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ON THE PROPER POSEDNESS OF TWO-POINT BOUNDARY-VALUE PROBLEM FOR SYSTEM WITH PSEUDODIFFERENTIAL OPERATORS

ПРО КОРЕКТНІСТЬ ДВОТОЧКОВОЇ КРАЙОВОЇ ЗАДАЧІ ДЛЯ СИСТЕМ ІЗ ПСЕВДОДИФЕРЕНЦІАЛЬНИМИ ОПЕРАТОРАМИ

The question on the proper posedness of boundary-value problem with nonlocal condition for a system of pseudodifferential equations of an arbitrary order is investigated. The equation and the boundary conditions contain the pseudodifferential operators which symbols are defined and continuous in some domain $H \subset \mathbb{R}^m_\sigma$. The criterion of the existence, uniqueness of solutions and of the continuously dependence of the solution on the boundary function is established.

Розглянуто питання про коректність крайової задачі з нелокальною умовою для системи псевдодиференціальних рівнянь довільного порядку. Рівняння та граничні умови містять псевдодиференціальні оператори із символами, що визначені та неперервні у деякій області $H \subset \mathbb{R}^m_\sigma$. Встановлено критерій існування та єдиності розв'язків, а також неперервної залежності розв'язку від граничної функції.

1. Introduction. The present paper generalizes and evolves the results of works [1, 2]. It investigates the question of the proper posedness of nonlocal boundary-value problem for system of equations, containing the pseudodifferential operators that symbols are defined and continuous in some domain $H \subset \mathbb{R}^m$. The solution of the problem is sought in functional spaces.

Consider in the infinite layer $\Pi = \mathbb{R}^m \times [0, T]$ the following two-point boundary problem

$$L\left(\frac{\partial}{\partial t}, -i\frac{\partial}{\partial x}\right)u(x,t) = \frac{\partial u}{\partial t} + P\left(-i\frac{\partial}{\partial x}\right)u = 0, \tag{1}$$

$$M\left(-i\frac{\partial}{\partial x}\right)u(x,t) = A\left(-i\frac{\partial}{\partial x}\right)u(x,0) + B\left(-i\frac{\partial}{\partial x}\right)u(x,T) = \varphi(x) \tag{2}$$

where

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_m}\right), \quad u = \operatorname{col}(u_1, u_2, \ldots, u_m),$$

$$\varphi(x) = \operatorname{col}(\varphi_1(x), \varphi_2(x), \ldots, \varphi_m(x));$$

$$P\left(-i\frac{\partial}{\partial x}\right) = \left\| P_{jk}\left(-i\frac{\partial}{\partial x}\right)\right\|_{j,k=\overline{1,l}}, \quad A\left(-i\frac{\partial}{\partial x}\right) = \left\| A_{jk}\left(-i\frac{\partial}{\partial x}\right)\right\|_{j,k=\overline{1,l}},$$

$$B\left(-i\frac{\partial}{\partial x}\right) = \left\| B_{jk}\left(-i\frac{\partial}{\partial x}\right)\right\|_{j,k=\overline{1,l}}$$

matrices that elements are pseudodifferential operators with symbols $P(\sigma)$, $A(\sigma)$, and $B(\sigma)$, respectively, continuous in some domain $H \subset \mathbb{R}^m_\sigma$.

Generally speaking, problem (1), (2) is improperly posed, even if $P_{jk}(\sigma)$, $A_{jk}(\sigma)$, and $B_{jk}(\sigma)$ are polynomials [3 – 6].

We shall be concerned with the question of the existence and uniqueness of solution of problem (1), (2), and the question of the continuously dependence of this solution on the boundary function $\varphi(x)$.

2. Notations and definitions. For the description of the spaces of solutions, we introduce the following notation:

W_Ω^∞ is the space of vector-functions $u(x) = (u_1(x), u_2(x), \ldots, u_l(x))$, $u_j(x) \in L_2(\mathbb{R}^m)$, such that the Fourier transform $\hat{u}_j(\sigma)$ is compactly supported in $\Omega \subset \mathbb{R}_\sigma^\infty$; the space W_Ω^∞ is invariant relatively to the pseudodifferential operator $\tilde{A}\left(-\frac{\partial}{\partial x}\right)$

$$\left(\tilde{A}\left(-i\frac{\partial}{\partial x}\right)u(x) = (2\pi)^{-m}\int_\Omega \tilde{A}(\sigma)\hat{u}(\sigma)\exp(ix\sigma)\,d\sigma, \quad u(x) \in W_\Omega^\infty\right)$$

with the matrix $\hat{A}(\sigma)$ continuous in Ω, moreover, $\tilde{A}\left(-i\frac{\partial}{\partial x}\right)\colon W_\Omega^\infty \to W_\Omega^\infty$ is a continuous application; $\left(W_\Omega^\infty\right)'$ is the space of generalized vector-functions on W_Ω^∞; this space is invariant relatively to $\hat{A}\left(i\frac{\partial}{\partial x}\right)$; $W_\Omega^{+\infty} = \left(W_\Omega^\infty\right)'$, $W_\Omega^{-\infty} = \left(W_{-\Omega}^\infty\right)'$, where $-\Omega = \{\sigma \in \mathbb{R}^m : -\sigma \in \Omega\}$; $C^k([0,T], W_\Omega^{\pm\infty})$ are spaces of vector-functions that for every $t \in [0,T]$ are functions of space $W_\Omega^{\pm\infty}$ respectively and continuously depend on t together with the derivatives up to order k.

According to I. G. Petrovski [7], introduce the following definition:

Definition. *We say that problem* (1), (2) *is properly posed in* $C^n([0,T], W_\Omega^{\pm\infty})$ *if, for every boundary function* $\varphi(x) \in W_\Omega^{\pm\infty}$, *problem* (1), (2) *should have in* $C^n([0,T], W_\Omega^{\pm\infty})$ *one and only one solution* $u(x,t)$, *continuously depending on* $\varphi(x)$.

We shall not be concerned with Cauchy problem (the proper posedness of the Cauchy problem for equation (1) is studied in [7]).

3. On the proper posedness of problem (1), (2). It is easily seen that the following two applications for every domain $\Omega \subset H$ are continuous:

$$L\left(\frac{\partial}{\partial t}, -i\frac{\partial}{\partial x}\right)\colon \quad C^n([0,T], W_\Omega^{\pm\infty}) \to C^0([0,T], W_\Omega^{\pm\infty}),$$

$$u \mapsto \frac{\partial u(x,t)}{\partial t} + P\left(-i\frac{\partial}{\partial x}\right)u(x,t)$$

and

$$M\left(-i\frac{\partial}{\partial x}\right)\colon \quad C^n([0,T], W_\Omega^{\pm\infty}) \to C^0([0,T], W_\Omega^{\pm\infty}),$$

$$u \mapsto A\left(-i\frac{\partial}{\partial x}\right)u(x,0) + B\left(-i\frac{\partial}{\partial x}\right)u(x,T).$$

Let us prove the inverse.

If we denote by $\hat{u}(\sigma, t)$ and $\hat{\varphi}(\sigma)$ the x-Fourier transforms of the solution $u(x,t)$ of problem (1), (2) and the boundary function $\varphi(x)$, respectively, it is easily seen that

$\hat{u}(\sigma, t)$ is a solution of the following boundary-value problem:

$$L\left(\frac{d}{dt}, \sigma\right)\hat{u}(\sigma, t) = \frac{d\hat{u}(\sigma, t)}{dt} + P(\sigma)\hat{u}(\sigma, t), \tag{3}$$

$$M(\sigma)\hat{u}(\sigma, t) = A(\sigma)\hat{u}(\sigma, 0) + B(\sigma)\hat{u}(\sigma, T) = \hat{\varphi}(\sigma). \tag{4}$$

Let us find the fundamental matrix of solutions $F(\sigma, t)$ of system (3). Because $P(\sigma)$ is a matrix continuous in H, the roots of characteristic equation

$$\det\| I\lambda + P(\sigma)\| = 0$$

are continuous functions of parameter σ. We denote by $\chi_j = \chi_j(\sigma)$ the multiplicity of the root $\lambda_j = \lambda_j(\sigma)$, $j = \overline{1, \nu}$, $\nu = \nu(\sigma)$. According to the theory of matrices [8], there exist two polynomial matrices $M(\lambda, \sigma)$ and $N(\lambda, \sigma) = \left\| N_{jk}(\lambda, \sigma)\right\|_{j,k=\overline{1,l}}$ such that $M(\lambda, \sigma) \times L(\lambda, \sigma) \times N(\lambda, \sigma) \equiv Q(\lambda, \sigma) \equiv \left\| h_j(\lambda, \sigma)\delta_{jk} \right\|_{j,k=\overline{1,l}}$, $\det N(\lambda, \sigma) \equiv$ $\equiv N(\sigma) \neq 0$, where $h_j(\lambda, \sigma) = \prod_{k=1}^{\nu} (\lambda - \lambda_k)^{q_{jk}(\sigma)}$, $j = \overline{1, l}$, where $\sum_{j=1}^{l} q_{jk}(\sigma) =$ $= \chi_k(\sigma)$, $k = \overline{1, l}$, and δ_{jk} are the Kronecker symbols.

It follows from the equality $M(\lambda, \sigma) \times L(\lambda, \sigma) \times N(\lambda, \sigma) \equiv Q(\lambda, \sigma)$ and condition $\det N(\lambda, \sigma) \equiv N(\sigma) \neq 0$ that $M(\lambda, \sigma) \times L(\lambda, \sigma) \equiv Q(\lambda, \sigma) \times N^{-1}(\lambda, \sigma)$, and after using the transformation $y(\sigma, t) = N^{-1}\left(\frac{d}{dt}, \sigma\right)\hat{u}(\sigma, t)$, system (3) takes the form

$$Q\left(\frac{d}{dt}, \sigma\right)y(\sigma, t) = 0,$$

that is,

$$h_j\left(\frac{d}{dt}, \sigma\right)y_j(\sigma, t) = 0, \quad j = \overline{1, l}. \tag{5}$$

The fundamental system of solutions of each of equations (5) reads as

$$y_{j,k\alpha}(\sigma, t) = \left(\frac{d}{d\lambda}\right)^{\alpha-1} \exp(\lambda t)\bigg|_{\lambda=\lambda_k}, \qquad \alpha = \overline{1, q_{jk}(\sigma)}, \quad k = \overline{1, \nu}.$$

Therefore, the fundamental matrix of solutions of system (3) reads as

$$F(\sigma, t) = \left\| \exp(\lambda_k t) \times (\alpha - 1)! \times \sum_{s=0}^{\alpha-1} \frac{t^s}{s!}\frac{1}{\rho!}\left(\frac{d}{d\lambda}\right)^{p} N_{j\beta}(\lambda, \sigma)\bigg|_{\lambda=\lambda_k} \right\|_{\substack{j,\beta=\overline{1,l},\ \alpha=\overline{1, q_{\beta k}},\\ k=\overline{1,\nu},\ \rho=\alpha-1-s}}.$$

Using the theorem of dependence of solutions on the parameter [9], we conclude that $F(\sigma, t)$ is continuous with respect to σ.

Therefore, the solution of system (3) reads as $\hat{u}(\sigma, t) = F(\sigma, t)C$, where $C =$ $= \operatorname{col}(C_1, C_2, \ldots, C_l)$. Using the boundary condition (4), we find that C is a solution of the following linear algebraic system

$$(A(\sigma)F(\sigma, 0) + B(\sigma)F(\sigma, T))C = \hat{\varphi}(\sigma),$$

whose determinant is

$$\Delta(\sigma) = \det D(\sigma) = \det\| A(\sigma)F(\sigma, 0) + B(\sigma)F(\sigma, T)\|,$$

where

$$D(\sigma) = A(\sigma)F(\sigma, 0) + B(\sigma)F(\sigma, T).$$

Let $\Omega = H \setminus N_\Delta$, $N_\Delta = \{\sigma \in \mathbb{R}^m\colon \Delta(\sigma) = 0\}$. For every $\sigma \in \Omega$, $C = C(\sigma) = D^{-1}(\sigma) \times \hat{\varphi}(\sigma)$ and the solution of problem (3), (4) reads as $\hat{u}(\sigma, t) = F(\sigma, t) \times D^{-1}(\sigma)\hat{\varphi}(\sigma)$ $(\forall \sigma \in \Omega)$. Let $W(\sigma, t) = F(\sigma, t) \times D^{-1}(\sigma)$.

Associate with matrix $W(\sigma, t)$ the pseudodifferential operator $W\left(-i\dfrac{\partial}{\partial x}, t\right)$ that acts continuously from $W_\Omega^{\pm\infty}$ respectively to $C^n([0, T], W_\Omega^{\pm\infty})$, that is,

$$W\left(-i\frac{\partial}{\partial x}, t\right)\colon \quad W_\Omega^{\pm\infty} \to C^n([0, T], W_\Omega^{\pm\infty}), \quad u(x) \mapsto W\left(-i\frac{\partial}{\partial x}, t\right)u(x)$$

is a continuous application on $W_\Omega^{\pm\infty}$.

Theorem. *In order that problem* (1), (2) *should be properly posed in* $C^n([0, T], W_\Omega^{\pm\infty})$, *it is necessary and sufficient that* $\Omega = H \setminus N_\Delta$.

Proof. It is clear that $\Omega \subset H$. We prove the theorem in the case of $C^n([0, T], W_\Omega^{+\infty})$ (the case of $C^n([0, T], W_\Omega^{-\infty})$ can be done by analogy with [8]).

Necessity. If $\sigma_0 \in \Omega \cap N_\Delta$, the homogeneous problem

$$L\left(\frac{d}{dt}, \sigma_0\right)\hat{u}(\sigma_0, t) = 0,$$

$$M(\sigma_0)\hat{u}(\sigma_0, t) = A(\sigma_0)\hat{u}(\sigma_0, 0) + B(\sigma_0)\hat{u}(\sigma_0, T) = 0$$

possesses more than one solution. Consequently, the solution (if it exists) of the homogeneous problem

$$L\left(\frac{\partial}{\partial t}, -i\frac{\partial}{\partial x}\right)u(x, t) = 0,$$

$$M\left(-i\frac{\partial}{\partial x}\right)u(x, t) = A\left(-i\frac{\partial}{\partial x}\right)u(x, 0) + B\left(-i\frac{\partial}{\partial x}\right)u(x, T) = 0$$

is not unique in $C^n([0, T], W_\Omega^{+\infty})$, and this implies the nonuniqueness of solutions of problem (1), (2) in $C^n([0, T], W_\Omega^{+\infty})$. The ill-posedness of problem (1), (2) in $C^n([0, T], W_\Omega^{+\infty})$ follows from the nonuniqueness of its solution in $C^n([0, T], W_\Omega^{+\infty})$.

Sufficiency. Let $\Omega = H \setminus N_\Delta$. For every $\varphi(x) \in W_\Omega^{+\infty}$,

$$u(x, t) = W\left(-i\frac{\partial}{\partial x}, t\right)\varphi(x)$$

is a solution of problem (1), (2) in $C^n([0, T], W_\Omega^{+\infty})$ and continuously depends on $\varphi(x)$, which implies the existence of solution of problem (1), (2) in $C^n([0, T], W_\Omega^{+\infty})$ that continuously depends on $\varphi(x)$. In order to prove the uniqueness of solution of problem (1), (2), let us notice that if $u(x, t) \in C^n([0, T], W_\Omega^{+\infty})$, its x-Fourier transform $\hat{u}(\sigma, t)$ will be a solution of problem (3), (4). Under the condition of the theorem, this problem (3), (4) possesses one and only one solution for every $\sigma \in \Omega$. If $\sigma \in \mathbb{R}^m \setminus \Omega$, then $\tilde{u}(\sigma, t) \equiv 0$. This implies the uniqueness of solution of problem (1), (2) in $C^n([0, T], W_\Omega^{+\infty})$, and the theorem is proved.

4. Application. Consider an infinite set of point-like particles, disposed on a string at the same distances l from each other; the mass of each particle is m, while F is the tension of the string. The values of F and m are supposed to be constant everywhere and independent of time. Particles are supposed to have only one degree of freedom. At each given time t, the motion of the j-th particle is completely defined in terms of the position of its adjacent particles, i.e., the $(j-1)$-th and $(j+1)$-th ones ($j = 0, \pm 1, \pm 2, \ldots$). Thus, the fundamental law of dynamics is given by

$$\ddot{v}_n(t) = \frac{a^2}{4}\big(v_{n+1}(t) + v_{n-1}(t) - 2v_n(t)\big), \quad n = 0, \pm 1, \pm 2, \ldots, \tag{6}$$

being $a = 2\sqrt{\dfrac{F}{ml}}$. We associate with (6) the boundary conditions

$$\begin{aligned} v_n(0) + v_n(T) &= \varphi_n, \\ \dot{v}_n(0) + \dot{v}_n(T) &= \psi_n. \end{aligned} \tag{7}$$

Consider the differentiable function $v(x, t)$ that takes the values of v_j at the nodes of the lattice, i.e., $v(jl, t) = v_j(t)$. Therefore, system (6) acquires the form of the difference-differential equation

$$\frac{\partial^2 v(x,t)}{\partial t^2} = \frac{a^2}{4}[v(x+l,t) - 2v(x,t) + v(x-l,t)], \quad 0 < t < T. \tag{8}$$

After using the equality $e^{i\left(-i2\frac{\partial}{\partial x}\right)} v(x,t) = v(x+l,t)$, equation (8) takes the form of the pseudodifferential equation:

$$\frac{\partial^2 v}{\partial t^2} + a^2 \sin^2\left(-i\frac{\partial}{\partial x}\right) v = 0, \quad 0 < t < T. \tag{9}$$

By setting $u = \operatorname{col}(u_1, u_2)$ with $u_1 = v$ and $u_2 = \dfrac{\partial v}{\partial t}$, equation (9) and the boundary conditions (7) become

$$\frac{\partial u}{\partial t} + P\left(-i\frac{\partial}{\partial x}\right) u = 0, \tag{10}$$

$$\begin{aligned} u_1(x,0) + u_1(x,T) &= \theta_1(x), \\ u_2(x,0) + u_2(x,T) &= \theta_2(x), \end{aligned} \tag{11}$$

respectively, where

$$P\left(-i\frac{\partial}{\partial x}\right) = \begin{pmatrix} 0 & -1 \\ a^2\sin^2\left(-i\frac{\partial}{\partial x}\right) & 0 \end{pmatrix}$$

and $\theta_1(x)$ and $\theta_2(x)$ are two differentiable functions that take the values of φ_n and ψ_n, respectively, at the nodes of the lattice. For this example, $F(\sigma, t)$ reads as

$$F(\sigma, t) = \begin{pmatrix} \cos(at\sin\sigma) & -\sin(at\sin\sigma) \\ -a\sin\sigma\sin(at\sin\sigma) & -a\sin\sigma\cos(at\sin\sigma) \end{pmatrix}$$

and

$$\Delta(\sigma) = -2a\sin\sigma[1 + \cos(aT\sin\sigma)].$$

Therefore,

$$N_{\Delta} = \{k\pi\}_{k \in \mathbb{Z}} \cup \left\{ \arcsin \frac{(2k+1)\pi}{aT}, \ -\frac{\pi + aT}{2\pi} \le k \le \frac{\pi + aT}{2\pi}, \ k \in \mathbb{Z} \right\}.$$

If we take $H = \mathbb{R}$, we conclude from the above theorem that problem (10), (11) is properly posed in $C^n([0, T], W_{\Omega}^{\pm\infty})$ with $\Omega = \mathbb{R} \setminus N_{\Delta}$.

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