will prove the theorem, supposing coefficient c to satisfy additional condition

$$\sup_{s \in \mathbb{R}^+,\, u \in \mathbb{R}^d,\, x \in \mathbb{R}^m} \|\nabla_x c(s,x,u)\| < 1. \tag{2.1}$$

Denote by $\{\mathcal{E}^t_r\}$ the $m \times m$-matrix valued process satisfying equation

$$\mathcal{E}^t_r = I_{\mathbb{R}^m} + \int\limits_r^t \nabla_x \tilde{a}(s, X(s)) \mathcal{E}^s_r\, ds +$$
$$+ \iint\limits_{[r,t]\times\mathbb{R}^d} \nabla_x c(s, X(s-), u) \mathcal{E}^{s-}_r\, \nu(ds, du).$$

Under condition (2.1) matrix $\mathcal{E}^t_r$ is a.s. invertible for every $r,\ t$.

The starting point in our proof is the following statement (see [8], Theorem 4.1). Denote by $p(\cdot)$ the point process corresponding to random point measure ν.

Proposition 2.1. *Denote by* S_t *the linear span of the set of vectors* $\Big\{[\mathcal{E}^{\tau-}_0]^{-1}\Delta(\tau, X(\tau-),\ p(\tau)),\ \tau \le t\Big\}$, *where* τ*'s are taken from the domain* $\mathcal{D}$ *of the process* $p(\cdot)$. *Let* $\Omega_t = \{\omega |\dim S_t(\omega) = m\}$, *then*

$$\mathrm{P}|_{\Omega_t} \circ [X(t)]^{-1} \ll \lambda^m.$$

Our aim is to show that under conditions of Theorem 1.1 the set Ω_t coincides with Ω almost surely. The method of proof is up to [8, 13] and is based on so called "time-stretching" transformations of the jump process, let us briefly give here necessary constructions. Denote $H = L_2(\mathbb{R}^+)$, $H_0 = L_\infty(\mathbb{R}^+) \cap L_2(\mathbb{R}^+)$, $Jh(\cdot) = \int_0^{\cdot} h(s)\, ds$, $h \in H$. For a fixed $h \in H_0$ define the family $\{T_h^t, t \in \mathbb{R}\}$ of transformations of the axis $\mathbb{R}^+$ by putting $T_h^t x$, $x \in \mathbb{R}^+$ equal to the value at the point $s = t$ of the solution of the Cauchy problem

$$z'_{x,h}(s) = Jh(z_{x,h}(s)), \qquad s \in \mathbb{R}, \quad z_{x,h}(0) = x.$$

The following properties hold:

a) $T_h^{s+t} = T_h^s \circ T_h^t$;

b) $\dfrac{d}{dt} T_h^t x|_{t=0} = Jh(x)$;

c) $T_h^t = T_{th}^1$.

We denote $T_h \equiv T_h^1$. Denote also $\Pi_{\text{fin}} = \big\{\Gamma \in \mathcal{B}(\mathbb{R}^d),\, \Pi(\Gamma) < +\infty\big\}$ and define for $h \in H_0, \Gamma \in \Pi_{\text{fin}}$ transformation T_h^Γ of the random measure ν by

$$\big[T_h^\Gamma \nu\big]([0,t] \times \Delta) =$$

$$= \nu\big([0, T_{-h}t] \times (\Delta \cap \Gamma)) + \nu([0,t] \times (\Delta \backslash \Gamma)\big), \qquad t \in \mathbb{R}^+, \quad \Delta \in \Pi_{\text{fin}}.$$

Transformation T_h^Γ is admissible for the distribution of ν in a sense that there exists function p_h^Γ (which can be given explicitly), such that for every $\{t_1, \ldots, t_n\} \subset \mathbb{R}^+$, $\{\Delta_1, \ldots, \Delta_n\} \subset \Pi_{\text{fin}}$ and Borel function $\varphi : \mathbb{R}^n \to \mathbb{R}$

$$\mathrm{E}\varphi\Big([T_h^\Gamma \nu]([0,t_1] \times \Delta_1), \ldots, [T_h^\Gamma \nu]([0,t_n] \times \Delta_n)\Big) =$$

$$= \mathrm{E} p_h^\Gamma \varphi\Big(\nu([0,t_1] \times \Delta_1), \ldots, \nu([0,t_n] \times \Delta_n)\Big).$$

This fact, under additional condition that σ-algebra of all random events is generated by ν, imply that T_h^Γ generates the corresponding transformation of random variables, we denote it also by T_h^Γ.

For a given $h \in H_0, \Gamma \in \Pi_{\text{fin}}$ and random variable f denote

$$\partial_h^\Gamma f = \lim_{\varepsilon \to 0} \frac{T_{\varepsilon h}^\Gamma f - f}{\varepsilon}, \tag{2.2}$$

the variable in the left hand side of (2.2) is defined on the set of such $\omega \in \Omega$, that the limit in the right-hand side exists. The key point in our considerations is the following simple statement.

Lemma 2.1. *Let f be a random variable, $h_1, \ldots, h_k \in H_0$, $\Gamma_1, \ldots, \Gamma_k \in \Pi_{\text{fin}}$ and*

$$A = \Big\{(\partial_{h_1}^{\Gamma_1}) \ldots (\partial_{h_k}^{\Gamma_k}) f \quad \textit{is defined and} \quad \neq 0\Big\},$$

then

$$\mathrm{P}|_A \circ f^{-1} \ll \lambda^1.$$

Sketch of the proof. One can verify that for a fixed h, Γ the transformation T_h^Γ generates measurable stratification of the initial space Ω, and therefore using stratification method (see [5], Chapter 2.5) one can show that

$$\mathrm{P}|_{\{\partial_h^\Gamma f\neq 0\}} \circ f^{-1} \ll \lambda^1, \tag{2.3}$$

i.e., the needed statement holds true for $k=1$. Statement (2.3) implies that $\mathrm{P}\Big(\{\partial_h^\Gamma f \neq \neq 0\} \cap \{f=0\}\Big) = 0$, which gives an opportunity to prove statement of the lemma by induction.

Let us prove first the statement A of Theorem 1.1, which is more simple. Let $\Upsilon(s,x,u)$ be a fixed vector-valued function, $S\subset\mathbb{R}^m$ be some subspace.

Lemma 2.2. *For every* $s<t$

$$\Big\{\exists\tau\in\mathcal{D}\cap(s,t):\,[\mathcal{E}_0^{\tau-}]^{-1}(\Lambda\Upsilon)(\tau,X(\tau-),p(\tau))\notin S\Big\}\subset$$
$$\subset\Big\{\exists\tau\in\mathcal{D}\cap(s,t):\,[\mathcal{E}_0^{\tau-}]^{-1}\Upsilon(\tau,X(\tau-),p(\tau))\notin S\Big\}$$

almost surely.

Proof. Denote by $l_1,\ldots,l_k$ some basis in $S^\perp$, and put

$$\Omega_{s,t,j,n}\equiv\Big\{\exists\tau\in\mathcal{D}_n\cap(s,t):\,\big([\mathcal{E}_0^{\tau-}]^{-1}(\Lambda\Upsilon)(\tau,X(\tau-),p(\tau)),l_j\big)_{\mathbb{R}^m}\neq 0\Big\}, \tag{2.4}$$

where $\mathcal{D}_n\equiv\left\{\tau\in\mathcal{D}|\|p(\tau)\|>\dfrac{1}{n}\right\}$. In order to prove the needed statement it is enough to show that for every $s<t,\ j\le k,\ n\ge 1$,

$$\Omega_{s,t,j,n}\subset\Big\{\exists\tau\in\mathcal{D}_n\cap(s,t):\,[\mathcal{E}_0^{\tau-}]^{-1}\Upsilon(\tau,X(\tau-),p(\tau))\notin S\Big\} \tag{2.5}$$

almost surely. Let $s<t,\ j\le k,\ n\ge 1$ be fixed, we define $\tilde\tau$ on the set $\Omega_{s,t,j,n}$ as *the first* point from $\mathcal{D}_n$, satisfying condition in the right-hand side of (2.4), and denote $\Psi=\Big([\mathcal{E}_0^{\tilde\tau-}]^{-1}\Upsilon(\tilde\tau,X(\tilde\tau-),p(\tilde\tau)),l_j\Big)_{\mathbb{R}^m}$. We shall prove that

$$\mathrm{P}|_{\Omega_{s,t,j,n}}\circ\Psi^{-1}\ll\lambda^1, \tag{2.6}$$

this will provide (2.5). For $N,r\in\mathbb{N}$ denote $\Omega_N^r=\left\{\mathcal{D}_n\cap\left(\dfrac{r-1}{N},\dfrac{r}{N}\right]=\{\tilde\tau\}\right\}$, one can see that $\mathrm{P}(\bigcup_{N,r}\Omega_N^r)=1$. Let us show that for $h_N^r=1\!\!\mathrm{I}_{(\frac{r-1}{N},\frac{r}{N}]}$, $\Gamma_n=\left\{u\,\big|\,\|u\|> > \dfrac{1}{n}\right\}$ almost surely on the set Ω_N^r there exist

$$\lim_{\varepsilon\to 0}\frac{T^{\Gamma_n}_{\varepsilon h_N^r}\Psi-\Psi}{\varepsilon}=-[Jh_N^r](\tilde\tau)\Big(\Big[\mathcal{E}_0^{\tilde\tau-}\Big]^{-1}(\Lambda\Upsilon)(\tilde\tau,X(\tilde\tau-),p(\tilde\tau)),l_j\Big)_{\mathbb{R}^m}. \tag{2.7}$$

This, together with Lemma 2.1, will give the needed statement, as soon as $[Jh_N^r](\tilde\tau)\neq 0$ on Ω_N^r. By the construction,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\Big[T^{\Gamma_n}_{\varepsilon h_N^r}\tilde\tau\Big]=-[Jh_N^r](\tilde\tau). \tag{2.8}$$

In order to find $\dfrac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\Big[T^{\Gamma_n}_{\varepsilon h_N^r}X(\tilde\tau-)\Big]$, let us note that on the set Ω_N^r for ε small enough $T^{\Gamma_n}_{\varepsilon h_N^r}X(\tilde\tau-)=\tilde X\Big(T^{\Gamma_n}_{\varepsilon h_N^r}\tilde\tau\Big)$, where $\tilde X$ is a solution of equation

$$\tilde X(v)=X(s)+\int_s^v\tilde a(z,\tilde X(z))\,dz+\int_s^v\int_{\|u\|\le\frac{1}{n}}c(z,\tilde X(z-),u)\nu(ds,du).$$

Under condition (1.1) almost every trajectory of the process $\tilde{X}$ is differentiable by v for almost all v w.r.t. Lebesgue measure on $[s,+\infty)$, and the corresponding derivative is equal to $\tilde{X}'(v)=\tilde{a}(v,\tilde{X}(v))$. Distribution of $\tilde{\tau}$ is absolutely continuous, $\tilde{\tau}$ and $\tilde{X}$ are independent. Therefore

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\Big[T^{\Gamma_n}_{\varepsilon h^r_N}\tilde{X}(\tilde{\tau})\Big]=-[Jh^r_N](\tilde{\tau})\tilde{a}(\tilde{\tau},X(\tilde{\tau}-)) \tag{2.9}$$

almost surely on Ω^r_N. The same considerations give that almost surely on Ω^r_N

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\Big[T^{\Gamma_n}_{\varepsilon h^r_N}\big[\mathcal{E}_0^{\tilde{\tau}-}\big]^{-1}\Big]=[Jh^r_N](\tilde{\tau})[\mathcal{E}_0^{\tilde{\tau}-}]^{-1}\nabla_x\tilde{a}(\tilde{\tau},X(\tilde{\tau}-)). \tag{2.10}$$

Equalities (2.8) – (2.10) together with the chain rule give (2.7).

The lemma is proved.

The end of the proof of statement A repeats the proof of Theorem 4.2 [8], let us give it here briefly. First let us give the following useful statement, which is a generalization of Lemma 4.3 [8].

Lemma 2.3. *Suppose that the following objects are chosen.*

1. *A measurable space* $(U,\mathcal{U})$ *with a measure* μ *on it and a compact metric space* Z.

2. *A sequence of functions* $\{f_n: Z\times U\to\mathbb{R}, n\in\mathbb{N}\}$, *such that every* f_n *is measurable w.r.t. second coordinate when the first one is fixed and is continuous w.r.t. first coordinate when the second one is fixed.*

3. *A sequence* $\{\alpha_r\}\subset\mathbb{R}^+$, *a sequence of open sets* $\{O_{n,k}\subset\mathbb{R},\ n,k\in\mathbb{N}\}$, *monotonously increasing by* k *for every fixed* n, *and a monotonously increasing sequence of measurable sets* $\{U_r\subset U\}$ *with* $\mu(U_r)<+\infty$ *and* $\cup_r U_r=U$.

Denote $O_n=\cup_k O_{n,k}$ *and suppose that for every* $z\in Z$

$$\sup_r\Big[\alpha_r\mu\{u\in U_r|\,\exists n\in\mathbb{N}:\ f_n(z,u)\in O_n\}\Big]=+\infty,$$

then

$$\lim_{n,k,r\to\infty}\inf_{z\in Z}\sup_{q\le r}\Big[\alpha_q\mu\{u\in U_q|\,\exists i\le n:\ f_i(z,u)\in O_{i,k}\}\Big]=+\infty.$$

Proof. Let us consider functions

$$\varphi_{n,k,r}(z)=\sup_{q\le r}\Big[\alpha_q\mu\{u\in U_q|\,\exists i\le n:\ f_i(z,u)\in O_{i,k}\}\Big],$$

due to conditions of the lemma for every $z\in Z$ $\varphi_{n,k,r}(z)$ tends to $+\infty$ and is monotonous w.r.t. every index n,k,r while others are fixed. Moreover, every function $\varphi_{n,k,r}$ is lower semicontinuous, i.e., for every sequence $z_j\to z$ we have the inequality $\varphi_{n,k,r}(z)\le\liminf_j\varphi_{n,k,r}(z_j)$. Therefore the needed statement holds true due to the correspondent version of the Dini theorem.

The lemma is proved.

***Corollary* 2.1.** *Let* K *be some compact subset in* $\mathbb{R}^m$, *take* $Z=[0,T]\times K\times\times\{\bar{l}\in\mathbb{R}^m:\|\bar{l}\|=1\}$, $\mu=\Pi$, $U_r=\Big\{u:\|u\|>\dfrac{1}{r}\Big\}$, $\alpha_r\equiv1$, $f_n(s,x,\bar{l},u)==\big(\Delta_0^n(s,x,u),\bar{l}\big)_{\mathbb{R}^m}$, $O_{n,k}\equiv\mathbb{R}\backslash\{0\}$. *Then due to lemma under condition* (1.4) *for every* $T<+\infty$ *and compact set* $K\subset\mathbb{R}^m$

$$\lim_{n,k\to+\infty}\ \inf_{s\le T,x\in K,\bar{l}\ne\bar{0}}\Pi\bigg\{u|\|u\|\ge\frac{1}{n},\ \exists j\le k:\ \bar{l}\ \textit{is not orthogonal to}\ \Delta_0^j(s,x,u)\bigg\}=+\infty. \tag{2.11}$$

Another corollary will be given below, in the proof of statement B (see (2.18)).

In order to shorten notations we suppose further that for some compact $K \subset \mathbb{R}^m$ $X(s) \in K$, $s \leq t$ a.s., the standard way to give rigorous basis for this supposition is the following one. Take the Markov moment ζ_K of the exit of $X(\cdot)$ from the set K and consider the new process $X^K(\cdot) = X(\cdot \wedge \zeta_K)$. For this process all estimates, given below, hold true, and for every given t the probability of the set $\{X|_{[0,t]} \neq X^K|_{[0,t]}\} \subset \subset \{\zeta_K > t\}$ can be made arbitrary small by an appropriate choice of K.

Let n, k be fixed, denote by τ_i^n the i-th point from $\mathcal{D}_n$, $S_t^n = \Big\langle \left[\mathcal{E}_0^{\tau-}\right]^{-1}\Delta(\tau, X(\tau-), p(\tau)),\ \tau \in \mathcal{D}_n,\ \tau \leq t \Big\rangle$. Due to Fubini theorem, for every $s \leq t$

$$\mathrm{P}\Big(\dim S_t^n = \dim S_s^n \,\Big|\, \dim S_s^n < m\Big) =$$

$$= \int\limits_{\{\dim S < m\}\times\mathbb{R}^m} \mathrm{P}(\forall \tau \in \mathcal{D}_n \cap (s,t) \times$$

$$\times Q\left[\mathcal{E}_s^{\tau-}\right]^{-1}\Delta(\tau, X(y, \tau-, s), \rho(\tau)) \in S\Big) \varkappa_{s,n}(dS, dy, dQ),$$

here we suppose that the space of all subspaces of $\mathbb{R}^m$ is parameterized in such a way that it becomes a Polish space, and $\varkappa_{s,n}$ is the joint distribution of $S_s^n, X(s)$ and $[\mathcal{E}_0^s]^{-1}$.

Due to Lemma 2.2 for every $S \neq \mathbb{R}^m$, $y \in \mathbb{R}^m$ one has

$$\mathrm{P}\Big(\forall \tau \in \mathcal{D}_n \cap (s,t),\ \ [\mathcal{E}_0^{\tau-}]^{-1}\Delta(\tau, X(y, \tau-, s), \rho(\tau)) \notin S\Big) =$$

$$= \mathrm{P}\Big(\forall \tau \in \mathcal{D}_n \cap (s,t),\ \ \exists j \leq k\ \ [\mathcal{E}_0^{\tau-}]^{-1}\Delta_0^j(\tau, X(y, \tau-, s), \rho(\tau)) \notin S\Big) \geq$$

$$\geq \inf_{\bar{l}\neq\bar{0}} \mathrm{P}\Big(\forall \tau \in \mathcal{D}_n \cap (s,t),\ \ \exists j \leq k\ \ [\mathcal{E}_0^{\tau-}]^{-1}\Delta_j(\tau, X(y, \tau-, s), \rho(\tau)) \not\perp \bar{l}\Big). \quad (2.12)$$

The variable $p(\tau_i^n)$ (the value of the i-th jump from $\mathcal{D}_n$) is independent from the values of others jumps, from the moments of all jumps and from $\left[\mathcal{E}_0^{\tau_i^n-}\right]^{-1}$. The distribution of $p(\tau_i^n)$ is equal $\dfrac{\Pi|_{\{u:\|u\|\geq\frac{1}{n}\}}}{\Pi\Big(\Big\{u : \|u\| \geq \dfrac{1}{n}\Big\}\Big)}$. Therefore, denoting

$$\gamma_{n,k} = \inf_{s\leq T, x\in K, \bar{l}\neq 0} \Pi\Big(u : \|u\| > \frac{1}{n},\ \ \exists j \leq k :\ \ (\Delta_j(x,s,u), \bar{l})_{\mathbb{R}^d} \neq 0\Big),$$

$$\lambda_n = \Pi\Big(\Big\{u : \|u\| > \frac{1}{n}\Big\}\Big),$$

$N_{s,t}^n = \#(\mathcal{D}_n \cap (s,t))$ (it has the Poisson distribution with intensity $\lambda_n(t-s)$), one can estimate the last term in (2.12) by

$$E\left(1 - \frac{\gamma_{n,k}}{\lambda_n}\right)^{N_{s,t}^n} = \exp\{-(t-s)\gamma_{n,k}\}.$$

This implies that

$$\mathrm{P}(\dim S_t = m) \geq \lim_{n\to+\infty} \prod_{r=1}^{m} \mathrm{P}\left(\dim S^n_{\frac{tr}{m}} > \dim S^n_{\frac{t(r-1)}{m}} \,\middle|\, \dim S^n_{\frac{t(r-1)}{m}} < m\right) \geq$$

$$\geq \lim_{n,k\to+\infty} \left(1 - \exp\left\{-\frac{t}{m}\gamma_{n,k}\right\}\right)^m = 1,$$

which gives the needed statement.

Now let us proceed with the proof of statement B. In order to shorten notations we will consider only the time-homogeneous case. Also, without loss of generality, we suppose that there are some compacts $K \subset \mathbb{R}^m$, $\tilde{K} \subset \mathbb{R}^{m\times m}$ such that $X(t) \in K$, $\left[\mathcal{E}_0^{t-}\right]^{-1} \in \in \tilde{K}$ a.s., $t \geq 0$.

Let us introduce some notations. For a given ordered set $\bar{t} \equiv \{t_0 < t_2 < \cdots < t_k\}$, $t_j \in \mathbb{Q}\cap\mathbb{R}^+$, $k \geq 1$ denote $d(\bar{t}) = \min_j(t_j - t_{j-1})$. For every such $\bar{t}$ and every $l \geq 1$ let us choose a sequence $\bar{h}^{\bar{t},l} = \left\{h_j^{\bar{t},l} \in H_0,\, j = 1,\dots,k\right\}$ such that

a) $\operatorname{supp} Jh_j^{\bar{t},l} \subset (t_j, t_{j-1})$;

b) $Jh_j^{\bar{t},l} = 1$ on $\left(t_j + \dfrac{d(\bar{t})}{3l}, t_{j-1} - \dfrac{d(\bar{t})}{3l}\right)$.

Next, for a given $\bar{t}$ and $n, l \in \mathbb{N}$ we put

$$\Omega^{\bar{t},n} = \bigcap_{j=0}^{k-1} \left\{\#\Big[\mathcal{D}_n \cap (t_j, t_{j+1})\Big] = 1\right\},$$

$$\Omega^{\bar{t},l,n} = \Omega^{\bar{t},n} \cap \bigcap_{j=0}^{k-1} \left\{\#\left[\mathcal{D}_n \cap \left(t_{j+1} + \frac{d(\bar{t})}{3l}, t_j - \frac{d(\bar{t})}{3l}\right)\right] = 1\right\}.$$

Denote $T_\varepsilon^{j,\bar{t},l,n} = T_{\varepsilon h_j^{\bar{t},l}}^{\{u\mid\|u\|\geq\frac{1}{n}\}}$, $j = 1,\dots,k$. The following properties hold true:

1) the set $\Omega^{\bar{t},n}$ is invariant w.r.t. every transformation $T_\varepsilon^{j,\bar{t},l,n}$;

2) for every $\varepsilon_{1,2}, j_{1,2}$ transformations $T_{\varepsilon_1}^{j_1,\bar{t},l,n}$ and $T_{\varepsilon_2}^{j_2,\bar{t},l,n}$ commute.

Denote $\partial_j^{\bar{t},l,n} = \dfrac{d}{d\varepsilon} T_\varepsilon^{j,\bar{t},l,n}\Big|_{\varepsilon=0}$, derivative is taken in an a.s sense. One can see that

$$\partial_i^{\bar{t},l,n} \tau_j^{\bar{t},l,n} = -\delta_{i,j} \quad \text{almost surely on} \quad \Omega^{\bar{t},l,n}, \quad i, j \leq k,$$

here by $\tau_j^{\bar{t},n}$ we denote the unique point from $\mathcal{D}_n \cap (t_j, t_{j+1})$, $\delta_{i,j}$ is the Kronecker symbol. In order to shorten notations we will further omit the subscripts $^{\bar{t},l,n}$, $^{\bar{t},n}$ over ∂_j, τ_j.

For a given $\bar{t}$, n consider processes $\tilde{X}$, $\tilde{\mathcal{E}}$ $\left(= \tilde{X}^{\bar{t},n}, \tilde{\mathcal{E}}^{\bar{t},n}\right)$, defined as the solutions of SDE's

$$\tilde{X}(t) = x + \int_0^t \tilde{a}(\tilde{X}(s))\,ds + \left[\int_0^{t\wedge t_0}\int_{\mathbb{R}^d} + \int_{t\wedge t_0}^{t} \int_{\{u\mid\|p(u)>\frac{1}{n}\}}\right] c(\tilde{X}(s-), u)\nu(ds, du),$$

$$\tilde{\mathcal{E}}_0^t = I + \int_0^t \nabla_x \tilde{a}(\tilde{X}(s))\tilde{\mathcal{E}}_0^s\,ds + \left[\int_0^{t\wedge t_0}\int_{\mathbb{R}^d} + \int_{t\wedge t_0}^{t} \int_{\{u\mid\|p(u)>\frac{1}{n}\}}\right] \times$$

$$\times \nabla_x c(\tilde{X}(s-), u)\,\tilde{\mathcal{E}}_0^{s-}\,\nu(ds, du).$$

Lemma 2.4. *Under condition* $\mathcal{B}$) *there exist functions* $V_k^N \in C(\mathbb{R}^+)$, $V_k^N(0) = 0$, $k,\ N \in \mathbb{N}$, *such that for every* $n,\ l \in \mathbb{N}$, $\bar{t} = \{t_0 < \cdots < t_k\}$ *and every* $i_1, \ldots, i_k \in \in \{0, \ldots, N\}$

i) $$\Big\|(\partial_k)^{i_k}(\partial_{k-1}+\partial_k)^{i_{k-1}}\ldots(\partial_1+\ldots+\partial_k)^{i_1}\Big([\mathcal{E}_0^{\tau_k-}]^{-1}\Delta(X(\tau_k-),p(\tau_k))\Big)-$$
$$-(\partial_k)^{i_k}(\partial_{k-1}+\partial_k)^{i_{k-1}}\ldots(\partial_1+\ldots+\partial_k)^{i_1}\Big([\tilde{\mathcal{E}}_0^{\tau_k-}]^{-1}\Delta(\tilde{X}(\tau_k-),p(\tau_k))\Big)\Big\| \le$$
$$\le V_k^N(\eta_{t_k}^{n,(k+1)N} - \eta_{t_0}^{n,(k+1)N}),$$

ii) $$\Big\|(\partial_k)^{i_k}(\partial_{k-1}+\partial_k)^{i_{k-1}}\ldots(\partial_1+\ldots+\partial_k)^{i_1}\Big([\tilde{\mathcal{E}}_0^{\tau_k-}]^{-1}\Delta(\tilde{X}(\tau_k-),p(\tau_k))\Big)-$$
$$-[\mathcal{E}_0^{\tau_1-}]\Delta_{k-1}^{i_1,\ldots,i_k}(X(\tau_1-),p(\tau_1),\ldots,p(\tau_k))\Big\|_{\mathbb{R}^m} \le V_k^N(t_k-t_0)$$

almost surely on $\Omega^{\bar{t},l,n}$, *where*

$$\eta_t^{n,r} = \int_0^t \int_{\{\|u\|\le\frac{1}{n}\}} \sup_{x\in K}(\|c(x,u)\|_{\mathbb{R}^m} + \ldots + \|(\nabla_x)^r c(x,u)\|_{(\mathbb{R}^m)^{\times r}})\nu(ds,du).$$

Proof. By the definition $\tilde{X} = X$ and $\tilde{\mathcal{E}} = \mathcal{E}$ on $[0,\tau_1)$. Due to (2.7)

$$(\partial_1)^{i_1}\Big([\mathcal{E}_0^{\tau_1-}]^{-1}\Delta(\tau_1,X(\tau_1-),p(\tau_1))\Big) = [\mathcal{E}_0^{\tau_1-}]^{-1}\Delta_0^{i_1}(X(\tau_1-),p(\tau_1))$$

almost surely on $\Omega^{\bar{t},l,n}$, which means that the case $k=1$ is already proved.

To proceed with the case $k>1$ we need two auxiliary technical results. Supposing n to be fixed, denote by $\Psi_{r,t}(x) \equiv \Psi_{r,t}^0(x)$ solution of SDE

$$X(t) = x + \int_r^t \tilde{a}(X(s))\,ds + \int_r^t \int_{\{u\|\|p(u)\|\le\frac{1}{n}\}} c(X(s-),u)\nu(ds,du), \quad t \ge r.$$

It follows from the general results about differentiability of the solution of differential equation w.r.t. initial value that functions $\Psi_{r,t}^j \equiv (\nabla_x)^j\Psi_{r,t}$ are well defined almost surely. We denote by $\Phi_{r,t}(x) \equiv \Phi_{r,t}^0(x)$ solution of ODE

$$X(t) = x + \int_r^t \tilde{a}(X(s))\,ds, \quad t \ge r,$$

and put $\Phi_{r,t}^j \equiv (\nabla_x)^j\Phi_{r,t}$.

Proposition 2.2. *For every* $N \in \mathbb{N}$ *there exists function* $W^N \in C(\mathbb{R}^+)$ *with* $W^N(0) = 0$ *such that for every* $j \le N, x \in \mathbb{R}^m, t > r$

$$\|\Psi_{r,t}^j(x) - \Phi_{r,t}^j(x)\|_{(\mathbb{R}^m)^{\times(j+1)}} \le W^N(\eta_{t_k}^{n,N} - \eta_{t_0}^{n,N}).$$

almost surely.

Sketch of the proof. One can write down iteratively differential equations both on $\Psi_{r,t}^j$ and $\Phi_{r,t}^j$ (stochastic for $\Psi_{r,t}^j$ and ordinary for $\Psi_{r,t}^j$). These equations are linear nonhomogeneous equations with free terms constructed (in a same regular manner) from functions a, c with their derivatives up to the order j and functions $\{\Psi_{r,t}^i, i<j\}$ or $\{\Phi_{r,t}^i, i<j\}$ correspondingly. Now the needed statement can be obtained by induction using condition $\mathcal{B}$) and Gronwall lemma.

The same considerations together with the fact that the process $\eta_t^{n,N}$ in every point t almost surely has derivative w.r.t. $t,$ equal to $0,$ provide the following statement.

Proposition 2.3. *The function* W^N *in previous proposition can be chosen in such a way that for every* $j_1,\, j_2,\, j_3 \le N,\, x\in\mathbb{R}^m,\, t>r$

$$\left\|\frac{\partial^{j_1}}{\partial r^{j_1}}\frac{\partial^{j_2}}{\partial t^{j_2}}\Psi^{j_3}_{r,t}(x)-\frac{\partial^{j_1}}{\partial r^{j_1}}\frac{\partial^{j_2}}{\partial t^{j_2}}\Phi^{j_3}_{r,t}(x)\right\|_{(\mathbb{R}^m)^{\times(j_3+1)}}\le W^N\Big(\eta^{n,N}_{t_k}-\eta^{n,N}_{t_0}\Big)$$

almost surely.

Now let us return to the proof of the lemma. In order to shorten notations we will consider only the case $k=2,$ the arguments for $k>2$ will be the same.

Using (2.7), we obtain that

$$(\partial_2)^{i_2}\Big([\mathcal{E}_0^{\tau_2-}]^{-1}\Delta(X(\tau_2-),p(\tau_2))\Big)=[\mathcal{E}_0^{\tau_2-}]^{-1}(\Lambda^{i_2}\Delta)(X(\tau_2-),p(\tau_2)).$$

Let us estimate $(\partial_1)^{i_1}\Big([\mathcal{E}_0^{\tau_2-}]^{-1}\Upsilon(X(\tau_2-),p(\tau_2))\Big)$ for a vector-valued function Υ. One can write down

$$[\mathcal{E}_0^{\tau_2-}]^{-1}\Upsilon(X(\tau_2-),p(\tau_2))=$$

$$=[\mathcal{E}_0^{\tau_1-}]^{-1}\Big[I+\nabla_x c(X(\tau_1-),p(\tau_1))\Big]^{-1}\Big[\Psi^1_{\tau_1,\tau_2}(X(\tau_1-)+c(X(\tau_1-),p(\tau_1)))\Big]^{-1}\times$$

$$\times\Upsilon\Big(\Psi_{\tau_1,\tau_2}(X(\tau_1-)+c(X(\tau_1-),p(\tau_1)),p(\tau_2))\Big), \tag{2.13}$$

$$[\tilde{\mathcal{E}}_0^{\tau_2-}]^{-1}\Upsilon(\tilde X(\tau_2-),p(\tau_2))=$$

$$=[\tilde{\mathcal{E}}_0^{\tau_1-}]^{-1}\Big[I+\nabla_x c(\tilde X(\tau_1-),p(\tau_1))\Big]^{-1}\Big[\Phi^1_{\tau_1,\tau_2}(\tilde X(\tau_1-)+c(\tilde X(\tau_1-),p(\tau_1))\Big]^{-1}\times$$

$$\times\Upsilon\Big(\Phi_{\tau_1,\tau_2}(\tilde X(\tau_1-)+c(\tilde X(\tau_1-),p(\tau_1))),p(\tau_2)\Big). \tag{2.14}$$

We know that almost surely on the set $\Omega^{\bar t,l,n}$

$$\partial_1\tau_1=-1,\qquad \partial_1[\mathcal{E}_0^{\tau_1-}]^{-1}=[\mathcal{E}_0^{\tau_1-}]^{-1}\nabla_x\tilde a(X(\tau_1-)),$$

$$\partial_1 X(\tau_1-)=-\tilde a(X(\tau_1-)),\qquad \partial_1[\tilde{\mathcal{E}}_0^{\tau_1-}]^{-1}=[\tilde{\mathcal{E}}_0^{\tau_1-}]^{-1}\nabla_x\tilde a(\tilde X(\tau_1-)), \tag{2.15}$$

$$\partial_1\tilde X(\tau_1-)=-\tilde a(\tilde X(\tau_1-)).$$

Taking iteratively ∂_1 from the right-hand sides of equalities (2.13), (2.14) and using (2.15) and Propositions 2.2, 2.3 we obtain statement i) of the lemma.

Now let us estimate the value $(\partial_1+\partial_2)^{i_1}(\partial_2)^{i_2}\Big([\tilde{\mathcal{E}}_0^{\tau_2-}]^{-1}\Delta(\tilde X(\tau_2-),p(\tau_2))\Big)$. As soon as function Φ is defined by a homogeneous equation, one has that for every $j\ge 0$ $\left(\dfrac{\partial}{\partial r}+\dfrac{\partial}{\partial t}\right)\Phi^j_{r,t}(x)=0.$ This together with (2.15) means that for $\varphi\in C^1$

$$(\partial_1+\partial_2)\Big[\Phi^j_{\tau_1,\tau_2}(\varphi(\tilde X(\tau_1-)))\Big]=-\Big(\Phi^{j+1}_{\tau_1,\tau_2}(\varphi(\tilde X(\tau_1-))),[\nabla\varphi\nabla_x\tilde a](X(\tau_1-))\Big)_{\mathbb{R}^m}. \tag{2.16}$$

Note that $\|\Phi^j_{\tau_1,\tau_2}(x)\|_{(\mathbb{R}^m)^{\times(j+1)}}=O(t_2-t_0)$ on $\Omega^{\bar t,l,n}$ for all $j\ge 2,$ thus iterating (2.16) we obtain that

$$\left\|(\partial_1+\partial_2)^i\Big[\Phi^1_{\tau_1,\tau_2}(\tilde X(\tau_1-)+c(\tilde X(\tau_1-),p(\tau_1)))\Big]^{-1}\right\|_{\mathbb{R}^{m\times m}}=O(t_2-t_0)\quad\text{on}\quad\Omega^{\bar t,l,n}.$$

The same considerations give that

$$\Big\|(\partial_1+\partial_2)^i\Upsilon(\Phi_{\tau_1,\tau_2}(\tilde X(\tau_1-)+c(\tilde X(\tau_1-),p(\tau_1))),p(\tau_2))-$$

$$-(\partial_1+\partial_2)^i\Upsilon(\tilde X(\tau_1-)+c(\tilde X(\tau_1-),p(\tau_1)),p(\tau_2))\Big\|_{\mathbb{R}^m}=O(t_2-t_0)\quad\text{on}\quad\Omega^{\bar t,l,n}.$$

Therefore, as soon as $\partial_2\tilde X(\tau_1-)=0$ and $\partial_2\tilde{\mathcal{E}}_0^{\tau_1-}=0,$ we have that, up to some $O(t_2-t_0)$ term, $(\partial_1+\partial_2)^i\Big(\big[\tilde{\mathcal{E}}_0^{\tau_2-}\big]^{-1}\Upsilon(\tilde X(\tau_2-),p(\tau_2))\Big)$ is equal to

$$(\partial_1)^i\Big\{\big[\tilde{\mathcal{E}}_0^{\tau_1-}\big]^{-1}\Big[I+\nabla_x c(\tilde X(\tau_1-),p(\tau_1))\Big]^{-1}\times$$

$$\times\,\Upsilon((\tilde X(\tau_1-)+c(\tilde X(\tau_1-),p(\tau_1)),p(\tau_2))\Big\}=$$

$$=[\tilde{\mathcal{E}}_0^{\tau_1-}]^{-1}\Big[\Lambda^i\Xi_{p(\tau_1)}\Upsilon(\cdot,p(\tau_2))\Big](\tilde X(\tau_1-)),$$

which gives the needed statement.

The lemma is proved.

For a given $s<t$ and $\bar l\neq 0$ let us consider the event

$$A_{s,t,\bar l}=\Big\{\exists l,\;\; n\in\mathbb{N},\;\;\bar t=\{t_0,\ldots,t_k\}\subset(s,t)\cap\mathbb{Q},\;\; i_0,\ldots i_k\geq 0:$$

$$(\partial_1^{\bar t,l,n})^{i_0}\ldots(\partial_1^{\bar t,l,n}+\ldots+\partial_k^{\bar t,l,n})^{i_k}\times$$

$$\times\Big(\Big[(\tilde{\mathcal{E}}_0^{\bar t,\bar n})^{\tau_k^{\bar t,n}-}\Big]\Delta\Big(\tilde X^{\bar t,n}(\tau_k^{\bar t,n}-),p(\tau_k^{\bar t,n})\Big),\bar l\Big)_{\mathbb{R}^m}\neq 0\Big\}.$$

Due to representation (2.14) and condition $\mathcal{A}$), for every $n,\ \bar t=\{t_0<\ldots$ $\ldots<t_k\}$ there exists a function

$$\varphi_{\bar t,l,n}:(\mathbb{R}^m)\times(\mathbb{R}^{m\times m})\times(\mathbb{R}^d)^k\times\{(v_1,\ldots,v_k):t_0\leq v_1\leq\ldots\leq v_k\}\to\mathbb{R}^m,$$

which is analytical in every point w.r.t. coordinates $v_1,\ldots,v_k$ with the radius of analyticity not less than $C,$ and such that

$$\Big[\Big(\tilde{\mathcal{E}}_0^{\bar t,\bar n}\Big)^{\tau_k^{\bar t,n}-}\Big]\Delta\Big(\tilde X^{\bar t,n}(\tau_k^{\bar t,n}-),p(\tau_k^{\bar t,n})\Big)=$$

$$=\varphi_{\bar t,l,n}\Big(X(t_0),\mathcal{E}_0^{t_0},p(\tau_1^{\bar t,n}),\ldots,p(\tau_k^{\bar t,n}),\tau_1^{\bar t,n},\ldots,\tau_k^{\bar t,n}\Big).$$

The following fact is well known: if some function is analytical on some subset of $\mathbb{R}^k$ and is not equal to 0 in some point, then it is not equal to 0 in almost every point w.r.t. λ^k. Variables $\tau_1^{\bar t,n},\ldots,\tau_k^{\bar t,n}$ are independent from $X(t_0)$, $\mathcal{E}_0^{t_0}$, $p(\tau_1^{\bar t,n}),\ldots,p(\tau_k^{\bar t,n})$ and their joint distribution is absolutely continuous w.r.t. λ^k. This together with statement ii) of Lemma 2.4 implies that for a given $n,\ l,\ \bar t$ and $\bar l\neq 0$ almost surely

$$\Omega^{\bar t,l,n}\cap\Big\{(\partial_1^{\bar t,l,n})^{i_0}\ldots(\partial_1^{\bar t,l,n}+\ldots+\partial_k^{\bar t,l,n})^{i_k}\times$$

$$\times\Big(\Big[(\tilde{\mathcal{E}}_0^{\bar t,\bar n})^{\tau_k^{\bar t,n}-}\Big]\Delta(\tilde X^{\bar t,n}(\tau_k^{\bar t,n}-),p(\tau_k^{\bar t,n})),\bar l\Big)_{\mathbb{R}^m}\neq 0\Big\}\supset$$

$$\supset\Omega^{\bar t,l,n}\cap\Big\{\Big(\Big[\mathcal{E}_0^{\tau_1^{\bar t,n}-}\Big]^{-1}\Delta_{k-1}^{i_0,\ldots,i_k}(X(\tau_1^{\bar t,n}-),p(\tau_1^{\bar t,n}),\ldots,p(\tau_k^{\bar t,n})),\bar l\Big)_{\mathbb{R}^m}\neq 0\Big\}.$$

This gives that almost surely

$$A_{s,t,\bar l} \supset B_{s,t,\bar l} \equiv \bigcup_{n,k\ge 1}\Big\{\exists j\ge 1:\ \ \tau_j^n,\dots,\tau_{j+k-1}^n \in (s,t)$$

$$\text{and}\quad \left(\left[\mathcal{E}_0^{\tau_j^n-}\right]^{-1}\right)^{*}\bar l\quad \text{is not orthogonal to}$$

$$\mathfrak{L}_k(X(\tau_j^n-),p(\tau_j^n),\dots,p(\tau_{j+k-1}^n))\Big\}.$$

Let us show that

$$\mathrm{P}(B_{s,t,\bar l})=1. \tag{2.17}$$

Denote by $\mathfrak{L}_k^M(s,x,u_0,\dots,u_k)$, $k\ge 0$ the linear span of the vectors

$\Big\{\Delta_j^{i_0,\dots,i_j}(s,x,u_{0+r},\dots,u_{j+r}),\ \ i_0,\dots,i_j\le M,\ \ r=0,\dots,k-j,\ \ j=0,\dots,k\Big\}.$

Let condition (1.5) to hold true with some given $k>0$. Then due to Lemma 2.3 for every R there exist n,M such that for every $x\in K,\bar b\ne 0$ there exists $N=N^{M,R}(x,\bar b)\le n$ satisfying condition

$$\delta_M^N(x,\bar b)\equiv \Pi^{\otimes(k+1)}\left(\left\{(u_0,\dots,u_k)\in\left\{\|u\|>\frac{1}{N}\right\}^{k+1}:\right.\right.$$

$$\left.\left.\bar b\quad\text{is not orthogonal to}\quad \mathfrak{L}_k^M(x,u_0,\dots,u_k)\right\}\right)\times$$

$$\times\left(\left\{\Pi\left(\left\{u\in\mathbb{R}^d|\,\|u\|>\frac{1}{N}\right\}\right)\right\}^{k}\right)^{-1}\ge R. \tag{2.18}$$

Note that $\lambda_N\equiv\Pi\left(\left\{\|u\|>\dfrac{1}{N}\right\}\right)$ is not less than R and therefore $\inf_{M,x,\bar b}N^{M,R}(x,\bar b)\to+\infty,\ R\to+\infty.$ Let us denote

$$B_{s,t,\bar l}^{n,M}\equiv\bigcup_{N\le n}\Big\{\exists j\ge 1:\quad \tau_j^N,\dots,\tau_{j+k-1}^N\in(s,t)\ \text{ and}$$

$$\left(\left[\mathcal{E}_0^{\tau_j^N-}\right]^{-1}\right)^{*}\bar l\quad\text{is not orthogonal to}\quad \mathfrak{L}_k^M\Big(X(\tau_j^N-),p(\tau_j^N),\dots,p(\tau_{j+k-1}^N)\Big)\Big\}$$

and estimate probability of $B_{s,t,\bar l}^{n,M}$. First we take constant $C=C(k)$ such that $e^{-C}<\ <\dfrac13$ and $\sum_{i\ge k}e^{-C}\dfrac{C^i}{i!}>\dfrac12$. Then we construct inductively a random covering of the interval $(s,+\infty)$ in the following way. Let us take the interval $\left(s,s+\dfrac{C}{\lambda_n}\right)$ and consider the set $\mathcal{D}_n\cap\left(s,s+\dfrac{C}{\lambda_n}\right)$. If this set is empty, we put $I_1=\left(s,s+\dfrac{C}{\lambda_n}\right]$, otherwise we take the first point θ from this set, define $\tilde N=N^{M,R}\Big(X(\theta-),\left(\left[\mathcal{E}_0^{\theta-}\right]^{-1}\right)^{*}\bar l\Big)$ and put $I_1=\left(s,\theta+\dfrac{C}{\lambda_{\tilde N}}\right]$. Then we take I_1 as the first set in the covering which we are going to construct, replace $(s,+\infty)$ by $(s,+\infty)\backslash I_1$ and repeat the preceding procedure. We obtain a countable covering of the interval $(s,+\infty)$ by a segments $\Big\{I_r=(v_{r-1},v_r],\ r\ge 1\Big\}$, which can be separated in two groups:

1) some segments of the length $\dfrac{C}{\lambda_n}$, we denote this group by $\mathcal{G}^1$;

2) some segments of the length $> \dfrac{C}{\lambda_n}$, we denote this group by $\mathcal{G}^{\in}$.

Note that the length of every segment is not greater than $\dfrac{2C}{R}$, we suppose that R is taken sufficiently large and $\dfrac{2C}{R} \le \dfrac{t-s}{3}$. Next, by the construction every v_r is a stopping time and v_{r+1} is independent from $\mathcal{F}_{v_r-}$, random event $\Big\{I_r = (v_{r-1}, v_r] \in \in \mathcal{G}^1\Big\}$ is independent from $\mathcal{F}_{v_r-}$ and its probability is equal to e^{-C}. Denote by $Z_{s,t}$ the total length of all segments I_r in the first group such that $v_{r-1} < t$, then

$$EZ_{s,t} = \frac{C}{\lambda_n} \sum_r \mathrm{P}(I_r \in \mathcal{G}^1, v_{r-1} < t) =$$

$$= \frac{Ce^{-C}}{\lambda_n} \sum_r \mathrm{P}(v_{r-1} < t) \le \left\{ \left[\frac{\lambda_n (t-s)}{C} \right] + 1 \right\} \frac{Ce^{-C}}{\lambda_n},$$

here we used the obvious fact that $\mathrm{P}(v_{r-1} < t) = 0,\ r > \left[\dfrac{\lambda_n(t-s)}{C}\right] + 1$. Analogously one can verify that

$$DZ_{s,t} \le \left\{ \left[\frac{\lambda_n (t-s)}{C} \right] + 1 \right\} \frac{C^2 (e^{-C} - e^{-2C})}{(\lambda_n)^2},$$

which means that

$$Z_{s,t} - EZ_{s,t} \xrightarrow{\mathrm{P}} 0 \quad \text{and} \quad \mathrm{P} - \lim \sup_{\lambda_n \to +\infty} Z_{s,t} \le e^{-C}(t-s) < \frac{t-s}{3}.$$

Therefore for every fixed $p \in (0,1)$ one can choose initial number R (and, consequently, number n) large enough to provide estimate

$$\mathrm{P}\left(Z_{s,t} \le \frac{t-s}{3} \right) \ge p. \tag{2.19}$$

Next, let us monotonously enumerate the second group, $\mathcal{G}^2 = \{J_j\}$. For a given j let θ_j be the first point from $\mathcal{D}_n \cap I_r$, $N_j = N^{M,R}\Big(X(\theta_j-), \theta_j, \Big([\mathcal{E}_0^{\theta_j-}]^{-1}\Big)^* \bar{l}\Big)$. Denote by D_j the event $\left\{\text{the segment } \left(\theta_j, \theta_j + \dfrac{C}{\lambda_{N_j}}\right] \text{ contains at least } k \text{ points from } \mathcal{D}_{N_j}\right\}$, $P(D_j) = \sum_{i \ge k} e^{-C} \dfrac{C^i}{i!} > \dfrac{1}{2}$. Denote the first k points from $\mathcal{D}_{N_j} \cap (\theta_j, +\infty)$ by $\theta_j^1, \ldots, \theta_j^k$. Due to the choice of N_j probability of the event

$$C_j = \left\{ \Big([\mathcal{E}_0^{\theta_j^k -}]^{-1} \Big)^* \bar{l} \quad \text{is not orthogonal to} \right.$$

$$\left. \mathfrak{L}_k^M \Big(X(\theta_j^k -), p(\theta_j), p(\theta_j^1, \ldots, p(\theta_j^k) \Big) \right\}$$

s not less than $\dfrac{R}{\lambda_{N_j}}$. Events C_j, D_j are independent both from $\mathcal{F}_{\theta_j-}$ and from each other, therefore

$$\mathrm{P}\Big(B^{n,M}_{s,t,\bar l}\Big) \ge 1 - \mathrm{P}\left(\bigcap_{j:\theta_j<\frac{s+2t}{3}} \big[\Omega\backslash(C_j\cap D_j)\big]\right) \ge$$

$$\ge 1 - E\prod_{j:J_j\subset(s,t)}\left(1-\frac{R}{2\lambda_{N_j}}\right) \ge 1 - E\exp\left[-\sum_{j:J_j\subset(s,t)}\frac{R}{2\lambda_{N_j}}\right].$$

The variable $W_{s,t} = \sum_{j:J_j\subset(s,t)}\dfrac{C}{\lambda_{N_j}}$ is just the total length of the intervals from the second group, which are contained in (s,t). One have that $W_{s,t} \ge \dfrac{2(t-s)}{3} - Z_{s,t}$, and under (2.19) we have that $W_{s,t} \ge \dfrac{t-s}{3}$ with probability $\ge p$, which gives that

$$\mathrm{P}\Big(B^{n,M}_{s,t,\bar l}\Big) \ge p - \exp\left[-\frac{R}{2C}(t-s)\right]. \tag{2.20}$$

Now we proceed in a following way: for a given $p\in(0,1)$ we take R_p such that (2.19) holds for every $R\ge R_p$, then take $R\to\infty$ in (2.20) and therefore obtain that $\mathrm{P}(B_{s,t,\bar l})\ge p$. At last, we take $p\uparrow 1$ and obtain (2.17).

Denote

$$A^{M,j,N}_{s,t,\bar l} = \Bigg\{\exists l,\ \ n\ge N,\ \ \bar t=\{t_0,\dots,t_k\}\subset(s,t)\cap\mathbb{Q},\ \ k,i_0,\dots i_k\le M:$$

$$\Big|(\partial_1^{\bar t,l,n})^{i_0}\dots(\partial_1^{\bar t,l,n}+\dots+\partial_k^{\bar t,l,n})^{i_k}\times$$

$$\times\Big(\Big[(\tilde{\mathcal{E}}_0^{\bar t,\bar n})^{\tau_k^{\bar t,n}-}\Big]\Delta\Big(\tilde X^{\bar t,n}(\tau_k^{\bar t,n}-),p(\tau_k^{\bar t,n})\Big),\bar l\Big)_{\mathbb{R}^m}\Big| > \frac{1}{j}\Bigg\},$$

by the construction $A^{M,j,N}_{s,t,\bar l}\subset A^{\tilde M,\tilde j,\tilde N}_{s,t,\bar l}$, $N\le\tilde N$, $M\le\tilde M$, $j\le\tilde j$, and due to (2.17) $\mathrm{P}\Big(A^{M,j}_{s,t,\bar l}\Big)\to 1$ as $M,\ j\to+\infty$ for every $\bar l\ne 0$, $s<t$. For a given $\varepsilon\in(0,1)$ let us take $N_*,\ j_*,\ M_*$ such that $\mathrm{P}\Big(A^{M_*,j_*,N}_{s,t,\bar l}\Big)\ge p$ for every $N\ge N_*$. Next, we take N^* such that for every $n\ge N^*$ $\mathrm{P}\left(V^M_M\Big(\eta_t^{n,M(M+1)}-\eta_s^{n,M(M+1)}\Big) > \dfrac{1}{j^*}\right)\le\varepsilon$. Now we can apply the statement ii) of Lemma 2.4 for $n\ge N_*\vee N^*$ and obtain that the probability of the event

$$C_{s,t,\bar l}\equiv\Bigg\{\exists l,\ \ n\in\mathbb{N},\ \ \bar t=\{t_0,\dots,t_k\}\subset(s,t)\cap\mathbb{Q},\ \ i_0,\dots i_k\ge 0:$$

$$(\partial_1^{\bar t,l,n})^{i_0}\dots(\partial_1^{\bar t,l,n}+\dots+\partial_k^{\bar t,l,n})^{i_k}\times$$

$$\times\Big(\Big[(\mathcal{E}_0^{\bar t,\bar n})^{\tau_k^{\bar t,n}-}\Big]\Delta\Big(X^{\bar t,n}(\tau_k^{\bar t,n}-),p(\tau_k^{\bar t,n})\Big),\bar l\Big)_{\mathbb{R}^m}\ne 0\Bigg\}$$

is not less than $1-2\varepsilon$ and therefore $\mathrm{P}(C_{s,t,\bar l})=1$. Using this fact and Lemma 2.1, we obtain analogously to the proof of statement A that $\mathrm{P}(\dim S_t=\dim S_s|\dim S_s< <m)=0$ for every $s<t$, which gives the needed statement.

The last thing, which we need to do, is to remove condition (2.1). Denote by ζ the moment of the first jump such that $\big\|\nabla_x c(\zeta,X(\zeta-),p(\zeta))\big\|\ge 1$. Considerations, analogous to those made before, imply that

$$\mathrm{P}|_{\{\zeta>t\}} \circ [X(t)]^{-1} \le \lambda^m.$$

Next, let $\Gamma_{R,t} = \Big\{u \,\Big|\, \sup_{s\le t, \|x\|\le R} \|\nabla_x c(s,x,u)\| \ge 1\Big\}$, $\zeta_{R,t-\delta} = \inf\big\{r \ge t-\delta | r \in \in \mathcal{D},\, p(r) \in \Gamma_{R,t}\big\}$. The same considerations, together with evolutionary property of the family $X(x,t,s)$, gives that for every R, δ

$$\mathrm{P}|_{\{\zeta_{R,t-\delta}>t\}\cap\{\sup_{s\le t}\|X(s)\|\le R\}} \circ [X(t)]^{-1} \le \lambda^m.$$

This means that the total mass of the singular part of the distribution of $X(t)$ can be estimated by

$$\mathrm{P}\left(\{\zeta_{R,t-\delta} < t\} \cup \left\{\sup_{s\le t} \|X(s)\| > R\right\}\right),$$

which can be made arbitrarily small by taking first R large enough and then δ small enough.

The theorem is proved.

3. Appendix: some improvements and unsolved problems. One can see from the proof of the Theorem 1.1 that conditions $\mathcal{A}$), $\mathcal{B}$) are a technical ones, which are used to calculate and estimate compositions of derivatives w.r.t. the first $k-1$ jumps in a given set of k jumps $\Big($derivatives $\partial_1^{\bar{t},l,n}, \ldots, \partial_{k-1}^{\bar{t},l,n}$, see notations before Lemma 2.4$\Big)$. This remark immediately gives the following version of statement B.

Proposition 3.1. *Denote by* $\tilde{\mathfrak{L}}_k(x, u_0, \ldots, u_k)$ *the span of the vectors*

$$\Big\{\Delta_j^{i,0,\ldots,0}(x, u_{0+r}, \ldots, u_{j+r}),\ j = 0,\ldots,k,\ r = 0,\ldots,k-j,\ i \ge 0\Big\}.$$

Suppose that for some $k > 0$ *for every* $x \in \mathbb{R}^m$, $s \in \mathbb{R}^+$, $\bar{l} \in \mathbb{R}^m\backslash\{0\}$

$$\Pi_{k+1}^*\Big\{(u_0,\ldots,u_k) \in [\Theta_{s,x}]^{k+1} :$$

$$\bar{l} \text{ is not orthogonal to } \tilde{\mathfrak{L}}_k(s,x,u_0,\ldots,u_k)\Big\} = +\infty. \tag{3.1}$$

Then for every $x \in \mathbb{R}^m$, $0 \le r < t$

$$\mathrm{P} \circ [X(x,t,r)]^{-1} \ll \lambda^m.$$

The proof is analogous to the proof of statement B and is omitted. Proposition 3.1 allows, in particular, to consider SDE's such that their drift coefficients have a rot of zeros.

***Example* 3.1. a)** Consider one-dimensional SDE

$$X(t,x) = x + \int_0^t a(X(s,x))\,ds + \eta_t, \tag{3.2}$$

where η_t is the Levy process with the Levy measure $\Pi = \sum_{k\ge1} \alpha_k \delta_{3^{-k}}$, where $\sum_k \alpha_k = +\infty$. Suppose that $a \in C^\infty(\mathbb{R})$ is such that in every point of the Cantor set $K \subset [0,1]$ function a together with all its derivatives is equal to zero, and $a \ne 0$ outside K. Theorem 1.1 can not be applied here. Indeed, as soon as supp $\Pi \subset K$, one has that $\Delta_0^i(0,u) = 0$ for every $u \in \text{supp}\,\Pi, i \ge 0$, which means that condition (1.4) fails and statement A. is not applicable. Statement B we can not apply because function a is not analytical.

On the other hand, for every point $x \in \mathbb{R}$ and every $i \neq j$ one has that at least one of the numbers $x + 3^{-i}$, $x + 3^{-j}$, $x + 3^{-i} + 3^{-j}$ does not belong to K. This means that for every x

$$\Pi_2^*\Big\{(u_0, u_1) : \tilde{\mathfrak{L}}_2(x, u_0, u_1) \neq \{0\}\Big\} \geq \sup_n \frac{\sum_{i<j\leq n} \alpha_i \alpha_j}{\sum_{j\leq n} \alpha_j} = +\infty,$$

and (3.1) holds true with $k = 1$. Therefore solution of (3.2) has absolutely continuous distribution.

It is worth to be mentioned that regularity properties of the solution of SDE of the type (3.2) can essentially depend on the mutual properties of the set of zeros of the function a and Levy measure of the process η_t.

***Example* 3.1. b)** Let a be equal to zero on the set $K_4^{1,0,1,1}$ of the points $y \in [0, 1]$ such that in their representations

$$y = \sum_{j=1}^{\infty} \frac{y_j}{4^j}, \quad y_j \in \{0, 1, 2, 3\}, \quad j \geq 1, \tag{3.3}$$

every digit y_j is not equal to 1 (note that the classical Cantor set from the previous example can be written in these notations as $K_3^{1,0,1}$). Let us consider SDE's of the type (3.2) with two different processes η_t in the right-hand side, having Levy measures equal $\Pi_\rho = \sum_{k\geq 1} k^\rho \delta_{4^{-k}}$, $\rho > 1$, and $\Pi_{-1} = \sum_{k\geq 1} \frac{1}{k}\delta_{4^{-k}}$ correspondingly. The first case can be treated analogously to the previous example. Namely, for every $x \in \mathbb{R}$ and every given $i > j$ there exist numbers $\varepsilon_1 \in \{0, 1, 2, 3\}$, $\varepsilon_2 \in \{0, 1\}$, not equal simultaneously to 0, such that $x + \varepsilon_1 4^{-i} + \varepsilon_2 4^{-j} \notin K_4^{1,0,1,1}$. This means that if in every point $x \notin K_4^{1,0,1,1}$ some derivative of a is not equal to zero, then

$$\Pi_4^*\Big\{(u_0, \ldots, u_4) : \tilde{\mathfrak{L}}_4(x, u_0, \ldots, u_4) \neq \{0\}\Big\} \geq \sup_n \frac{\sum_{j<i\leq n} i^{3\rho} j^\rho}{4! \cdot [\sum_{i\leq n} i^\rho]^3} = +\infty,$$

and solution of (3.2) has absolutely continuous distribution.

On the other hand, process $\eta.$ with Levy measure Π_{-1} on the interval $(0, 1)$ does not have multiply jumps (i.e., all its jumps have different values) with probability

$$p^* = \prod_{n=1}^{\infty} e^{-\frac{1}{n}} \left(1 + \frac{1}{n}\right) = e^{-\gamma^*} > 0,$$

here $\gamma^* = 0{,}577215\ldots$ is the Euler's constant. This means that with probability p^* the value η_s in every point $s \leq 1$ has in its representation (3.3) all digits equal to either 0 or 1. Let us take by starting point $x = \frac{2}{3}$, all its digits in (3.3) are equal 2, and therefore with probability p^* all digits of $\frac{2}{3} + \eta_s$ for every $s \leq 1$ are equal 2 or 3, which means that $\frac{2}{3} + \eta_s \in K_4^{1,0,1,1}$. If $a = 0$ on $K_4^{1,0,1,1}$, then with the same probability $X\left(1, \frac{2}{3}\right) = \frac{2}{3} + \eta_1 \in K_4^{1,0,1,1}$. Remind that $\lambda^1(K_4^{1,0,1,1}) = 0$, and this together with the preceding arguments gives that the distribution of $X\left(1, \frac{2}{3}\right)$ has a nontrivial singular component.

At this time we can not give any general condition, say, in the terms of the entropy of the set of zeroes of the drift coefficient in (3.2), sufficient for solution to have an absolutely continuous distribution.

At the end, let us give another improvement of statement B. One can see that the constant $C = C(k)$ in the proof can be chosen in the form $C(k) = C^*k.$ Repeating the rest of the proof, we obtain that statement B holds true with the condition (1.5) replaced by the weaker condition

$$\frac{1}{k} \inf_{x,s,\bar{l}\neq 0} \Pi^*_{k+1}\Big\{(u_0,\ldots,u_k) \in [\Theta_{s,x}]^{k+1} :$$

$$\bar{l} \quad \text{is not orthogonal to} \quad \mathfrak{L}_k(s,x,u_0,\ldots,u_k)\Big\} \to +\infty, \quad k \to +\infty. \tag{3.4}$$

The question whether the term $\dfrac{1}{k}$ in the left hand side of (3.4) is sharp or it can be replaced by some term, increasing more slowly (or maybe removed at all), is still open.

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