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AN EXTENDED STOCHASTIC INTEGRAL AND THE WICK CALCULUS ON THE CONNECTED WITH THE GAMMA-MEASURE SPACES OF REGULAR GENERALIZED FUNCTIONS*

РОЗШИРЕНИЙ СТОХАСТИЧНИЙ ІНТЕГРАЛ ТА ВІКІВСЬКЕ ЧИСЛЕННЯ НА ПРОСТОРАХ РЕГУЛЯРНИХ УЗАГАЛЬНЕНИХ ФУНКЦІЙ, ЩО ПОВ'ЯЗАНІ З ГАММА-МІРОЮ

This paper is dedicated to Professor Yu. M. Berezansky,
who is one of my mentors

We introduce and study an extended stochastic integral, a Wick product and Wick versions of holomorphic functions on the Kondratiev-type spaces of regular generalized functions. These spaces are connected with the Gamma-measure on some generalization of the Schwartz distributions space $\mathcal{S}'$. As examples we consider stochastic equations with Wick-type nonlinearity.

Вводиться та вивчається розширений стохастичний інтеграл, віківське множення та віківські версії голоморфних функцій на просторах (типу Кондратьєва) регулярних узагальнених функцій. Ці простори пов'язані з гамма-мірою на певному узагальненні простору узагальнених функцій Шварца $\mathcal{S}'$. Як приклади розглядаються стохастичні рівняння з нелінійностями віківського типу.

Introduction. In the paper [1] the Gamma-measure μ as a particular case of the compound Poisson measure on the Schwartz distributions space $\mathcal{S}'$ was considered and elements of the corresponding white noise analysis were studied. In particular, orthogonal polynomials in the space $L^2(\mathcal{S}', \mu)$ of square integrable with respect to μ functions on $\mathcal{S}'$ (the so-called generalized Laguerre polynomials — a particular case of the generalized Appell polynomials) were constructed. But it was found that as distinguished from the Gaussian and Poisson cases the orthogonality relation contains the special scalar product connected with a nature of μ. This fact, so as an absence of the chaotic representation property in the "Gamma-analysis" (see, e.g., [2]), led to the situation when an extended stochastic integral connected with the Gamma-measure on $\mathcal{S}'$ can not be constructed by analogy with the Gaussian or Poisson analysis.

In the paper [3] the author offered a natural construction of an extended stochastic integral on $L^2(\mathcal{S}'_\sigma, \mu)$ (where $\mathcal{S}'_\sigma$ is some generalization of $\mathcal{S}'$) and on the corresponding Kondratiev-type space of *nonregular* generalized functions $(\mathcal{S}')'$ (more exactly, integrable functions have values in $L^2(\mathcal{S}'_\sigma, \mu)$ and in $(\mathcal{S}')'$ correspondingly). The space $(\mathcal{S}')'$ was selected because its properties are well studied and it is very simple to introduce a Wick product and Wick versions of holomorphic functions on this space; this is very important for construction of the informative integral theory. But, on the other hand, $(\mathcal{S}')'$ is too wide space and kernels from the natural orthogonal decompositions of elements of $(\mathcal{S}')'$ belong to the distributions spaces without "good" properties. This is inconvenient for applications.

The main aim of this paper is to move main results of [3] on the so-called Kondratiev-type space of *regular* generalized functions $(L^2)^{-1}$. This space in

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narrower than $(\mathcal{S}')'$ and here is no the mentioned problem with orthogonal decompositions in $\left(L^2\right)^{-1}$. At the same time some difficulties with the Wick calculus on $\left(L^2\right)^{-1}$ have a technological nature and were successfully overcame. As an additional argument in behalf of study of an extended stochastic integral and the Wick calculus on $\left(L^2\right)^{-1}$ we note that solutions of many stochastic equations with Wick-type nonlinearity lie in $\left(L^2\right)^{-1}$ (as an example we consider the classical Verhulst-type stochastic equation; its solution X_t, as it well known, does not lie in $L^2\left(\mathcal{S}', \mu\right)$ for $X_0 = \frac{1}{2}$, but if follows from our results that $X_t \in \left(L^2\right)^{-1}$ for all $X_0 \in \left(L^2\right)^{-1}$).

The paper is organized in the following manner. In the first section we recall some elements of the “Gamma-analysis”. In the second section we introduce and study an extended stochastic integral on $\left(L^2\right)^{-1}$. The third section devoted to the Wick calculus and its interconnection with a stochastic integration. In the end of the paper we consider examples of stochastic equations with Wick-type nonlinearity.

Finally we note that some questions connected with a stochastic integration in the “Gamma-analysis” were studied in [4].

1. Preliminaries. Let σ be a nonatomic positive regular σ-finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ satisfying some additional condition, see Remark 1.1 for details (here and below the symbol $\mathcal{B}$ denotes the Borel σ-algebra). We denote $\mathcal{H} := L^2(\mathbb{R}, \sigma)$ (the space of square integrable with respect to σ functions on $\mathbb{R}$). Let $\mathcal{S}$ be the Schwartz test functions space on $\operatorname{supp}\sigma$ (if, e.g., σ is the Lebesgue measure then $\mathcal{S}$ is the usual Schwartz space of rapidly decreasing infinitely differentiable functions). As it well known, there exist Hilbert spaces $\mathcal{H}_p \equiv \mathcal{H}_p(\mathbb{R}) \subset \mathcal{H}$, $p \in \mathbb{N}$, such that we have the nuclear chain

$$\mathcal{S}' = \operatorname*{ind\,lim}_{p' \in \mathbb{N}} \mathcal{H}_{-p'} \supset \mathcal{H}_{-p} \supset \mathcal{H} \equiv \mathcal{H}_0 \supset \mathcal{H}_p \supset \operatorname*{pr\,lim}_{p' \in \mathbb{N}} \mathcal{H}_{p'} = \mathcal{S}, \tag{1.1}$$

where $\mathcal{H}_{-p}$, $p \in \mathbb{N}$, $\mathcal{S}'$ are the dual spaces to $\mathcal{H}_p$, $\mathcal{S}$ with respect to the zero space $\mathcal{H}$. Note that one can select spaces $\mathcal{H}_p$, $p \in \mathbb{N}$, such that for each $p > p'$ it will be $|\cdot|_p \geq |\cdot|_{p'}$ (where $|\cdot|_p$ denotes the norm in $\mathcal{H}_p$, $p \in \mathbb{Z}$, in particular, $|\cdot|_0 = |\cdot|_{\mathcal{H}}$). We preserve the notation $|\cdot|_p$ for norms in tensor powers and complexifications of $\mathcal{H}_p$, $p \in \mathbb{Z}$.

Remark **1.1.** Let us describe the construction of the spaces $\mathcal{H}_p$, $p \in \mathbb{N}$, in details, following [5]. Let $(e_j)_{j=0}^{\infty}$ be the system of Hermite functions on $\mathbb{R}$. For each $p \geq 1$ we denote by $\tilde{\mathcal{H}}_p \equiv \tilde{\mathcal{H}}_p(\mathbb{R})$ the Hilbert space constructed by the orthogonal basis $\left(e_j(2j+2)^{-p}\right)_{j=0}^{\infty}$, and assume that the measure σ is such that for some $\varepsilon \geq 0$ the space $\tilde{\mathcal{H}}_{1+\varepsilon}$ is continuously embedded into $\mathcal{H} = L^2(\mathbb{R}, \sigma)$. Further, let $O_p \colon \tilde{\mathcal{H}}_p \to \mathcal{H}$ be the embedding operator. Without loss of generality one can suppose that for ε defined above $O_{1+\varepsilon}$ is the operator of Hilbert – Schmidt type (for example, if σ is the Lebesgue measure then one can put $\varepsilon = 0$). Now we can put $\mathcal{H}_p := \tilde{\mathcal{H}}_{p+\varepsilon}\Big|_{KER\, O_{p+\varepsilon}}$ (the Hilbert factor space).

Let us denote by the subindex “$\mathbb{C}$” complexifications of spaces. Let $\langle \cdot, \cdot \rangle$ denote the generated by the scalar product in $\mathcal{H}$ (real) dual pairing between elements of $\mathcal{S}'_{\mathbb{C}}$ and $\mathcal{S}_{\mathbb{C}}$ (and also $\mathcal{H}_{-p,\mathbb{C}}$ and $\mathcal{H}_{p,\mathbb{C}}$); this notation will be preserved for pairings in tensor powers of spaces. Let F be the σ-algebra on $\mathcal{S}'$ generated by cylinder sets.

Definition 1.1. *The measure* μ *on the measurable space* $(\mathcal{S}', F)$ *with the Laplace transform*

$$l_\mu(\lambda) = \int_{\mathcal{S}'} e^{\langle x,\lambda\rangle}\mu(dx) = \exp\{-\langle 1, \log(1-\lambda)\rangle\}, \quad 1 > \lambda \in \mathcal{S}, \tag{1.2}$$

is called the Gamma-measure.

Remark **1.2.** Strictly speaking, one can not apply the Minlos theorem to (1.2) in order to prove existence and uniqueness of the measure μ, because λ in (1.2) is not an arbitrary element of $\mathcal{S}_{\mathbb{C}}$. But as it was proved in [1] the Gamma-measure is the particular case of the compound Poisson measure. So, this is the well-defined probability measure on $\mathcal{S}'$ with the holomorphic at zero Laplace transform.

Remark **1.3.** The term "Gamma-measure" is connected with the fact that μ is the measure of the so-called *Gamma-white noise*. Let us explain this in more details, following [1]. If σ is the Lebesgue measure m, then for each $t > 0$ the Laplace transform

$$l_{\mu^m}\left(\lambda 1_{[0,t]}\right) = \exp\{-t\log(1-\lambda)\} = (1-\lambda)^{-t}, \quad 1 > \lambda \in \mathbb{R}$$

(here $1_{[0,t]}$ denotes the indicator of the set $[0,t]$) coincides with the Laplace transform $l_{\xi(t)}(\lambda)$ of a random variable $\xi(t)$ having the so-called Gamma-distribution, i.e., the density of the distribution function has the form

$$p_t(x) = \frac{x^{t-1}e^{-x}}{\Gamma(t)} 1_{\{x>0\}}, \quad t > 0.$$

The process $\{\xi(t), t > 0;\ \xi(0) := 0\}$ is known as the *Gamma-process*. Thus the triple $\left(\mathcal{S}', F, \mu^m\right)$ is a direct representation of the generalized stochastic process $\{\dot\xi(t), t \ge 0\}$ that is a distributional derivative of the Gamma-process.

Now by $\left(L^2\right) \equiv L^2\left(\mathcal{S}', \mu\right)$ we denote the space of square integrable with respect to μ functions on $\mathcal{S}'$ and construct orthogonal polynomials in $\left(L^2\right)$. Let $\alpha : \mathcal{S}_{\mathbb{C}} \to \mathcal{S}_{\mathbb{C}}$ be the function defined on some neighbourhood of $0 \in \mathcal{S}_{\mathbb{C}}$ by the formula $\alpha(\lambda) := \frac{\lambda}{\lambda - 1}$. We define the so-called *Wick exponential* (a generating function of the orthogonal polynomials)

$$:\!\exp(x;\lambda)\!: \overset{\text{df}}{=} \frac{\exp\{\langle x, \alpha(\lambda)\rangle\}}{l_\mu(\alpha(\lambda))} = \exp\left\{\left\langle x, \frac{\lambda}{\lambda-1}\right\rangle - \langle 1, \log(1-\lambda)\rangle\right\}, \tag{1.3}$$

where $\lambda \in \mathcal{U}_0 \subset \mathcal{S}_{\mathbb{C}}$, $x \in \mathcal{S}'$, $\mathcal{U}_0$ is some neighbourhood of $0 \in \mathcal{S}_{\mathbb{C}}$.

Remark **1.4.** Note that (1.3) is the infinite-dimensional analogs of the generating functions of the one-dimensional Laguerre polynomials. These polynomials are orthogonal "with respect to the one-dimensional Gamma-measure", see, e.g., [6].

It is clear that $:\!\exp(x;\cdot)\!:$ is a holomorphic at zero function on $\mathcal{S}_{\mathbb{C}}$ for each $x \in \mathcal{S}'$. So, using the Cauchy inequalities (see, e.g., [7]) and the kernel theorem (see, e.g., [8, p. 46]) one can obtain the representation

$$:\!\exp(x;\lambda)\!: = \sum_{n=0}^{\infty} \frac{1}{n!}\left\langle L_n(x), \lambda^{\otimes n}\right\rangle, \qquad L_n(x) \in \mathcal{S}'^{\hat\otimes n}_{\mathbb{C}}, \quad \lambda \in \mathcal{S}_{\mathbb{C}},$$

where $\hat{\otimes}$ denotes a symmetric tensor product, $\lambda^{\otimes 0} = 1$ even for $\lambda \equiv 0$. (Note that actually for $x \in \mathcal{S}'$ $L_n(x) \in \mathcal{S}'^{\hat{\otimes} n}$.)

Definition 1.2. *The polynomials* $\langle L_n(x), f^{(n)} \rangle$, $f^{(n)} \in \mathcal{S}_{\mathbb{C}}^{\hat{\otimes} n}$, *are called the generalized Laguerre polynomials.*

In order to formulate a statement on an orthogonality of $\langle L_n(x), f^{(n)} \rangle$ we need the following definition.

Definition 1.3. *We define the scalar product* $\langle \cdot, \cdot \rangle_{\mathrm{ext}}$ *on* $\mathcal{S}_{\mathbb{C}}^{\hat{\otimes} n}$ *by the formula*

$$\langle f^{(n)}, g^{(n)} \rangle_{\mathrm{ext}} = \sum_{\substack{k, l_j, s_j \in \mathbb{N}:\, j = 1, \ldots, k,\, l_1 > l_2 > \ldots > l_k, \\ l_1 s_1 + \ldots + l_k s_k = n}} \frac{n!}{l_1^{s_1} \ldots l_k^{s_k}\, s_1! \ldots s_k!} \times$$

$$\times \int_{\mathbb{R}^{s_1 + \ldots + s_k}} f^{(n)}(\underbrace{\tau_1, \ldots, \tau_1}_{l_1}, \ldots, \underbrace{\tau_{s_1}, \ldots, \tau_{s_1}}_{l_1}, \ldots, \underbrace{\tau_{s_1 + \ldots + s_k}, \ldots, \tau_{s_1 + \ldots + s_k}}_{l_k}) \times$$

$$\times\, g^{n}(\underbrace{\tau_1, \ldots, \tau_1}_{l_1}, \ldots, \underbrace{\tau_{s_1}, \ldots, \tau_{s_1}}_{l_1}, \ldots, \underbrace{\tau_{s_1 + \ldots + s_k}, \ldots, \tau_{s_1 + \ldots + s_k}}_{l_k}) \times$$

$$\times\, \sigma(d\tau_1) \ldots \sigma(d\tau_{s_1 + \ldots + s_k}). \tag{1.4}$$

By $|\cdot|_{\mathrm{ext}}$ we denote the corresponding norm, i.e., $|f^{(n)}|_{\mathrm{ext}}^2 = \langle f^{(n)}, \overline{f^{(n)}} \rangle_{\mathrm{ext}}$.

Example **1.1.** It follows from (1.4) that for $n = 1$ $\langle f^{(1)}, g^{(1)} \rangle_{\mathrm{ext}} = \langle f^{(1)}, g^{(1)} \rangle$. Further, for $n = 2$

$$\langle f^{(2)}, g^{(2)} \rangle_{\mathrm{ext}} = \langle f^{(2)}, g^{(2)} \rangle + \int_{\mathbb{R}} f^{(2)}(\tau, \tau) g^{(2)}(\tau, \tau) \sigma(d\tau).$$

Theorem 1.1 [1]. *The generalized Laguerre polynomials are orthogonal in* (L^2) *in the sense that*

$$\int_{\mathcal{S}'} \langle L_n(x), f^{(n)} \rangle \langle L_m(x), g^{(m)} \rangle \mu(dx) = \delta_{mn}\, n! \langle f^{(n)}, g^{(n)} \rangle_{\mathrm{ext}}.$$

By $\mathcal{H}_{\mathrm{ext}}^{(n)}$, $n \in \mathbb{N}$, we denote the closure of $\mathcal{S}_{\mathbb{C}}^{\hat{\otimes} n}$ with respect to the norm $|\cdot|_{\mathrm{ext}}$ (see (1.4)), $\mathcal{H}_{\mathrm{ext}}^{(0)} := \mathbb{C}$. For $f^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)}$ we define $(L^2) \ni \langle L_n, f^{(n)} \rangle :=$ $:= \lim_{k \to \infty} \langle L_n, f_k^{(n)} \rangle$ in (L^2), where $\mathcal{S}_{\mathbb{C}}^{\hat{\otimes} n} \ni f_k^{(n)} \to f^{(n)}$ (as $k \to \infty$) in $\mathcal{H}_{\mathrm{ext}}^{(n)}$ (the correctness of this definition can be proved by analogy with the classical Gaussian case, see also [3, 9]). The following statement from results of [5] follows.

Theorem 1.2. *A function* $f \in (L^2)$ *if and only if there exists a sequence of kernels* $(f^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)})_{n=0}^{\infty}$ *such that* f *can be presented in the form*

$$f = \sum_{n=0}^{\infty} \langle L_n, f^{(n)} \rangle, \tag{1.5}$$

where the series converges in (L^2), *i.e., the* (L^2)*-norm of* f

$$\| f \|_{(L^2)}^2 = \sum_{n=0}^{\infty} n! |f^{(n)}|_{\mathrm{ext}}^2 < \infty.$$

Furthermore, the system $\{\langle L_n, f^{(n)}\rangle,\ f^{(n)}\in\mathcal{H}_{\mathrm{ext}}^{(n)},\ n\in\mathbb{Z}_+\}$ *plays a role of an orthogonal basis in* (L^2) *in the sense that for* $f, g\in(L^2)$

$$(f,g)_{(L^2)} = \sum_{n=0}^{\infty} n!\langle f^{(n)}, g^{(n)}\rangle_{\mathrm{ext}},$$

where $f^{(n)}, g^{(n)}$ *are the kernels from decompositions* (1.5) *for* f, g.

Now let us introduce the Kondratiev-type spaces of regular test and generalized functions. First we consider the set $\mathcal{P} := \{f = \sum_{n=0}^{N_f}\langle L_n, f^{(n)}\rangle,\ f^{(n)}\in\mathcal{H}_{\mathrm{ext}}^{(n)},\ N_f\in\mathbb{Z}_+\}\subset(L^2)$ of polynomials and $\forall q\in\mathbb{N}$ introduce on this set the scalar product $(\cdot,\cdot)_q$, putting for $f = \sum_{n=0}^{N_f}\langle L_n, f^{(n)}\rangle$, $g = \sum_{n=0}^{N_g}\langle L_n, g^{(n)}\rangle$

$$(f,g)_q := \sum_{n=0}^{\min(N_f,N_g)} (n!)^2 2^{qn}\langle f^{(n)}, g^{(n)}\rangle_{\mathrm{ext}}.$$

Let $\|\cdot\|_q$ be the corresponding norm: $\|f\|_q = \sqrt{(f,\bar f)_q} = \sqrt{\sum_{n=0}^{N_f}(n!)^2 2^{qn}|f^{(n)}|_{\mathrm{ext}}^2}$.

Definition 1.4. *We define the Kondratiev-type spaces of* ("*regular*") *test functions* $(L^2)_q^1$, $q\in\mathbb{N}$, *as the closures of* $\mathcal{P}$ *with respect to the norms* $\|\cdot\|_q$, $(L^2)^1 := \operatorname{pr\,lim}_{q\in\mathbb{N}}(L^2)_q^1$.

It is not difficult to see that $f\in(L^2)_q^1$ if and only if f can be presented in form (1.5) with

$$\|f\|_q^2 = \sum_{n=0}^{\infty}(n!)^2 2^{qn}|f^{(n)}|_{\mathrm{ext}}^2 < \infty,$$

therefore the generalized Laguerre polynomials play a role of an orthogonal basis in $(L^2)_q^1$.

It is obvious that $\forall q\in\mathbb{N}$ $\|\cdot\|_{(L^2)}\le\|\cdot\|_q$. Further, let a sequence $(f_k\in\mathcal{P})_{k=0}^{\infty}$ be a Cauchy one in $(L^2)_q^1$ and tends to zero in (L^2), and let $f := \lim_{k\to\infty} f_k$ in $(L^2)_q^1$. We have

$$\|f\|_{(L^2)} = \|f - f_k + f_k\|_{(L^2)} \le \|f - f_k\|_{(L^2)} + \|f_k\|_{(L^2)} \le$$
$$\le \|f - f_k\|_q + \|f_k\|_{(L^2)} \to 0$$

as $k\to\infty$, so $\|f\|_{(L^2)} = 0$. But it follows from here that all coefficients $f^{(n)}\in\mathcal{H}_{\mathrm{ext}}^{(n)}$ from decomposition (1.5) for f are equal to zero whence $\|f\|_q = 0$. Therefore $f_k\to\to 0$ (as $k\to\infty$) in $(L^2)_q^1$. Thus (see, e.g., [8, p. 51]) $(L^2)_q^1 \hookrightarrow (L^2)$. Moreover, because l_μ is a holomorphic at zero function, this embedding is dense (see [10]). Therefore one can consider the chain

$$(L^2)^{-1} := \operatorname{ind\,lim}_{\tilde q\in\mathbb{N}}(L^2)_{-\tilde q}^{-1} \supset (L^2)_{-q}^{-1} \supset (L^2) \supset (L^2)_q^1 \supset (L^2)^1,$$

where $(L^2)_{-q}^{-1}$, $(L^2)^{-1}$ are the spaces dual to $(L^2)_q^1$, $(L^2)^1$ with respect to (L^2) correspondingly.

Definition 1.5. *The spaces* $(L^2)^{-1}_{-q}$, $(L^2)^{-1}$ *are called the Kondratiev-type spaces of regular generalized functions* (*cf.* [11]).

It is easy to see that $F \in (L^2)^{-1}_{-q}$ if and only if F can be presented as the formal series

$$F = \sum_{m=0}^{\infty} \left\langle L_m, F^{(m)} \right\rangle, \quad F^{(m)} \in \mathcal{H}^{(m)}_{\text{ext}} \tag{1.6}$$

with

$$\| F \|^2_{-q} := \sum_{m=0}^{\infty} 2^{-qm} \left| F^{(m)} \right|^2_{\text{ext}} < \infty.$$

Moreover, the generalized Laguerre polynomials play a role of an orthogonal basis in $(L^2)^{-1}_{-q}$ in the sense that for F, $H \in (L^2)^{-1}_{-q}$ presented in form (1.6) we have $(F, H)_{-q} = \sum_{m=0}^{\infty} 2^{-qm} \left\langle F^{(m)}, H^{(m)} \right\rangle_{\text{ext}}$ (here $(\cdot, \cdot)_{-q}$ denotes the (real) scalar product in $\left(L^2\right)^{-1}_{-q}$, $\| F \|_{-q} = \sqrt{\left(F, \overline{F}\right)_{-q}}$).

By $\langle\langle \cdot, \cdot \rangle\rangle$ we denote the dual pairing between elements of $(L^2)^{-1}_{-q}$ and $(L^2)^{1}_{q}$ (correspondingly $(L^2)^{-1}$ and $(L^2)^{1}$), this pairing is generated by the scalar product in $\left(L^2\right)$. If $F \in (L^2)^{-1}_{-q}$ and $f \in (L^2)^{1}_{q}$ we have

$$\langle\langle F, f \rangle\rangle = \sum_{n=0}^{\infty} n! \left\langle F^{(n)}, f^{(n)} \right\rangle_{\text{ext}},$$

where $F^{(n)}$, $f^{(n)} \in \mathcal{H}^{(n)}_{\text{ext}}$ are the kernels from decompositions (1.6), (1.5) for F and f respectively.

Finally, in order to compare results of this paper with the corresponding results in a "*nonregular*" case (see [3]) we have to recall the corresponding definitions and statements. It was proved in [3] (see also [5]) that for all p, $n \in \mathbb{N}$ the continuous embeddings $\mathcal{H}^{\hat{\otimes} n}_{p,\mathbb{C}} \hookrightarrow \mathcal{H}^{(n)}_{\text{ext}}$, $\mathcal{S}^{\hat{\otimes} n}_{\mathbb{C}} \hookrightarrow \mathcal{H}^{(n)}_{\text{ext}}$ hold. Hence one can consider the chains

$$\mathcal{S}'^{(n)}_{\mathbb{C}} \supset \mathcal{H}^{(n)}_{-p,\text{ext}} \supset \mathcal{H}^{(n)}_{\text{ext}} \supset \mathcal{H}^{\hat{\otimes} n}_{p,\mathbb{C}} \supset \mathcal{S}^{\hat{\otimes} n}_{\mathbb{C}}, \quad n \in \mathbb{Z}_+, \quad p \in \mathbb{N}, \tag{1.7}$$

where $\mathcal{S}'^{(n)}_{\mathbb{C}}$ (provided by the inductive limit topology), $\mathcal{H}^{(n)}_{-p,\text{ext}}$ are the spaces dual to $\mathcal{S}^{\hat{\otimes} n}_{\mathbb{C}}$, $\mathcal{H}^{\hat{\otimes} n}_{p,\mathbb{C}}$ with respect to the zero space $\mathcal{H}^{(n)}_{\text{ext}}$ correspondingly. For the (real) dual pairings between elements of $\mathcal{S}'^{(n)}_{\mathbb{C}}$ and $\mathcal{S}^{\hat{\otimes} n}_{\mathbb{C}}$ we preserve the notation $\langle \cdot, \cdot \rangle_{\text{ext}}$.

***Remark* 1.5.** Of course, for $n = 1$ chain (1.7) has the form

$$\mathcal{S}'_{\mathbb{C}} \supset \mathcal{H}_{-p,\mathbb{C}} \supset \mathcal{H}_{\mathbb{C}} \supset \mathcal{H}_{p,\mathbb{C}} \supset \mathcal{S}_{\mathbb{C}},$$

i.e., this chain coincides with the complexification of chain (1.1). But for $n > 1$ chain (1.7) *is not a tensor power* of a chain of type (1.1).

***Remark* 1.6.** It was proved in [4] that *the space* $\mathcal{H}^{(n)}_{\text{ext}}$, $n > 1$, is *the orthogonal sum of* $\mathcal{H}^{\hat{\otimes} n}_{\mathbb{C}}$ *and some another Hilbert spaces.* In this sense $\mathcal{H}^{\hat{\otimes} n}_{\mathbb{C}}$ can be considered as a subspace of $\mathcal{H}^{(n)}_{\text{ext}}$.

Let $\mathcal{P}(\mathcal{S}')$ be the set of all continuous polynomials on $\mathcal{S}'$. It follows from results of [12 – 14] that any element of $\mathcal{P}(\mathcal{S}')$ can be presented in the form

$$f(x) = \sum_{n=0}^{N_f} \left\langle L_n(x), f^{(n)} \right\rangle, \quad f^{(n)} \in \mathcal{S}_{\mathbb{C}}^{\hat{\otimes} n}. \tag{1.8}$$

We define on $\mathcal{P}(\mathcal{S}')$ a family of the scalar products, putting for $f, g \in \mathcal{P}(\mathcal{S}')$ presented in form (1.8), $p, q \in \mathbb{N}$

$$(f, g)_{p,q} = \sum_{n=0}^{\min(N_f, N_g)} (n!)^2 2^{qn} \left\langle f^{(n)}, g^{(n)} \right\rangle_p,$$

where by $\langle \cdot, \cdot \rangle_p$ the scalar product in $\mathcal{H}_p^{\hat{\otimes} n}$ denoted. The corresponding norms are denoted by $\|\cdot\|_{p,q}$, i.e., for $f \in \mathcal{P}(\mathcal{S}')$ of form (1.8) we have

$$\|f\|_{p,q}^2 = \left(f, \bar{f}\right)_{p,q} = \sum_{n=0}^{\mathbb{N}_f} (n!)^2 2^{qn} \left| f^{(n)} \right|_p^2.$$

Definition 1.6. *We define the Kondratiev-type test functions spaces* $\left(\mathcal{H}_p\right)_q$, p, $q \in \mathbb{N}$, *as the closures of* $\mathcal{P}(\mathcal{S}')$ *with respect to the norms* $\|\cdot\|_{p,q}$; $(\mathcal{S}) :=$ $:= \operatorname{pr} \lim_{p, q \in \mathbb{N}} \left(\mathcal{H}_p\right)_q$.

It is clear that $f \in \left(\mathcal{H}_p\right)_q$ if and only if f can be presented in the form

$$f = \sum_{n=0}^{\infty} \left\langle L_n, f^{(n)} \right\rangle, \quad f^{(n)} \in \mathcal{S}_{\mathbb{C}}^{\hat{\otimes} n}, \tag{1.9}$$

where the series converges in the sense that

$$\|f\|_{p,q}^2 = \sum_{n=0}^{\infty} (n!)^2 2^{qn} \left| f^{(n)} \right|_p^2 < \infty. \tag{1.10}$$

Further, it follows from Definition 1.6 that $f \in (\mathcal{S})$ if and only if f can be presented in form (1.9) and norm (1.10) is finite for all $p, q \in \mathbb{N}$.

***Remark* 1.7.** Let $f, g \in \left(\mathcal{H}_p\right)_q$ and presented in form (1.9). Then

$$(f, g)_{p,q} = \sum_{n=0}^{\infty} (n!)^2 2^{qn} \left\langle f^{(n)}, g^{(n)} \right\rangle_p,$$

so the system of the generalized Laguerre polynomials plays the role of an orthogonal basis in $\left(\mathcal{H}_p\right)_q$.

Proposition 1.1 [3]. *There exists* $p_0 \in \mathbb{N}$ *such that for each* $p \geq p_0$ *there exists* $q_0(p)$ *such that for each* $q > q_0(p)$ *the dense and continuous embedding* $\left(\mathcal{H}_p\right)_q \hookrightarrow \left(L^2\right)$ *holds.*

So, for $p, q \in \mathbb{N}$ sufficiently large we can consider the following chain:

$$(\mathcal{S}')' = \operatorname{ind} \lim_{\tilde{p}, \tilde{q} \in \mathbb{N}} \left(\mathcal{H}_{-\tilde{p}}\right)_{-\tilde{q}} \supset \left(\mathcal{H}_{-p}\right)_{-q} \supset \left(L^2\right) \supset \left(\mathcal{H}_p\right)_q \supset (\mathcal{S}) =$$
$$= \operatorname{pr} \lim_{\tilde{p}, \tilde{q} \in \mathbb{N}} \left(\mathcal{H}_{\tilde{p}}\right)_{\tilde{q}},$$

where $\left(\mathcal{H}_{-p}\right)_{-q}$, $(\mathcal{S}')'$ are the dual with respect to $\left(L^2\right)$ spaces to $\left(\mathcal{H}_p\right)_q$, $(\mathcal{S})$ correspondingly.

Definition 1.7. *The spaces* $(\mathcal{H}_{-p})_{-q}$, $(\mathcal{S}')'$ *are called the Kondratiev-type spaces of nonregular generalized functions.*

Note that Kondratiev-type spaces of regular and nonregular functions are connected as follows:

$$(\mathcal{S}')' \supset (\mathcal{H}_{-p})_{-q} \supset (L^2)_{-q}^{-1} \supset (L^2) \supset (L^2)_q^1 \supset (\mathcal{H}_p)_q \supset (\mathcal{S}),$$

$$(\mathcal{S}')' \supset (L^2)^{-1} \supset (L^2) \supset (L^2)^1 \supset (\mathcal{S}).$$

Unfortunately, the spaces $(\mathcal{H}_p)_q$ and $(L^2)^1$ (so as $(\mathcal{H}_{-p})_{-q}$ and $(L^2)^{-1}$) do not included one to another. (More exactly there exists $f \in (L^2)^1$ such that the coefficients from decomposition (1.5) $f^{(n)} \notin \mathcal{S}_{\mathbb{C}}^{\hat{\otimes} n}$, so $f \notin (\mathcal{H}_p)_q$; and there exists $g \in (\mathcal{H}_p)_q$ such that $\|g\|_{q+1} = +\infty$, so $g \notin (L^2)^1$.)

Now let us construct an orthogonal basis in $(\mathcal{H}_{-p})_{-q}$ and introduce some another notions which will be necessary below.

First we note that because in the complexification of tensor power n of chain (1.1) and in chain (1.7) the test functions spaces are the same, the spaces $\mathcal{S}_{\mathbb{C}}'^{(n)}$ and $\mathcal{S}_{\mathbb{C}}'^{\hat{\otimes}(n)}$, $n \in \mathbb{Z}_+$ are isomorphic (we remind that $\mathcal{S}_{\mathbb{C}}'^{(0)} = \mathcal{S}_{\mathbb{C}}'^{\hat{\otimes}(0)} = \mathbb{C}$, $\mathcal{S}_{\mathbb{C}}'^{(1)} = \mathcal{S}_{\mathbb{C}}'^{\hat{\otimes}(1)} = \mathcal{S}_{\mathbb{C}}'$). So, *there exists the family of bijective operators* $U_n \colon \mathcal{S}_{\mathbb{C}}'^{(n)} \to \mathcal{S}_{\mathbb{C}}'^{\hat{\otimes}(n)}$, $n \in \mathbb{Z}_+$ *such that for any* $n \in \mathbb{Z}_+$, *for each* $F_{\text{ext}}^{(n)} \in \mathcal{S}_{\mathbb{C}}'^{(n)}$

$$\left\langle F_{\text{ext}}^{(n)}, f^{(n)} \right\rangle_{\text{ext}} \equiv \left\langle U_n F_{\text{ext}}^{(n)}, f^{(n)} \right\rangle \quad \forall\, f^{(n)} \in \mathcal{S}_{\mathbb{C}}^{\hat{\otimes} n}. \tag{1.11}$$

***Remark* 1.8.** Unfortunately, *the restriction of* U_n, $n > 1$, *on* $\mathcal{H}_{\text{ext}}^{(n)}$ *is not an isomorphism between* $\mathcal{H}_{\text{ext}}^{(n)}$ *and* $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ (see [3] for details).

Further, let us define on $\mathcal{P}(\mathcal{S}')$ the operator $\left\langle F_{\text{ext}}^{(m)}, :D:^{\otimes m} \right\rangle_{\text{ext}}$ with constant coefficients $F_{\text{ext}}^{(m)} \in \mathcal{S}_{\mathbb{C}}'^{(m)}$, putting on the "monomials" $\left\langle L_n, f^{(n)} \right\rangle$, $f^{(n)} \in \mathcal{S}_{\mathbb{C}}^{\hat{\otimes} n}$

$$\left\langle F_{\text{ext}}^{(m)}, :D:^{\otimes(m)} \right\rangle_{\text{ext}} \left\langle L_n, f^{(n)} \right\rangle := 1_{\{n \ge m\}} \frac{n!}{(n-m)!} \left\langle L_{n-m}, \left\langle F_{\text{ext}}^{(m)}, f^{(n)} \right\rangle_{\text{ext}} \right\rangle$$

and continue by linearity. Here $\left\langle F_{\text{ext}}^{(m)}, f^{(n)} \right\rangle_{\text{ext}} \in \mathcal{S}_{\mathbb{C}}^{\hat{\otimes}(n-m)}$ with $n > m$ is defined for $f^{(n)} = \lambda^{\otimes n}$ by the formula

$$\left\langle F_{\text{ext}}^{(m)}, \lambda^{\otimes n} \right\rangle_{\text{ext}} := \left\langle F_{\text{ext}}^{(m)}, \lambda^{\otimes m} \right\rangle_{\text{ext}} \lambda^{\otimes(n-m)}$$

and for a general $f^{(n)} \in \mathcal{S}_{\mathbb{C}}^{\hat{\otimes} n}$ by the corresponding limit $\left(\left\{\lambda^{\otimes n} \colon \lambda \in \mathcal{S}_{\mathbb{C}}\right\}\right.$ is a total set in $\left.\mathcal{S}_{\mathbb{C}}^{\hat{\otimes}(n)}\right)$.

It follows from results of [3] (see also [12]) that the operator $\left\langle F_{\text{ext}}^{(m)}, :D:^{\otimes m} \right\rangle_{\text{ext}}$ can be continued to the linear continuous operator on $(\mathcal{S})$.

Now let us consider the dual to $\left\langle F_{\text{ext}}^{(m)}, :D:^{\otimes m} \right\rangle_{\text{ext}}$ with respect to (L^2) operator $\left\langle F_{\text{ext}}^{(m)}, :D:^{\otimes m} \right\rangle_{\text{ext}}^{*} \colon (\mathcal{S}')' \to (\mathcal{S}')'$ defined by the formula

$$\left\langle\!\left\langle \left\langle F_{\text{ext}}^{(m)}, :D^{:\otimes m} \right\rangle_{\text{ext}}^{*} H, f \right\rangle\!\right\rangle \equiv \left\langle\!\left\langle H, \left\langle F_{\text{ext}}^{(m)}, :D^{:\otimes m} \right\rangle_{\text{ext}} f \right\rangle\!\right\rangle$$

$$\forall\, H \in (\mathcal{S}')', \quad \forall\, f \in (\mathcal{S}).$$

Definition 1.8. *For each* $F_{\text{ext}}^{(m)} \in \mathcal{S}_{\mathbb{C}}'^{(m)}$ *we define the generalized function* $\left\langle L_m, F_{\text{ext}}^{(m)} \right\rangle \in (\mathcal{S}')'$, *putting*

$$\left\langle L_m, F_{\text{ext}}^{(m)} \right\rangle := \left\langle F_{\text{ext}}^{(m)}, :D^{:\otimes m} \right\rangle_{\text{ext}}^{*} 1.$$

Theorem 1.3 [3]. *The generalized functions* $\left\langle L_m, F_{\text{ext}}^{(m)} \right\rangle$, $F_{\text{ext}}^{(m)} \in \mathcal{S}_{\mathbb{C}}'^{(m)}$, $m \in \mathbb{Z}_+$, *are orthogonal to the generalized Laguerre polynomials in the sense that*

$$\left\langle\!\left\langle \left\langle L_m, F_{\text{ext}}^{(m)} \right\rangle, \left\langle L_n, f^{(n)} \right\rangle \right\rangle\!\right\rangle = \delta_{mn}\, n! \left\langle F_{\text{ext}}^{(n)}, f^{(n)} \right\rangle_{\text{ext}}, \quad n \in \mathbb{Z}_+, \quad f^{(n)} \in \mathcal{S}_{\mathbb{C}}^{\hat{\otimes} n}. \tag{1.12}$$

***Remark* 1.9.** It follows from (1.12) and Theorem 1.1 that for a "regular" $F_{\text{ext}}^{(m)} \in \mathcal{S}_{C}^{\hat{\otimes} m}$ $\left\langle L_m, F_{\text{ext}}^{(m)} \right\rangle$ is the generalized Laguerre polynomial. So, the designation accepted in Definition 1.8 is natural.

Theorem 1.4 [3]. $F \in \left(\mathcal{H}_{-p}\right)_{-q}$ *if and only if there exists a sequence of generalized kernels* $\left(F_{\text{ext}}^{(m)} \in \mathcal{S}_{\mathbb{C}}'^{(m)}\right)_{m=0}^{\infty}$ *such that* F *can be presented in the form*

$$F = \sum_{m=0}^{\infty} \left\langle L_m, F_{\text{ext}}^{(m)} \right\rangle \tag{1.13}$$

and

$$\| F \|_{-p,-q}^2 := \sum_{m=0}^{\infty} 2^{-qm} \left| U_m F_{\text{ext}}^{(m)} \right|_{-p}^2 < \infty.$$

Moreover, for F, $H \in \left(\mathcal{H}_{-p}\right)_{-q}$ *presented in form* (1.13) *the scalar product in* $\left(\mathcal{H}_{-p}\right)_{-q}$ *has the form*

$$(F, H)_{-p,-q} = \sum_{m=0}^{\infty} 2^{-qm} \left\langle U_m F_{\text{ext}}^{(m)}, U_m H_{\text{ext}}^{(m)} \right\rangle_{-p},$$

where $\langle \cdot, \cdot \rangle_{-p}$ *denotes the* (*real*) *scalar product in tensor powers of* $\mathcal{H}_{-p,\mathbb{C}}$. *So, the generalized functions* $\left\langle L_m, F_{\text{ext}}^{(m)} \right\rangle$, $F_{\text{ext}}^{(m)} \in \mathcal{S}_{\mathbb{C}}'^{(m)}$ *play the role of an orthogonal basis in* $\left(\mathcal{H}_{-p}\right)_{-q}$.

It is easy to see that $F \in (\mathcal{S}')'$ if and only if F can be presented in form (1.13) and there exist p_0, $q_0 \in \mathbb{N}$ such that for all $p, q \in \mathbb{N}$, $p \geq p_0$, $q \geq q_0$ $\| F \|_{-p,-q} < \infty$.

2. An extended stochastic integral on spaces of regular generalized functions. The natural extended stochastic integral on the square integrable functions space $L^2(\mathcal{S}, \mu) = (L^2)$ and on the corresponding Kondratiev-type space of *nonregular* generalized functions $(\mathcal{S}')'$ was introduced and studied in [3]. In this section we consider this integral on the spaces of *regular* generalized functions.

First, let us recall the classical definition of the extended stochastic integral. Let γ be the Gaussian measure on the usual Schwartz distributions space $\mathcal{S}'$, i.e., the probability measure with the Laplace transform

$$l_\gamma(\lambda) = \int_{\mathcal{S}'} \exp\{\langle x, \lambda\rangle\}\gamma(dx) = \exp\left\{\frac{1}{2}\langle\lambda, \lambda\rangle\right\},$$

where the dual pairing $\langle\cdot,\cdot\rangle$ is generated by the scalar product in $L^2(\mathbb{R}, dt)$.

By the Wiener – Itô chaos decomposition theorem (see, e.g., [15, 16]) we can write any function $f \in L^2(\mathcal{S}', \gamma)$ in the form

$$f = \sum_{n=0}^{\infty} \int f_n \, dW^{\otimes n}, \tag{2.1}$$

where $f_n \in \widehat{L^2}\left(\mathbb{R}_+^n, m\right)$ (m is the Lebesgue measure), i.e., $f_n \in L^2\left(\mathbb{R}_+^n\right)$ and f_n is a *symmetric* function $\Big($in the sense that $f_n\left(\cdot_{\pi(1)}, \ldots, \cdot_{\pi(n)}\right) = f_n(\cdot_1, \ldots, \cdot_n)$ for all permutations π of $\{1, \ldots, n\}$$\Big)$, and

$$\int f_n \, dW^{\otimes n} = \int_{\mathbb{R}_+^n} f_n(u) \, dW_n^{\otimes n} =$$

$$= n! \int_0^\infty \int_0^{u_n} \ldots \int_0^{u_3} \int_0^{u_2} f_n(u_1, \ldots, u_n) \, dW_{u_1} \, dW_{u_2} \ldots dW_{u_{n-1}} \, dW_{u_n}$$

for $n \geq 1$, while $n = 0$ term in (2.1) is just a constant f_0. Here W_u. is the standard Wiener process.

Now suppose that $f \in L^2(\mathcal{S}', \mu) \otimes L^2(\mathbb{R}_+)$ is $\mathcal{F} \times \mathcal{B}(\mathbb{R}_+)$-measurable stochastic process. Then for almost all $s \geq 0$ there exist $f_n(s; \cdot) \in \widehat{L^2}\left(\mathbb{R}_+^n\right)$, $n \in \mathbb{N}$, $f_0(s) \in \mathbb{C}$ such that

$$F_s(x) = \sum_{n=0}^{\infty} \int_{\mathbb{R}_+^n} f_n(s; u) \, dW_u(x)^{\otimes n}. \tag{2.2}$$

Fix $t \in [0, +\infty]$. Let $\hat{f}_{n,t}$ be the symmetrization of $f_n(s; \cdot) 1_{\{s \in [0, t)\}}$ with respect to $n + 1$ variables. Suppose

$$\sum_{n=0}^{\infty} (n+1)! \left| \hat{f}_{n,t} \right|_{L^2\left(\mathbb{R}_+^{n+1}\right)} < \infty.$$

Then the extended stochastic integral of $F.$ is defined by

$$\int_0^t F_s \, \hat{d}W_s := \sum_{n=0}^{\infty} \int_{\mathbb{R}_+^{n+1}} \hat{f}_{n,t}(s; u) \, dW_{(s;u)}^{\otimes(n+1)}. \tag{2.3}$$

Proposition 2.1 [17, 16]. *Extended stochastic integral* (2.3) *is an extension of the Itô integral in the following sense: if* F_s *is adapted with respect to the flow of* σ-*algebras generated by the Wiener process and* $\mathbf{E}\left[\int_0^t |F_s|^2 \, ds\right] < \infty$ $\Big($*here* $\mathbf{E}$ *denotes the expectation*$\Big)$ *then* F *is integrable in the extended sense and by Itô* $\Big($*we denote the corresponding Itô integral by* $\int_0^t F_s \, dW_s$$\Big)$ *and*

$$\int_0^t F_s \,\hat{d}W_s = \int_0^t F_s \,dW_s.$$

On the other hand, it is well known (see, e.g., [16]) that one can complete a definition of f_n from (2.2) for negative arguments, putting $f_n = 0$ if at least one its argument is negative, and identify the multiple stochastic integral with the corresponding generalized Hermite polynomial, i.e.,

$$\int_{\mathbb{R}_+^n} f_n(s;u)\,dW_u(x)^{\otimes n} = \langle H_n(x), f_n(s;\cdot)\rangle,$$

where $H_n(x) \in \mathcal{S}_{\mathbb{C}}'^{\hat{\otimes} n}$, $n \in \mathbb{Z}_+$, is the kernel of the Hermite polynomial of power n from the decomposition

$$:\!\exp(x;\lambda)\!: = \exp\left\{\langle x, \lambda\rangle - \frac{1}{2}\langle \lambda, \lambda\rangle\right\} = \sum_{n=0}^{\infty} \frac{1}{n!}\left\langle H_n(x), \lambda^{\otimes n}\right\rangle, \quad \lambda \in \mathcal{S}_{\mathbb{C}}.$$

Thus one can write the integrand F_s and the stochastic integral $\int_0^t F_s \,\hat{d}W_s$ in the form

$$F_s = \sum_{n=0}^{\infty} \langle H_n, f_n(s;\cdot)\rangle$$

and

$$\int_0^t F_s \,\hat{d}W_s = \sum_{n=0}^{\infty} \left\langle H_{n+1}, \hat{f}_{n,t}\right\rangle$$

correspondingly.

If instead of the space $L^2(\mathcal{S}', \gamma)$ with the Gaussian measure γ we use the space $(L^2) = L^2(\mathcal{S}', \mu)$ with the Gamma-measure μ (the main probability space now is $(\mathcal{S}', F, \mu)$), then the full analog of the construction of the extended stochastic integral recalled above can not be obtained. In the first place, as it well known (see, e.g., [2]), the Gamma-measure has no the chaotic representation property (CRP), i.e., there is no a full analog of the Wiener – Itô chaos decomposition theorem, and therefore we can not present *any* element $f \in (L^2)$ in form (2.1) with the corresponding stochastic process. In the second place, an attempt to "go around" the absence of the CRP leads to use of $\mathcal{H}_{\text{ext}}^{(n)}$ instead of $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$, see [5] (we recall that $\mathcal{H}_{\mathbb{C}} = L^2(\mathbb{R}, \sigma)_{\mathbb{C}}$). But because the spaces $\mathcal{H}_{\text{ext}}^{(n)}$, $n > 1$, are not tensor powers of some Hilbert spaces, it is impossible to construct the kernels $\hat{f}_{n,t}$ (see above) by analogy with the Gaussian case. So, in order to construct a *natural* extended stochastic integral connected with the Gamma-measure, we need a modification of the classical scheme described above. The idea of such modification is very simple: in order to construct $\hat{f}_{n,t} \in \mathcal{H}_{\text{ext}}^{(n+1)}$ starting from $f_n(s) \in \mathcal{H}_{\text{ext}}^{(n)}$ we "exclude a diagonal of $\hat{f}_{n,t}$", i.e., (nonstrictly speaking) we symmetrize the function

$$\tilde{f}_{n,t}(\tau_1, \ldots, \tau_n; s) := \begin{cases} f_n(\tau_1, \ldots, \tau_n; s)1_{[0,t)}(s), & \text{if} \quad s \neq \tau_1, \ldots, s \neq \tau_n, \\ 0, & \text{in other cases} \end{cases}$$

(see Lemma 2.1 bellow).

Now let us pass to construction of an extended stochastic integral. By analogy with the classical Gaussian analysis one can consider the compensated Gamma-process $G_s = \langle L_1, 1_{[0,s)} \rangle \in (L^2)$, $s \in \mathbb{R}_+$, on the probability space $(\mathcal{S}', F, \mu)$ $\big($from this point of view μ is the measure of the Gamma-white noise G'_s, *formally* $G'_s = \langle L_1, \delta_s \rangle$, where δ_s is the delta-function$\big)$. Let $F \in (L^2)^{-1}_{-q} \otimes \mathcal{H}^+_{\mathbb{C}}$, where $\mathcal{H}^+_{\mathbb{C}} := L^2(\mathbb{R}_+)_{\mathbb{C}} \subset \subset \mathcal{H}_{\mathbb{C}}$, $q \in \mathbb{N}$. Then (see (1.6))

$$F_\cdot = \sum_{m=0}^{\infty} \langle L_m, F_\cdot^{(m)} \rangle, \quad F_\cdot^{(m)} \in \mathcal{H}^{(m)}_{\text{ext}} \otimes \mathcal{H}^+_{\mathbb{C}}. \tag{2.4}$$

Lemma 2.1 [3]. *For given* $F_\cdot^{(m)} \in \mathcal{H}^{(m)}_{\text{ext}} \otimes \mathcal{H}^+_{\mathbb{C}}$ *and* $t \in [0, +\infty]$ *we construct the element* $\hat{F}^{(m)}_{[0,t)} \in \mathcal{H}^{(m+1)}_{\text{ext}}$ *by the following way. Let us consider a sequence* $\left\{ f^{(m)}_{i,\cdot} \in \mathcal{S}^{\hat{\otimes} m}_{\mathbb{C}} \otimes \mathcal{S}_{\mathbb{C}} \right\}_{i=1}^{\infty}$ *such that* $F_\cdot^{(m)} = \lim_{i \to \infty} f^{(m)}_{i,\cdot}$ *in* $\mathcal{H}^{(m)}_{\text{ext}} \otimes \mathcal{H}^+_{\mathbb{C}}$ *and put*

$$\tilde{f}^{(m)}_{[0,t),i}(\tau_1, \ldots, \tau_m, \tau) := \begin{cases} f^{(m)}_{i,\tau}(\tau_1, \ldots, \tau_m) 1_{[0,t)}(\tau), & \text{if} \quad \tau \neq \tau_1, \ldots, \tau \neq \tau_m, \\ 0, & \text{in other cases} \end{cases}$$

$\hat{f}^{(m)}_{[0,t),i} := P \tilde{f}^{(m)}_{[0,t),i}$, *where* $1_{[0,t)}(\tau)$ *denotes the indicator of* $\{\tau \in [0,t)\}$, P *is the symmetrization operator. Then* $\hat{F}^{(m)}_{[0,t)} := \lim_{i \to \infty} \hat{f}^{(m)}_{[0,t),i}$ *in* $\mathcal{H}^{(m+1)}_{\text{ext}}$. *This limit does not depend on the sequence* $\left\{ f^{(m)}_{i,\cdot} \right\}_{i=1}^{\infty}$ *and the estimate*

$$\left| \hat{F}^{(m)}_{[0,t)} \right|_{\text{ext}} \leq \left| F_\cdot^{(m)} \right|_{\mathcal{H}^{(m)}_{\text{ext}} \otimes \mathcal{H}^+_{\mathbb{C}}}$$

holds.

Definition 2.1. *Let* $F \in (L^2)^{-1}_{-q} \otimes \mathcal{H}^+_{\mathbb{C}}$, $q \in \mathbb{N}$. *For each* $t \in [0, +\infty]$ *we define the extended stochastic integral* $\int_0^t F_s \, \hat{d} G_s \in (L^2)^{-1}_{-q}$, *putting*

$$\int_0^t F_s \, \hat{d} G_s := \sum_{m=0}^{\infty} \langle L_{m+1}, \hat{F}^{(m)}_{[0,t)} \rangle, \tag{2.5}$$

where the kernels $\hat{F}^{(m)}_{[0,t)}$ *are defined as in Lemma* 2.1 *starting from the kernels* $F^{(m)}$ *from decomposition* (2.4) *for* F.

Because

$$\left\| \int_0^t F_s \, \hat{d} G_s \right\|^2_{-q} = \sum_{m=0}^{\infty} 2^{-q(m+1)} \left| \hat{F}^{(m)}_{[0,t)} \right|^2_{\text{ext}} \leq$$

$$\leq 2^{-q} \sum_{m=0}^{\infty} 2^{-qm} \left| F^{(m)} \right|^2_{\mathcal{H}^{(m)}_{\text{ext}} \otimes \mathcal{H}^+_{\mathbb{C}}} = 2^{-q} \| F \|^2_{(L^2)^{-1}_{-q} \otimes \mathcal{H}^+_{\mathbb{C}}} < \infty,$$

$\int_0^t F_s \, \hat{d} G_s$ is well-defined.

Note that our definition of $\int_0^t F_s \, \hat{d} G_s$ formally coincides with the definition of the extended stochastic integral on $(L^2) \otimes \mathcal{H}^+_{\mathbb{C}}$ given in [3].

Let now $F \in (\mathcal{S}')' \otimes \mathcal{H}^+_{\mathbb{C}}$. Then (see (1.13)) F can be presented in the form

$$F_{\cdot} = \sum_{m=0}^{\infty} \left\langle L_m, F_{\text{ext},\cdot}^{(m)} \right\rangle, \quad F_{\text{ext},\cdot}^{(m)} \in \mathcal{S}_{\mathbb{C}}'^{(m)} \otimes \mathcal{H}_{\mathbb{C}}^{+},$$

$$\| F \|^2_{(\mathcal{H}_{-p})_{-q} \otimes \mathcal{H}_{\mathbb{C}}^{+}} = \sum_{m=0}^{\infty} 2^{-qm} \left| U_m F_{\text{ext},\cdot}^{(m)} \right|^2_{\mathcal{H}_{-p,\mathbb{C}}^{\hat{\otimes} m} \otimes \mathcal{H}_{\mathbb{C}}^{+}} < \infty$$

for some $p, q \in \mathbb{N}$ (see (1.11) for the definition of U_m).

Definition 2.2 [3]. *For a generalized function* $F \in (\mathcal{S}')' \otimes \mathcal{H}_{\mathbb{C}}^{+}$ *and* $t \in [0, +\infty]$ *we define an analog of an extended stochastic integral, putting*

$$\int_0^t F_s \, \hat{\hat{d}} G_s := \sum_{m=0}^{\infty} \left\langle L_{m+1}, U_{m+1}^{-1} F_t^{\wedge(m)} \right\rangle,$$

where $F_t^{\wedge(m)} \in \mathcal{S}_{\mathbb{C}}'^{\hat{\otimes}(m+1)}$ *is the "symmetrization of* $\left(U_m F_{\text{ext},\cdot}^{(m)}\right) 1_{[0,t)}(\cdot) \in \mathcal{S}_{\mathbb{C}}'^{\hat{\otimes}(m)} \otimes \mathcal{H}_{\mathbb{C}}^{+}$ *with respect to* $m+1$ *variables".*

It was proved in [3] that for $F_{\text{ext},\cdot}^{(m)} \in \mathcal{H}_{\text{ext}}^{(m)} \otimes \mathcal{H}_{\mathbb{C}}^{+}$ and $t \in [0, +\infty]$ $U_{m+1}^{-1} F_t^{\wedge(m)}$ coincides with $\hat{F}_{[0,t)}^{(m)}$ constructed in Lemma 2.1. Therefore we have the following statement.

Proposition 2.2. *The restriction of the analog of an extended stochastic integral* $\int_0^t \circ \, \hat{\hat{d}} G_s$ *on* $\left(L^2\right)_{-q}^{-1} \otimes \mathcal{H}_{\mathbb{C}}^{+}$ *coincides with* $\int_0^t \circ \, \hat{d} G_s$, *i.e., for* $F \in \left(L^2\right)_{-q}^{-1} \otimes \mathcal{H}_{\mathbb{C}}^{+}$ $\int_0^t F_s \, \hat{\hat{d}} G_s = \int_0^t F_s \, \hat{d} G_s$ $\Big($*thus below we will denote all stochastic integrals by* $\int_0^t \circ \, \hat{d} G_s$ $\Big)$.

The following statement (the "Gamma-analog" of Proposition 2.1) explains that our generalization of the stochastic integral is natural.

Theorem 2.1. *Let* $F \in \left(L^2\right) \otimes \mathcal{H}_{\mathbb{C}}^{+}$ *be an* $\mathcal{F} \times \mathcal{B}(\mathbb{R}_+)$ *-measurable stochastic process adapted with respect to the flow of* σ*-algebras generated by the compensated Gamma-process* G_s, *and* $\mathbf{E} \int_0^{\infty} |F_s|^2 \sigma(ds) < \infty$ *(here as above* $\mathbf{E}$ *denotes the expectation). Then for each* $t \in [0, +\infty]$ F *is integrable on the interval* $[0, t)$ *in the extended sense and by Itô with respect to* G_s *(in the sense of the so-called* L^2*-theory) and*

$$\int_0^t F_s \, \hat{d} G_s = \int_0^t F_s \, dG_s,$$

where by $\int_0^t F_s \, dG_s$ *the Itô integral denoted.*

Proof. It follows directly from Definition 2.1 that the restriction of $\int_0^t \circ \, \hat{d} G_s$ on $\left(L^2\right) \otimes \mathcal{H}_{\mathbb{C}}^{+}$ coincides with the extended stochastic integral on $\left(L^2\right) \otimes \mathcal{H}_{\mathbb{C}}^{+}$ constructed in [3]. But for the last integral the statement of the theorem was proved in [3, 9].

Finally, let us explain that the extended stochastic integral $\int_0^t \circ \, \hat{d} G_s$ can be described as the operator dual to the stochastic differentiation operator (see, e.g., [16] for a detailed description of such approach in the Gaussian analysis).

For $n \in \mathbb{N}$, $f^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$ and $g \in \mathcal{H}_{\mathbb{C}}$ we define $\left\langle f^{(n)}, g \right\rangle_{\text{ext}} \in \mathcal{H}_{\text{ext}}^{(n-1)}$ by the formula

$$\left\langle \left\langle f^{(n)}, g \right\rangle_{\text{ext}}, h^{(n-1)} \right\rangle_{\text{ext}} \equiv \left\langle f^{(n)}, U_n^{-1}\left(\left(U_{n-1}h^{(n-1)}\right) \hat{\otimes}\, g\right)\right\rangle_{\text{ext}} \quad \forall\, h^{(n-1)} \in \mathcal{H}_{\text{ext}}^{(n-1)}.$$

The well-definiteness of $\left\langle f^{(n)}, g \right\rangle_{\text{ext}}$ was proved in [9], see also [3].

Proposition 2.3 [9]. *Let* $f^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$, $n \in \mathbb{N}$. *Then there exists a unique* $f^{(n)}(\cdot) \in \mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}$ *such that*

$$\int_{\mathbb{R}} f^{(n)}(s)\, g(s)\, \sigma(ds) = \left\langle f^{(n)}, g \right\rangle_{\text{ext}} \quad \forall\, g \in \mathcal{H}_{\mathbb{C}} \tag{2.6}$$

and

$$\left| f^{(n)}(\cdot) \right|_{\mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}} \le \left| f^{(n)} \right|_{\text{ext}}.$$

Here the integral in the left-hand side of (2.6) *is understanding in the sense that for each* $h^{(n-1)} \in \mathcal{H}_{\text{ext}}^{(n-1)}$

$$\left\langle \int_{\mathbb{R}} f^{(n)}(s)\, g(s)\, \sigma(ds), h^{(n-1)} \right\rangle_{\text{ext}} \equiv \int_{\mathbb{R}} \left\langle f^{(n)}(s), h^{(n-1)} \right\rangle_{\text{ext}} g(s)\, \sigma(ds).$$

Definition 2.3. *Let* $f \in \left(L^2\right)_q^1$, $q \in \mathbb{N}$. *We define the stochastic derivative* $\partial_\cdot f \in \left(L^2\right)_q^1 \otimes \mathcal{H}_{\mathbb{C}}$ *putting*

$$\partial_\cdot f := \sum_{n=1}^{\infty} n \left\langle L_{n-1}, f^{(n)}(\cdot) \right\rangle,$$

where the kernels $f^{(n)}(\cdot) \in \mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}$ *are defined as in Proposition* 2.3 *starting from the kernels* $f^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$ *from decomposition* (1.5) *for* f.

It is easy to see that

$$\|\partial_\cdot f\|^2_{(L^2)_q^1 \otimes \mathcal{H}_{\mathbb{C}}} = \sum_{n=0}^{\infty} (n!)^2 2^{qn} (n+1)^2 \left| f^{(n+1)}(\cdot) \right|^2_{\mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}} \le$$

$$\le 2^{-q} \sum_{n=0}^{\infty} ((n+1)!)^2 2^{q(n+1)} \left| f^{(n+1)}(\cdot) \right|^2_{\text{ext}} \le 2^{-q} \|f\|_q^2 < \infty,$$

therefore $\partial_\cdot$ is well-defined. Note that *formally* $\partial_\cdot = \langle \delta_\cdot, :D: \rangle_{\text{ext}}$, where $\delta_\cdot$ denotes the δ-function.

Theorem 2.2. *Let* $f \in \left(L^2\right)_q^1$, $F \in \left(L^2\right)_{-q}^{-1} \otimes \mathcal{H}_{\mathbb{C}}^{+}$, $q \in \mathbb{N}$. *Then*

$$\left\langle\!\left\langle \int_0^t F_s\, \hat{d}G_s, f \right\rangle\!\right\rangle = \int_0^t \langle\langle F_s, \partial_s f \rangle\rangle \sigma(ds) \quad \forall\, t \in [0, +\infty].$$

Proof. First we note that

$$\left\langle\!\left\langle \int_0^t F_s\, \hat{d}G_s, f \right\rangle\!\right\rangle = \sum_{n=0}^{\infty} (n+1)! \left\langle \hat{F}_{[0,t)}^{(n)}, f^{(n+1)} \right\rangle_{\text{ext}},$$

where $\hat{F}^{(n)}_{[0,t)} \in \mathcal{H}^{(n+1)}_{\text{ext}}$ are from decomposition (2.5) for $\int_0^t F_s \,\hat{d} G_s$, $f^{(n+1)} \in \mathcal{H}^{(n+1)}_{\text{ext}}$ are from decomposition (1.5) for f. Further, $\partial.f = \sum_{n=0}^{\infty}(n+1)\langle L_n, f^{(n+1)}(\cdot)\rangle$, so we have

$$\langle\langle F., \partial.f\rangle\rangle = \sum_{n=0}^{\infty}(n+1)!\langle F.^{(n)}, f^{(n+1)}(\cdot)\rangle_{\text{ext}}$$

(here $F.^{(n)} \in \mathcal{H}^{(n)}_{\text{ext}} \otimes \mathcal{H}^{+}_{\mathbb{C}}$ are the kernels from decomposition (2.4) for $F.$, $f^{(n+1)}(\cdot) \in \mathcal{H}^{(n)}_{\text{ext}} \otimes \mathcal{H}^{+}_{\mathbb{C}}$ are defined in Proposition 2.3). Therefore in order to finish the proof it is sufficient to show that

$$\langle \hat{F}^{(n)}_{[0,t)}, f^{(n+1)}\rangle_{\text{ext}} = \int_0^t \langle F^{(n)}_{\tau}, f^{(n+1)}(\tau)\rangle_{\text{ext}}\, \sigma(d\tau). \tag{2.7}$$

Let $F.^{(n)} \in \mathcal{H}^{(n)}_{\text{ext}} \otimes \mathcal{H}^{+}_{\mathbb{C}}$ be the kernels from decomposition (2.4) for F. We consider a sequence $\mathcal{S}^{\hat{\otimes} n}_{\mathbb{C}} \otimes \mathcal{S}_{\mathbb{C}} \ni F^{(n)}_{\cdot,i} \to F.^{(n)}$ (as $i \to \infty$) in $\mathcal{H}^{(n)}_{\text{ext}} \otimes \mathcal{H}^{+}_{\mathbb{C}}$ and construct as in Lemma 2.1 $\mathcal{S}^{\hat{\otimes}(n+1)}_{\mathbb{C}} \ni \hat{F}^{(n)}_{[0,t),i} \to \hat{F}^{(n)}_{[0,t)}$ (as $i \to \infty$) in $\mathcal{H}^{(n+1)}_{\text{ext}}$. Let also $\mathcal{S}^{\hat{\otimes}(n+1)}_{\mathbb{C}} \ni f^{(n+1)}_i \to f^{(n+1)}$ (as $i \to \infty$) in $\mathcal{H}^{(n+1)}_{\text{ext}}$. Now we have (see (1.4))

$$\langle \hat{F}^{(n)}_{[0,t),i}, f^{(n+1)}_i\rangle_{\text{ext}} = \sum_{\substack{k,\, l_j,\, s_j \in \mathbb{N}:\ j=1,\ldots,k,\ l_1>l_2>\ldots>l_k,\\ l_1 s_1+\ldots+l_k s_k = n+1}} \frac{(n+1)!}{l_1^{s_1}\ldots l_k^{s_k}\, s_1!\ldots s_k!}\, \times$$

$$\times \int_{\mathbb{R}^{s_1+\ldots+s_k}} \hat{F}^{(n)}_{[0,t),i}(\underbrace{\tau_1,\ldots,\tau_1}_{l_1},\ldots,\underbrace{\tau_{s_1+\ldots+s_k},\ldots,\tau_{s_1+\ldots+s_k}}_{l_k})\, \times$$

$$\times f^{(n+1)}_i(\underbrace{\tau_1,\ldots,\tau_1}_{l_1},\ldots,\underbrace{\tau_{s_1+\ldots+s_k},\ldots,\tau_{s_1+\ldots+s_k}}_{l_k})\, \sigma(d\tau_1)\ldots\sigma(d\tau_{s_1+\ldots+s_k}) =$$

$$= \sum_{\substack{k,\, l_j,\, s_j \in \mathbb{N}:\ j=1,\ldots,k,\ l_1>l_2>\ldots>l_k,\\ l_1 s_1+\ldots+l_k s_k = n+1}} \frac{n!}{l_1^{s_1}\ldots l_k^{s_k}\, s_1!\ldots s_k!}\, \times$$

$$\times \Bigg[\int_{\mathbb{R}^{s_1+\ldots+s_k}} \tilde{F}^{(n)}_{[0,t),i}(\underbrace{\tau_1,\ldots,\tau_1}_{l_1},\ldots,\underbrace{\tau_{s_1+\ldots+s_k},\ldots,\tau_{s_1+\ldots+s_k}}_{l_k})\, \times$$

$$\times f^{(n+1)}_i(\underbrace{\tau_1,\ldots,\tau_1}_{l_1},\ldots,\underbrace{\tau_{s_1+\ldots+s_k},\ldots,\tau_{s_1+\ldots+s_k}}_{l_k})\, \sigma(d\tau_1)\ldots\sigma(d\tau_{s_1+\ldots+s_k}) +$$

$$+ \int_{\mathbb{R}^{s_1+\ldots+s_k}} \tilde{F}^{(n)}_{[0,t),i}(\tau_{s_1+\ldots+s_k}, \underbrace{\tau_1,\ldots,\tau_1}_{l_1},\ldots,\underbrace{\tau_{s_1+\ldots+s_k},\ldots,\tau_{s_1+\ldots+s_k}}_{l_k-1})\, \times$$

$$\times f^{(n+1)}_i(\underbrace{\tau_1,\ldots,\tau_1}_{l_1},\ldots,\underbrace{\tau_{s_1+\ldots+s_k},\ldots,\tau_{s_1+\ldots+s_k}}_{l_k})\, \sigma(d\tau_1)\ldots\sigma(d\tau_{s_1+\ldots+s_k}) + \ldots$$

$$\ldots + \int_{\mathbb{R}^{s_1+\ldots+s_k}} \tilde{F}^{(n)}_{[0,t),i}(\underbrace{\tau_1,\ldots,\tau_1}_{l_1-1},\ldots,\underbrace{\tau_{s_1+\ldots+s_k},\ldots,\tau_{s_1+\ldots+s_k}}_{l_k},\tau_1)\times$$

$$\times f_i^{(n+1)}(\underbrace{\tau_1,\ldots,\tau_1}_{l_1},\ldots,\underbrace{\tau_{s_1+\ldots+s_k},\ldots,\tau_{s_1+\ldots+s_k}}_{l_k})\,\sigma(d\tau_1)\ldots\sigma(d\tau_{s_1+\ldots+s_k})\Bigg] =$$

$$= \sum_{\substack{k,l_j,s_j\in\mathbb{N}:\,j=1,\ldots,k-1,\,l_1>\ldots>l_{k-1}>1,\\ l_1s_1+\ldots+l_{k-1}s_{k-1}+(s_k-1)=n}} \frac{n!}{l_1^{s_1}\ldots l_{k-1}^{s_{k-1}}\,s_1!\ldots s_{k-1}!(s_k-1)!}\times$$

$$\times\int_{\mathbb{R}^{s_1+\ldots+s_k}} \tilde{F}^{(n)}_{[0,t),i}(\underbrace{\tau_1,\ldots,\tau_1}_{l_1},\ldots,\underbrace{\tau_{s_1+\ldots+s_{k-1}},\ldots,\tau_{s_1+\ldots+s_{k-1}}}_{l_{k-1}},\tau_{s_1+\ldots+s_{k-1}+1},\ldots$$

$$\ldots,\tau_{s_1+\ldots+s_k-1},\tau)\, f_i^{(n+1)}(\underbrace{\tau_1,\ldots,\tau_1}_{l_1},\ldots,\underbrace{\tau_{s_1+\ldots+s_{k-1}},\ldots,\tau_{s_1+\ldots+s_{k-1}}}_{l_{k-1}},$$

$$\tau_{s_1+\ldots+s_{k-1}+1},\ldots\,\tau_{s_1+\ldots+s_k-1},\tau)\,\sigma(d\tau_1)\ldots\sigma(d\tau_{s_1+\ldots+s_k-1})\sigma(d\tau) =$$

$$= \int_0^t \left\langle F^{(n)}_{\tau,i}, f_i^{(n+1)}(\tau)\right\rangle_{\text{ext}} \sigma(d\tau)$$

(a nonatomicity of σ used). Approaching the limit as $i\to\infty$ we obtain (2.7).

The theorem is proved.

3. Elements of the Wick calculus and stochastic equations. In this section we introduce a Wick product and Wick versions of holomorphic functions on the Kondratiev-type space of *regular* generalized functions $\left(L^2\right)^{-1}$. Then we study the interconnection of these objects with an extended stochastic integral and consider some stochastic equations with Wick-type nonlinearity.

First we recall elements of the Wick calculus on the space $(\mathcal{S}')'$ of *nonregular* generalized functions.

Definition 3.1. *For* $F\in(\mathcal{S}')'$ *we define an integral* S*-transform* $(SF)(\lambda)$, λ *belongs to some neighbourhood of zero in* $\mathcal{S}_{\mathbb{C}}$, *putting* (*see* (1.3))

$$(SF)(\lambda) := \langle\langle F, :\exp(\cdot,\lambda):\rangle\rangle.$$

This definition is correct because for each $F\in(\mathcal{S}')'$ there exist $p,\ q\in\mathbb{N}$ such that $F\in\left(\mathcal{H}_{-p}\right)_{-q}$; and for $\lambda\in\mathcal{S}_{\mathbb{C}}$ such that $2^q|\lambda|_p^2<1$ we have $:\exp(\cdot,\lambda):\in$ $\in\left(\mathcal{H}_p\right)_q$.

***Remark* 3.1.** We note that if $F\in(\mathcal{S}')'$ is presented in form (1.6) then $(SF)(\lambda) =$ $=\sum_{m=0}^{\infty}\left\langle F^{(m)},\lambda^{\otimes m}\right\rangle_{\text{ext}}$. In particular, $(SF)(0) = F^{(0)}$, $S1=1$.

Theorem 3.1 [18, 12, 13]. *An* S*-transform is a topological isomorphism between the space* $(\mathcal{S}')'$ *and the algebra* Hol_0 *of germs of holomorphic at zero functions on* $\mathcal{S}_{\mathbb{C}}$.

Definition 3.2. *For* $F,\ H\in(\mathcal{S}')'$ *we define the Wick product* $F\ \Diamond\ H\in(\mathcal{S}')'$, *putting*

$$F\Diamond H\ :=\ S^{-1}(SFSH).$$

***Remark* 3.2.** If generalized functions $F,\ H\in(\mathcal{S}')'$ presented in form (1.13) then

$$F\Diamond H\ =\ \sum_{k=0}^{\infty}\left\langle L_k,\ \sum_{n=0}^{k}F_{\text{ext}}^{(n)}\Diamond H_{\text{ext}}^{(k-n)}\right\rangle,$$

where for $F_{\text{ext}}^{(n)}\in\mathcal{S}_{\mathbb{C}}^{\prime(n)}$, $H_{\text{ext}}^{(m)}\in\mathcal{S}_{\mathbb{C}}^{\prime(m)}$ the element $F_{\text{ext}}^{(n)}\Diamond H_{\text{ext}}^{(m)}\in\mathcal{S}_{\mathbb{C}}^{\prime(n+m)}$ is defined by (see (1.11)) $F_{\text{ext}}^{(n)}\Diamond H_{\text{ext}}^{(m)}:=U_{n+m}^{-1}\Big(U_nF_{\text{ext}}^{(n)}\hat{\otimes}\,U_mH_{\text{ext}}^{(m)}\Big)$ (see (3.13) below).

Definition 3.3. *For* $F\in(\mathcal{S}')'$ *and a function* $h:\ \mathbb{C}\to\mathbb{C}$ *holomorphic at* $(SF)(0)$ *we define the Wick version* $h^{\Diamond}(F)\in(\mathcal{S}')'$ *putting*

$$h^{\Diamond}(F)\ :=\ S^{-1}h(SF).$$

The correctness of Definitions 3.2, 3.3 from Theorem 3.1 follows.

***Remark* 3.3.** It is easy to see that if h from Definition 3.3 presented in the form $h(u)=\sum_{n=0}^{\infty}h_n(u-(SF)(0))^n$ then $h^{\Diamond}(F)=\sum_{n=0}^{\infty}h_n(F-(SF)(0))^{\Diamond n}$, where $F^{\Diamond n}:=\underbrace{F\Diamond\ldots\Diamond F}_{n\text{ times}}$.

Because the space $\left(L^2\right)^{-1}$ of *regular* generalized functions in a subspace of $(\mathcal{S}')'$, the Wick product $F\Diamond H$ and the Wick versions of a holomorphic function $h^{\Diamond}(F)$ are well-defined for $F,\ H\in\left(L^2\right)^{-1}$; but as elements of $(\mathcal{S}')'$. In order to prove that actually now $F\Diamond H,\ h^{\Diamond}(F)\in\left(L^2\right)^{-1}$ we need the following statement (in a sense this is a generalization of Lemma 2.1).

Lemma 3.1. *Let* $F^{(n)}\in\mathcal{H}_{\text{ext}}^{(n)}$, $H^{(m)}\in\mathcal{H}_{\text{ext}}^{(m)}$. *Then one can extend* $F^{(n)}\Diamond H^{(m)}=U_{n+m}^{-1}\Big(U_nF^{(n)}\hat{\otimes}\,U_mH^{(m)}\Big)\in\mathcal{S}_{\mathbb{C}}^{\prime(n+m)}$ *(see (1.11)) to a linear continuous functional on* $\mathcal{H}_{\text{ext}}^{(n+m)}$ $\Big($*more exactly,* $\exists\,\widehat{F^{(n)}H^{(m)}}\in\mathcal{H}_{\text{ext}}^{(n+m)}$ *such that* $\forall\,g^{(n+m)}\in\mathcal{S}_{\mathbb{C}}^{\hat{\otimes}(n+m)}$ $\left\langle\widehat{F^{(n)}H^{(m)}},g^{(n+m)}\right\rangle_{\text{ext}}=\left\langle F^{(n)}\Diamond H^{(m)},g^{(n+m)}\right\rangle_{\text{ext}}\Big)$. *Identifying* $F^{(n)}\Diamond H^{(m)}$ *with this functional one can reckon that* $F^{(n)}\Diamond H^{(m)}=\widehat{F^{(n)}H^{(m)}}\in\mathcal{H}_{\text{ext}}^{(n+m)}$, *in this case*

$$\left|F^{(n)}\Diamond H^{(m)}\right|_{\text{ext}}\ \le\ \left|F^{(n)}\right|_{\text{ext}}\left|H^{(m)}\right|_{\text{ext}}. \tag{3.1}$$

One can construct the element $\widehat{F^{(n)}H^{(m)}}$ *as follows. Let* $\mathcal{S}_{\mathbb{C}}^{\hat{\otimes}n}\ni f_v^{(n)}\to F^{(n)}$ *(as* $v\to\infty$*) in* $\mathcal{H}_{\text{ext}}^{(n)}$, $\mathcal{S}_{\mathbb{C}}^{\hat{\otimes}m}\ni h_v^{(m)}\to H^{(m)}$ *(as* $v\to\infty$*) in* $\mathcal{H}_{\text{ext}}^{(m)}$. *We put*

$$\left(\widetilde{f^{(n)}h^{(m)}}\right)_v(t_1,\ldots,t_n;t_{n+1},\ldots,t_{n+m})\ \equiv\ \widetilde{f_v^{(n)}(t_1,\ldots,t_n)h_v^{(m)}}(t_{n+1},\ldots,t_{n+m})\ :=$$

$$:=\begin{cases}f_v^{(n)}(t_1,\ldots,t_n)h_v^{(m)}(t_{n+1},\ldots,t_{n+m}), & \textit{if }\forall\,i\in\{1,\ldots,n\},\forall\,j\in\{n+1,\ldots,n+m\},\,t_i\ne t_j,\\ 0, & \textit{in other cases}\end{cases} \tag{3.2}$$

$\left(\widehat{f^{(n)}h^{(m)}}\right)_v := P\left(\widetilde{f^{(n)}h^{(m)}}\right)_v$, *where P is the symmetrization operator. Then* $\widehat{F^{(n)}H^{(m)}} = \lim_{v\to\infty}\left(\widehat{f^{(n)}h^{(m)}}\right)_v$ *in* $\mathcal{H}_{\text{ext}}^{(n+m)}$ $\Big($*this limit does not depend on a choice of sequences* $\left(f_v^{(n)}\right)_{v\geq 1}$, $\left(h_v^{(m)}\right)_{v\geq 1}\Big)$.

***Remark* 3.4.** Note that nonstrictly speaking $\widehat{F^{(n)}H^{(m)}}$ is the symmetrization of the functions

$$\widetilde{F^{(n)}H^{(m)}}\left(t_1,\ldots,t_n;t_{n+1},\ldots,t_{n+m}\right) :=$$

$$:= \begin{cases} F^{(n)}(t_1,\ldots,t_n)\,H^{(m)}(t_{n+1},\ldots,t_{n+m}), & \text{if } \forall\, i\in\{1,\ldots,n\},\ \forall\, j\in\{n+1,\ldots,n+m\},\ t_i\neq t_j,\\ 0, & \text{in other cases}\end{cases}$$

with respect to $n+m$ variables.

Proof of the lemma. First we prove that $\widehat{F^{(n)}H^{(m)}}$ is well-defined in $\mathcal{H}_{\text{ext}}^{(n+m)}$, independent on a choice of approximating sequences $\left(f_v^{(n)}\right)_{v\geq 1}$, $\left(h_v^{(m)}\right)_{v\geq 1}$, and

$$\left|\widehat{F^{(n)}H^{(m)}}\right|_{\text{ext}} \leq \left|F^{(n)}\right|_{\text{ext}}\left|H^{(m)}\right|_{\text{ext}}. \tag{3.3}$$

Let us consider sequences $\left(f_v^{(n)}\right)_{v\geq 1}$, $\left(h_v^{(m)}\right)_{v\geq 1}$ introduced in the lemma. We may assume, without loss of generality, that $m\geq n$. It follows from the symmetry of $f_v^{(n)}$ and $h_v^{(m)}$ that

$$\left(\widehat{f^{(n)}h^{(m)}}\right)_v\left(t_1,\ldots,t_n;t_{n+1},\ldots,t_{n+m}\right) =$$

$$= \frac{n!m!}{(n+m)!}\sum_{\substack{1\leq p_1,\ldots,p_n\leq n,\ n+1\leq q_1,\ldots,q_m\leq n+m,\ 0\leq r\leq n\\ p_1<\ldots<p_r,\ p_{r+1}<\ldots<p_n,\ q_1<\ldots<q_{n-r},\ q_{n-r+1}<\ldots<q_m}}\left(\widetilde{f^{(n)}h^{(m)}}\right)_v\Big(t_{p_1},\ldots,t_{p_r},$$

$$t_{q_1},\ldots,t_{q_{n-r}};t_{p_{r+1}},\ldots,t_{p_n},t_{q_{n-r+1}},\ldots,t_{q_m}\Big) \tag{3.4}$$

$\Big($here for $r=n$ the argument in the right-hand side of (3.4) is $\left(t_{p_1},\ldots,t_{p_r};\ t_{q_1},\ldots,t_{q_m}\right)$; for $r=0$ this argument is $\left(t_{q_1},\ldots,t_{q_n};\ t_{p_1},\ldots,t_{p_n},\ t_{q_{n+1}},\ldots,t_{q_m}\right)\Big)$. To put it in another way, arguments of $\left(\widetilde{f^{(n)}h^{(m)}}\right)_v$ in this sum are t_j with all $j\in\{1,\ldots,n+m\}$, but subindexes of first n arguments and last m arguments ("before" and "after" ';') must be (independently) arranged in an ascending order.

Let us estimate $\left|\left(\widehat{f^{(n)}h^{(m)}}\right)_v\right|_{\text{ext}}$. In accordance with the definition of $|\cdot|_{\text{ext}}$ we have

$$\left|\left(\widehat{f^{(n)}h^{(m)}}\right)_v\right|_{\text{ext}}^2 = \sum_{\substack{k,l_j,s_j\in\mathbb{N}:\, j=1,\ldots,k,\, l_1>l_2>\ldots>l_k,\\ l_1s_1+\ldots+l_ks_k=n+m}} \frac{(n+m)!}{l_1^{s_1}\ldots l_k^{s_k}\, s_1!\ldots s_k!} \times$$

$$\times \int_{\mathbb{R}^{s_1+\ldots+s_k}} \left|\left(\widehat{f^{(n)}h^{(m)}}\right)_v (\underbrace{t_1,\ldots,t_1}_{l_1},\ldots,\underbrace{t_{s_1+\ldots+s_k},\ldots,t_{s_1+\ldots+s_k}}_{l_k})\right|^2 \times$$

$$\times\, \sigma(dt_1)\ldots\sigma(dt_{s_1+\ldots+s_k}). \tag{3.5}$$

We say that collections of equal arguments (like $\underbrace{t_1,\ldots,t_1}_{l_1}$) are called *processions* (we need this term below).

Now we can substitute expression (3.4) for $\left(\widehat{f^{(n)}h^{(m)}}\right)_v$ in (3.5) and use the well-known estimate $\left|\sum_{l=1}^{p} a_l\right|^2 \le p\sum_{l=1}^{p}|a_l|^2$. Because, as it is easy to see, the right-hand side of (3.4) contains $\dfrac{(n+m)!}{n!m!}$ terms, we have the estimate

$$\left|\left(\widehat{f^{(n)}h^{(m)}}\right)_v\right|_{\text{ext}}^2 \le \sum_{\substack{k,l_j,s_j\in\mathbb{N}:\, j=1,\ldots,k,\, l_1>l_2>\ldots>l_k,\\ l_1s_1+\ldots+l_ks_k=n+m}} \frac{(n+m)!}{l_1^{s_1}\ldots l_k^{s_k}\, s_1!\ldots s_k!} \times$$

$$\times \left[\int_{\mathbb{R}^{s_1+\ldots+s_k}} \left|\left(\widetilde{f^{(n)}h^{(m)}}\right)_v (\underbrace{t_1,\ldots,t_1}_{l_1},\ldots,\underbrace{t_{s_1+\ldots+s_k},\ldots,t_{s_1+\ldots+s_k}}_{l_k})\right|^2 \times\right.$$

$$\left.\times\, \sigma(dt_1)\ldots\sigma(dt_{s_1+\ldots+s_k}) + \ldots\right]. \tag{3.6}$$

The terms in the "interior" sum with processions "separated by ';' (see (3.4)) are equal to zero by the definition of $\left(\widetilde{f^{(n)}h^{(m)}}\right)_v$. The rest terms (if exist for given k, j, l_j, s_j) fall into groups of equal summands. These equal summands are obtained by rearrangements of processions of equal length "before" ';' and "after" ';'. (Note that because the subindexes of arguments in sums (3.4) and (3.5) are ordered, the processions "before" ';' (so as "after" ';') in (3.6) do not fail and do not rearrange, and elements inside of processions do not rearrange.) Furthermore, if "before" ';' there are s' processions of length l and "after" ';' there are s'' processions of length l then by means of rearrangements of these processions one obtains $\dfrac{(s'+s'')!}{s'!s''!}$ equal summands (here $s',\ s''\in\mathbb{Z}_+,\ s'+s''\in\mathbb{N}$). Thus the nonzero terms in the full sum in the right-hand side of (3.6) are "connected" with the expression

$$l_1s_1 + \ldots + l_ks_k = n + m \tag{3.7}$$

that can be presented in the form

$$l_1's_1' + \ldots + l_{k'}'s_{k'}' = n, \qquad l_1''s_1'' + \ldots + l_{k''}''s_{k''}'' = m,$$
$$k',\ k'',\ l_1',\ldots,l_{k'}',\ s_1',\ldots,s_{k'}',\ l_1'',\ldots,l_{k''}'',\ s_1'',\ldots,s_{k''}''\in\mathbb{N}, \tag{3.8}$$
$$l_1' > \ldots > l_{k'}', \quad l_1'' > \ldots > l_{k''}''.$$

Now for every s_j from (3.7) either $\exists\, s'_i = s_j$ $(l'_i = l_j)$ or $\exists\, s''_i = s_j$ $(l''_i = l_j)$ or $\exists\, s'_i$, s''_w such that $s'_i + s''_w = s_j$ $(l'_i = l''_w = l_j)$. Inequalities for l', l'' in (3.8) from the inequalities $l_1 > \ldots > l_k$ and from the ordering of subindexes of arguments in (3.4) follow (most "long" processions have least subindexes of arguments). Let us replace every group of equal terms in the right-hand side of (3.6) by a one representative multiplied by the quantity of terms in the group. It is easy to see that summands in the obtained sum depend on a structure of processions "before" ';' and "after" ';' but independend on subindexes of arguments (note that now processions are invariant with respect to all rearrangements). Therefore taking into account that $l^{s'+s''} = l^{s'} l^{s''}$ one can rewrite the sum in the right-hand side of (3.6) in the form

$$\sum_{\substack{l'_1 s'_1 + \ldots + l'_{k'} s'_{k'} = n,\, l''_1 s''_1 + \ldots + l''_{k''} s''_{k''} = m,\\ k', k'', l'_1, \ldots, l'_{k'}, s'_1, \ldots, s'_{k'}, l''_1, \ldots, l''_{k''}, s''_1, \ldots, s''_{k''} \in \mathbb{N},\\ l'_1 > \ldots > l'_{k'},\, l''_1 > \ldots > l''_{k''}}} \frac{n!m!}{l'^{\,s'_1}_1 \ldots l'^{\,s'_{k'}}_{k'} s'_1! \ldots s'_{k'}! l''^{\,s''_1}_1 \ldots l''^{\,s''_{k''}}_{k''} s''_1! \ldots s''_{k''}!} \times$$

$$\times \int_{\mathbb{R}^{s'_1 + \ldots + s'_{k'} + s''_1 + \ldots + s''_{k''}}} \Bigg| \left(\widetilde{f^{(n)} h^{(m)}}\right)_v (\underbrace{t_1, \ldots, t_1}_{l'_1}, \ldots, \underbrace{t_{s'_1 + \ldots + s'_{k'}}, \ldots, t_{s'_1 + \ldots + s'_{k'}}}_{l'_{k'}};$$

$$\underbrace{t_{n+1}, \ldots, t_{n+1}}_{l''_1}, \ldots, \underbrace{t_{n + s''_1 + \ldots + s''_{k''}}, \ldots, t_{n + s''_1 + \ldots + s''_{k''}}}_{l''_{k''}}) \Bigg|^2 \times$$

$$\times\, \sigma(dt_1) \ldots \sigma\left(dt_{s'_1 + \ldots + s'_{k'}}\right) \sigma(dt_{n+1}) \ldots \sigma\left(dt_{n + s''_1, + \ldots + s''_{k''}}\right). \tag{3.9}$$

Because the measure σ is nonatomic, one can replace $\left(\widetilde{f^{(n)} h^{(m)}}\right)_v$ in this sum by the product of $f_v^{(n)}$ and $h_v^{(m)}$. Therefore sum (3.9) is equal to $\left| f_v^{(n)} \right|^2_{\text{ext}} \left| h_v^{(m)} \right|^2_{\text{ext}}$, whence

$$\left| \left(\widehat{f^{(n)} h^{(m)}}\right)_v \right|_{\text{ext}} \le \left| f_v^{(n)} \right|_{\text{ext}} \left| h_v^{(m)} \right|_{\text{ext}}. \tag{3.10}$$

Actually, we proved that $\forall\, \varphi^{(n)} \in \mathcal{S}_C^{\hat{\otimes} n}$, $\forall\, \psi^{(m)} \in \mathcal{S}_{\mathbb{C}}^{\hat{\otimes} m}$

$$\left| P\left(\widetilde{\varphi^{(n)} \psi^{(m)}}\right) \right|_{\text{ext}} \equiv \left| \widehat{\varphi^{(n)} \psi^{(m)}} \right|_{\text{ext}} \le \left| \varphi^{(n)} \right|_{\text{ext}} \left| \psi^{(m)} \right|_{\text{ext}} \tag{3.11}$$

(here P is a symmetrization operator).

Further, $\forall\, v, w \in \mathbb{N}$ we have

$$\left(\widehat{f^{(n)} h^{(m)}}\right)_v - \left(\widehat{f^{(n)} h^{(m)}}\right)_w = P\left(f_v^{(n)} \widetilde{h_v^{(m)}} - f_w^{(n)} \widetilde{h_w^{(m)}}\right) =$$

$$= P\left(f_v^{(n)} \widetilde{h_v^{(m)} - f_w^{(n)}} h_w^{(m)}\right) = P\left(f_v^{(n)} h_v^{(m)} - f_v^{(n)} \widetilde{h_w^{(m)} + f_v^{(n)}} h_w^{(m)} - f_w^{(n)} h_w^{(m)}\right) =$$

$$= P\left(f_v^{(n)}\left(\widetilde{h_v^{(m)} - h_w^{(m)}}\right)\right) + P\left(\left(f_v^{(n)} - \widetilde{f_w^{(n)}}\right)h_w^{(m)}\right) =$$

$$= f_v^{(n)}\left(\widehat{h_v^{(m)} - h_w^{(m)}}\right) + \left(f_v^{(n)} - \widehat{f_w^{(n)}}\right)h_w^{(m)}$$

whence using (3.11) we obtain

$$\left|\left(\widehat{f^{(n)}h^{(m)}}\right)_v - \left(\widehat{f^{(n)}h^{(m)}}\right)_w\right|_{\text{ext}} \leq$$

$$\leq \left|f_v^{(n)}\left(\widehat{h_v^{(m)} - h_w^{(m)}}\right)\right|_{\text{ext}} + \left|\left(f_v^{(n)} - \widehat{f_w^{(n)}}\right)h_w^{(m)}\right|_{\text{ext}} \leq$$

$$\leq \left|f_v^{(n)}\right|_{\text{ext}}\left|h_v^{(m)} - h_w^{(m)}\right|_{\text{ext}} + \left|f_v^{(n)} - f_w^{(n)}\right|_{\text{ext}}\left|h_w^{(m)}\right|_{\text{ext}} \underset{v,w\to\infty}{\to} 0.$$

So, the sequence $\left(\left(\widehat{f^{(n)}h^{(m)}}\right)_v\right)_{v\geq 1}$ is a Cauchy one in $\mathcal{H}_{\text{ext}}^{(n+m)}$ and therefore there exists $\widehat{F^{(n)}H^{(m)}} := \lim_{v\to\infty}\left(\widehat{f^{(n)}h^{(m)}}\right)_v \in \mathcal{H}_{\text{ext}}^{(n+m)}$. This limit is independent on a choice of sequences $\left(f_v^{(n)}\right)_{v\geq 1}$, $\left(h_v^{(m)}\right)_{v\geq 1}$, this can be proved by a standard way. Namely, let us consider another sequences $\mathcal{S}_{\mathbb{C}}^{\hat{\otimes} n} \ni f_v'^{(n)} \underset{v\to\infty}{\to} F^{(n)}$ in $\mathcal{H}_{\text{ext}}^{(n)}$, $\mathcal{S}_{\mathbb{C}}^{\hat{\otimes} m} \ni h_v'^{(m)} \underset{v\to\infty}{\to} H^{(m)}$ in $\mathcal{H}_{\text{ext}}^{(n)}$, and put $\widehat{F^{(n)}H^{(m)}}' := \lim_{v\to\infty}\left(\widehat{f'^{(n)}h'^{(m)}}\right)_v \in$ $\in \mathcal{H}_{\text{ext}}^{(n+m)}$. Then for "mixed" sequences $\left(f_1^{(n)}, f_1'^{(n)}, f_2^{(n)}, f_2'^{(n)}, \ldots\right)$ and $\left(h_1^{(m)}, h_1'^{(m)}, h_2^{(m)}, h_2'^{(m)}, \ldots\right)$ the corresponding "final result" coincides with $\widehat{F^{(n)}H^{(m)}}$ and with $\widehat{F^{(n)}H^{(m)}}'$, therefore $\widehat{F^{(n)}H^{(m)}} = \widehat{F^{(n)}H^{(m)}}'$. Estimate (3.3) follows from (3.10) by passing to a limit.

Let us prove now that $F^{(n)} \lozenge H^{(m)}$ can be identified with $\widehat{F^{(n)}H^{(m)}}$. First we establish that $\forall \lambda \in \mathcal{S}_{\mathbb{C}}$

$$\left\langle \widehat{F^{(n)}H^{(m)}}, \lambda^{\otimes(n+m)}\right\rangle_{\text{ext}} = \left\langle F^{(n)} \lozenge H^{(m)}, \lambda^{\otimes(n+m)}\right\rangle_{\text{ext}}. \tag{3.12}$$

It follows directly from the definition of $F^{(n)} \lozenge H^{(m)}$ that (see (1.11))

$$\left\langle F^{(n)} \lozenge H^{(m)}, \lambda^{\otimes(n+m)}\right\rangle_{\text{ext}} = \left\langle U_{n+m}U_{n+m}^{-1}\left(U_nF^{(n)} \hat{\otimes} U_mH^{(m)}\right), \lambda^{\otimes(n+m)}\right\rangle =$$

$$= \left\langle U_nF^{(n)}, \lambda^{\otimes n}\right\rangle\left\langle U_mH^{(m)}, \lambda^{\otimes m}\right\rangle = \left\langle F^{(n)}, \lambda^{\otimes n}\right\rangle_{\text{ext}}\left\langle H^{(m)}, \lambda^{\otimes m}\right\rangle_{\text{ext}}. \tag{3.13}$$

On the other hand, let us consider the scalar product

$$\left\langle\left(\widehat{f^{(n)}h^{(m)}}\right)_v, \lambda^{\otimes(n+m)}\right\rangle_{\text{ext}} = \sum_{\substack{k,\, l_j,\, s_j \in \mathbb{N}:\, j=1,\ldots,k,\, l_1>l_2>\ldots>l_k,\\ l_1s_1+\ldots+l_ks_k=n+m}} \frac{(n+m)!}{l_1^{s_1}\ldots l_k^{s_k}\, s_1!\ldots s_k!} \times$$

$$\times \int\limits_{\mathbb{R}^{s_1+\ldots+s_k}} \left(\widehat{f^{(n)}h^{(m)}}\right)_v (\underbrace{t_1,\ldots,t_1}_{l_1},\ldots,\underbrace{t_{s_1+\ldots+s_k},\ldots,t_{s_1+\ldots+s_k}}_{l_k}) \times$$

$$\times\, \lambda^{l_1}(t_1)\ldots\lambda^{l_k}\left(t_{s_1+\ldots+s_k}\right)\sigma(dt_1)\ldots\sigma(dt_{s_1+\ldots+s_k}) =$$

$$= \sum_{\substack{k,\,l_j,\,s_j\in\mathbb{N}:\, j=1,\ldots,k,\, l_1>l_2>\ldots>l_k,\\ l_1s_1+\ldots+l_ks_k=n+m}} \frac{(n+m)!}{l_1^{s_1}\ldots l_k^{s_k}\, s_1!\ldots s_k!} \times$$

$$\times \int\limits_{\mathbb{R}^{s_1+\ldots+s_k}} \left(\widehat{f_\lambda^{(n)}h_\lambda^{(m)}}\right)_v (\underbrace{t_1,\ldots,t_1}_{l_1},\ldots,\underbrace{t_{s_1+\ldots+s_k},\ldots,t_{s_1+\ldots+s_k}}_{l_k})\,\sigma(dt_1)\ldots\sigma(dt_{s_1+\ldots+s_k}) \tag{3.14}$$

(we used the previous notation), where $\left(\widehat{f_\lambda^{(n)}h_\lambda^{(m)}}\right)_v$ is obtained by formulas (3.2), (3.4) starting from $f_{\lambda,v}^{(n)}(t_1,\ldots,t_n) = f_v^{(n)}(t_1,\ldots,t_n)\,\lambda(t_1)\ldots\lambda(t_n)$, $h_{\lambda,v}^{(m)}(t_1,\ldots,t_m) = h_v^{(m)}(t_1,\ldots,t_m)\,\lambda(t_1)\ldots\lambda(t_m)$. Substituting in (3.14) expression (3.4) for $\left(\widehat{f_\lambda^{(n)}h_\lambda^{(m)}}\right)_v$, by analogy with the proof that the sum in the right-hand side of (3.6) has form (3.9) we can transform the last sum in (3.14) to the form

$$\sum_{\substack{l_1's_1'+\ldots+l_{k'}'s_{k'}'=n,\; l_1''s_1''+\ldots+l_{k''}''s_{k''}''=m,\\ k',k'',l_1',\ldots,l_{k'}',s_1',\ldots,s_{k'}',l_1'',\ldots,l_{k''}'',s_1'',\ldots,s_{k''}''\in\mathbb{N},\\ l_1'>\ldots>l_{k'}',\; l_1''>\ldots>l_{k''}''}} \frac{n!m!}{l_1'^{s_1'}\ldots l_{k'}'^{s_{k'}'}s_1'!\ldots s_{k'}'!\,l_1''^{s_1''}\ldots l_{k''}''^{s_{k''}''}s_1''!\ldots s_{k''}''!} \times$$

$$\times \int\limits_{\mathbb{R}^{s_1'+\ldots+s_{k'}'+s_1''+\ldots+s_{k''}''}} \left(\widehat{f_\lambda^{(n)}h_\lambda^{(m)}}\right)_v (\underbrace{t_1,\ldots,t_1}_{l_1'},\ldots,\underbrace{t_{s_1'+\ldots+s_{k'}'},\ldots,t_{s_1'+\ldots+s_{k'}'}}_{l_{k'}'};$$

$$\underbrace{t_{n+1},\ldots,t_{n+1}}_{l_1''},\ldots,\underbrace{t_{n+s_1''+\ldots+s_{k''}''},\ldots,t_{n+s_1''+\ldots+s_{k''}''}}_{l_{k''}''}) \times$$

$$\times\, \sigma(dt_1)\ldots\sigma\left(dt_{s_1'+\ldots+s_{k'}'}\right)\sigma(dt_{n+1})\ldots\sigma\left(dt_{n+s_1''+\ldots+s_{k''}''}\right) =$$

$$= \sum_{\substack{l_1's_1'+\ldots+l_{k'}'s_{k'}'=n,\; l_1''s_1''+\ldots+l_{k''}''s_{k''}''=m,\\ k',k'',l_1',\ldots,l_{k'}',s_1',\ldots,s_{k'}',l_1'',\ldots,l_{k''}'',s_1'',\ldots,s_{k''}''\in\mathbb{N},\\ l_1'>\ldots>l_{k'}',\; l_1''>\ldots>l_{k''}''}} \frac{n!m!}{l_1'^{s_1'}\ldots l_{k'}'^{s_{k'}'}s_1'!\ldots s_{k'}'!\,l_1''^{s_1''}\ldots l_{k''}''^{s_{k''}''}s_1''!\ldots s_{k''}''!} \times$$

$$\times \int\limits_{\mathbb{R}^{s_1'+\ldots+s_{k'}'}} f_v^{(n)}(\underbrace{t_1,\ldots,t_1}_{l_1'},\ldots,\underbrace{t_{s_1'+\ldots+s_{k'}'},\ldots,t_{s_1'+\ldots+s_{k'}'}}_{l_{k'}'}) \times$$

$$\times\, \lambda^{l_1'}(t_1)\ldots\lambda^{l_{k'}'}\left(t_{s_1'+\ldots+s_k'}\right)\sigma(dt_1)\ldots\sigma(dt_{s_1'+\ldots+s_k'}) \times$$

$$\times \int\limits_{\mathbb{R}^{s_1''+\ldots+s_{k''}''}} f_v^{(n)}(\underbrace{t_{n+1},\ldots,t_{n+1}}_{l_1''},\ldots,\underbrace{t_{n+s_1''+\ldots+s_{k''}''},\ldots,t_{n+s_1''+\ldots+s_{k''}''}}_{l_{k''}''}) \times$$

$$\times\, \lambda^{l''_1}(t_{n+1})\ldots\lambda^{l''_{k''}}\Big(t_{n+s''_1+\ldots+s''_{k''}}\Big)\sigma(dt_{n+1})\ldots\sigma\Big(dt_{n+s''_1+\ldots+s''_{k''}}\Big) =$$

$$= \left\langle f_v^{(n)}, \lambda^{\otimes n}\right\rangle_{\text{ext}} \left\langle h_v^{(m)}, \lambda^{\otimes m}\right\rangle_{\text{ext}}$$

(here a nonatomicity of σ used). By passing to a limit as $v \to \infty$ we obtain $\left\langle \widehat{F^{(n)}H^{(m)}}, \lambda^{\otimes(n+m)}\right\rangle_{\text{ext}} = \left(F^{(n)}, \lambda^{\otimes n}\right)_{\text{ext}} \left(H^{(m)}, \lambda^{\otimes m}\right)_{\text{ext}}$. From here taking into account (3.13) we obtain (3.12).

Further, the restriction of $\widehat{F^{(n)}H^{(m)}}$ (as a linear functional) on $\mathcal{S}_{\mathbb{C}}^{\hat{\otimes}(n+m)}$ is a *linear continuous* functional on $\mathcal{S}_{\mathbb{C}}^{\hat{\otimes}(n+m)}$. This functional coincides with $F^{(n)} \lozenge H^{(m)}$ on the total in $\mathcal{S}_{\mathbb{C}}^{\hat{\otimes}(n+m)}$ set $\left\{\lambda^{\otimes(n+m)}\colon \lambda \in \mathcal{S}_{\mathbb{C}}\right\}$, therefore $\forall\, g^{(n+m)} \in \mathcal{S}_{\mathbb{C}}^{\hat{\otimes}(n+m)}$ $\left\langle \widehat{F^{(n)}H^{(m)}}, g^{(n+m)}\right\rangle_{\text{ext}} = \left\langle F^{(n)} \lozenge H^{(m)}, g^{(n+m)}\right\rangle_{\text{ext}}$. Thus $F^{(n)} \lozenge H^{(m)}$ can be extended to a linear continuous functional on $\mathcal{H}_{\text{ext}}^{(n+m)}$ by the formula $F^{(n)} \lozenge H^{(m)} := \widehat{F^{(n)}H^{(m)}}$ (it is natural to preserve the old notation for $F^{(n)} \lozenge H^{(m)}$).

The lemma is proved.

***Remark* 3.5.** Note that for $m = 0$ (or $n = 0$) $F^{(n)} \lozenge H^{(0)} = F^{(n)} \cdot H^{(0)}$ (because $H^{(0)} \in \mathbb{C}$) and estimate (3.1) is obvious.

Theorem 3.2. *For* $F,\ H \in \left(L^2\right)^{-1}$ *and a holomorphic at* $(SF)(0)$ *function* h: $\mathbb{C} \to \mathbb{C}$ *we have* $F \lozenge H \in \left(L^2\right)^{-1}$ *and* $h^{\lozenge}(F) \in \left(L^2\right)^{-1}$.

Proof. Actually, we shall prove somewhat more than we need for the present. First, we establish that for $F_1, \ldots, F_m \in \left(L^2\right)^{-1}$ and $q \in \mathbb{N}$ sufficiently large

$$\| F_1 \lozenge \ldots \lozenge F_m \|_{-q} \ \le\ C(m-1) \| F_1 \|_{-(q-1)} \ldots \| F_m \|_{-(q-1)},$$

where $C(m) := \sqrt{\max_{n \in \mathbb{N}} \left\{2^{-n}(n+1)^m\right\}}$. Let $F_j^{(k)} \in \mathcal{H}_{\text{ext}}^{(k)}$ be the kernels from decomposition (1.6) for $F_j,\ j \in \{1, \ldots, m\}$. It follows directly from Definition 3.2 that

$$F_1 \lozenge \ldots \lozenge F_m \ =\ \sum_{n=0}^{\infty} \left\langle L_n, \sum_{k_1,\ldots,k_m \in \mathbb{Z}_+ \colon \sum_{l=1}^m k_1 = n} F_1^{(k_1)} \lozenge \ldots \lozenge F_m^{(k_m)} \right\rangle,$$

therefore using (3.1) one can estimate as follows:

$$\| F_1 \lozenge \ldots \lozenge F_m \|_{-q}^2 \ =\ \sum_{n=0}^{\infty} 2^{-qn} \left| \sum_{k_1,\ldots,k_m \in \mathbb{Z}_+ \colon \sum_{l=1}^m k_l = n} F_1^{(k_1)} \lozenge \ldots \lozenge F_m^{(k_m)} \right|_{\text{ext}}^2 =$$

$$= \sum_{n=0}^{\infty} 2^{-qn} \left| \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \ldots \sum_{k_{m-1}=0}^{n-k_1-\ldots-k_{m-2}} F_1^{(k_1)} \lozenge \ldots \lozenge F_{m-1}^{(k_{m-1})} \lozenge F_m^{(n-k_1-\ldots-k_{m-1})} \right|_{\text{ext}}^2 \le$$

$$\le \sum_{n=0}^{\infty} 2^{-qn}(n+1) \times$$

$$\times \sum_{k_1=0}^{n} \left| \sum_{k_2=0}^{n-k_1} \ldots \sum_{k_{m-1}=0}^{n-k_1-\ldots-k_{m-2}} F_1^{(k_1)} \lozenge \ldots \lozenge F_{m-1}^{(k_{m-1})} \lozenge F_m^{(n-k_1-\ldots-k_{m-1})} \right|_{\text{ext}}^2 \leq \ldots$$

$$\ldots \leq \sum_{n=0}^{\infty} 2^{-qn} (n+1)^{(m-1)} \times$$

$$\times \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \ldots \sum_{k_{m-1}=0}^{n-k_1-\ldots-k_{m-2}} \left| F_1^{(k_1)} \lozenge \ldots \lozenge F_{m-1}^{(k_{m-1})} \lozenge F_m^{(m-k_1-\ldots-k_{m-1})} \right|_{\text{ext}}^2 =$$

$$= \sum_{n=0}^{\infty} \left(2^{-n}(n+1)^{(m-1)}\right) 2^{-(q-1)n} \times$$

$$\times \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \ldots \sum_{k_{m-1}=0}^{n-k_1-\ldots-k_{m-2}} \left| F_1^{(k_1)} \lozenge \ldots \lozenge F_{m-1}^{(k_{m-1})} \lozenge F_m^{(m-k_1-\ldots-k_{m-1})} \right|_{\text{ext}}^2 \leq$$

$$\leq [C(m-1)]^2 \sum_{k_1=0}^{\infty} 2^{(q-1)k_1} \left| F_1^{(k_1)} \right|_{\text{ext}}^2 \sum_{n=k_1}^{\infty} \sum_{k_2=0}^{n-k_1} \ldots \sum_{k_{m-1}=0}^{n-k_1-\ldots-k_{m-2}} 2^{-(q-1)k_2} \left| F_2^{(k_2)} \right|_{\text{ext}}^2 \ldots$$

$$\ldots 2^{-(q-1)k_{m-1}} \left| F_{m-1}^{(k_{m-1})} \right|_{\text{ext}}^2 2^{-(q-1)(n-k_1-\ldots-k_{m-1})} \left| F_m^{(n-k_1-\ldots-k_{m-1})} \right|_{\text{ext}}^2 =$$

$$= [C(m-1)]^2 \| F_1 \|_{-(q-1)}^2 \sum_{n=0}^{\infty} \sum_{k_2=0}^{n} \ldots \sum_{k_{m-1}=0}^{n-k_2-\ldots-k_{m-2}} 2^{-(q-1)k_2} \left| F_2^{(k_2)} \right|_{\text{ext}}^2 \ldots$$

$$\ldots 2^{-(q-1)k_{m-1}} \left| F_{m-1}^{(k_{m-1})} \right|_{\text{ext}}^2 2^{-(q-1)(n-k_2-\ldots-k_{m-1})} \left| F_m^{(n-k_2-\ldots-k_{m-1})} \right|_{\text{ext}}^2 = \ldots$$

$$\ldots = [C(m-1)]^2 \| F_1 \|_{-(q-1)}^2 \ldots \| F_m \|_{-(q-1)}^2 .$$

It follows directly from here that $F_1 \lozenge \ldots \lozenge F_m \in \left(L^2\right)^{-1}$ and in particular for F, $H \in \left(L^2\right)^{-1}$ $F \lozenge H \in \left(L^2\right)^{-1}$.

Further, let $F \in \left(L^2\right)^{-1}$ and $h: \mathbb{C} \to \mathbb{C}$ be a holomorphic at $(SF)(0)$ function. Let $F^{(m)} \in \mathcal{H}_{\text{ext}}^{(m)}$ be the kernels from decomposition (1.6) for F, $h(u) = \sum_{n=0}^{\infty} h_n \left(u - F^{(0)}\right)^n$. Because $SF \in \text{Hol}_0$ and $S1 = 1$ we have

$$h^{\lozenge}(F) = S^{-1}\left[h_0 + \sum_{n=1}^{\infty} h_n \left(SF - F^{(0)}\right)^n\right] =$$

$$= S^{-1}\left[h_0 + \sum_{n=1}^{\infty} h_n \left(\sum_{m=1}^{\infty} \left\langle F^{(m)}, \cdot^{\otimes m} \right\rangle_{\text{ext}}\right)^n\right] =$$

$$= S^{-1}\left[h_0 + \sum_{n=1}^{\infty} h_n \sum_{m_1,\ldots,m_n=1}^{\infty} \left\langle F^{(m_1)} \lozenge \ldots \lozenge F^{(m_n)}, \cdot^{\otimes(m_1+\ldots+m_n)} \right\rangle_{\text{ext}}\right] =$$

$$= S^{-1}\left[h_0 + \sum_{n=1}^{\infty} h_n \sum_{s=n}^{\infty} \left\langle \sum_{m_1,\ldots,m_n \in \mathbb{N}: \sum_{k=1}^{n} m_k = s} F^{(m_1)} \lozenge \ldots \lozenge F^{(m_n)}, \cdot^{\otimes s} \right\rangle_{\text{ext}}\right] =$$

$$= S^{-1}\left[h_0 + \sum_{s=1}^{\infty}\left\langle \sum_{n=1}^{s} h_n \sum_{m_1,\ldots,m_n \in \mathbb{N}:\, \sum_{k=1}^{n} m_k = s} F^{(m_1)} \lozenge \ldots \lozenge F^{(m_n)}, \cdot^{\otimes s}\right\rangle_{\text{ext}}\right] =$$

$$= h_0 + \sum_{s=1}^{\infty}\left\langle L_s, \sum_{n=1}^{s} h_n \sum_{m_1,\ldots,m_n \in \mathbb{N}:\, \sum_{k=1}^{n} m_k = s} F^{(m_1)} \lozenge \ldots \lozenge F^{(m_n)}\right\rangle.$$

Because $F \in \left(L^2\right)^1_{-q}$ for some $q \in \mathbb{N}$, we have $\|F\|^2_{-q} = \sum_{m=0}^{\infty} 2^{-qm}\left|F^{(m)}\right|^2_{\text{ext}} < \infty$, whence $\left|F^{(m)}\right|_{\text{ext}} \le 2^{qm/2}\|F\|_{-q}$; and because of holomorphy of h there exists $\tilde{q} \in \mathbb{N}$ such that $|h_n| < 2^{\tilde{q}n}$ for all $n \in \mathbb{Z}_+$. So, taking into account that $\sum_{m_1,\ldots,m_n \in \mathbb{N}:\, \sum_{l=1}^{n} m_l = s} 1 = C_{s-1}^{n-1} \le 2^{s-1}$, using (3.1) we obtain

$$\left\|h^{\lozenge}(F)\right\|^2_{-q'} = |h_0|^2 + \sum_{s=1}^{\infty} 2^{-q's}\left|\sum_{n=1}^{s} h_n \sum_{m_1,\ldots,m_n \in \mathbb{N}:\, \sum_{l=1}^{n} m_l = s} F^{(m_1)} \lozenge \ldots \lozenge F^{(m_n)}\right|^2_{\text{ext}} \le$$

$$\le |h_0|^2 + \sum_{s=1}^{\infty} 2^{-q's}\left(\sum_{n=1}^{s} |h_n| \sum_{m_1,\ldots,m_n \in \mathbb{N}:\, \sum_{l=1}^{n} m_l = s} \prod_{l=1}^{n}\left|F^{(m_l)}\right|_{\text{ext}}\right)^2 \le$$

$$\le |h_0|^2 + \sum_{s=1}^{\infty} 2^{-q's}\left(\sum_{n=1}^{s} 2^{\tilde{q}n} \sum_{m_1,\ldots,m_n \in \mathbb{N}:\, \sum_{l=1}^{n} m_l = s} 2^{qs/2}\|F\|^n_{-q}\right)^2 =$$

$$= |h_0|^2 + \sum_{s=1}^{\infty} 2^{(q-q')s}\left(\sum_{n=1}^{s} 2^{\left(\tilde{q} + \log_2\left(\|F\|_{-q}\right)\right)n} C_{s-1}^{n-1}\right)^2 \le$$

$$\le |h_0|^2 + \sum_{s=1}^{\infty} 2^{(q-q'+2)s-2}\left(\sum_{n=1}^{s} 2^{\left(\tilde{q} + \log_2\left(\|F\|_{-q}\right)\right)n}\right)^2 \le$$

$$\le |h_0|^2 + \sum_{s=1}^{\infty} 2^{(q-q'+2)s-2}\left(\sum_{n=1}^{s} 2^{\left(\tilde{q} + \left|\log_2\left(\|F\|_{-q}\right)\right|\right)n}\right)^2 \le$$

$$\le |h_0|^2 + C\sum_{s=1}^{\infty} 2^{\left(q-q'+2+2\tilde{q}+2\left|\log_2\left(\|F\|_{-q}\right)\right|\right)s-2} < \infty,$$

if $q' \in \mathbb{N}$ is sufficiently large $\left(\text{here } C := \dfrac{2^{2\tilde{q}+2\left|\log_2\left(\|F\|_{-q}\right)\right|}}{\left(2^{\tilde{q}+\left|\log_2\left(\|F\|_{-q}\right)\right|} - 1\right)^2}\right)$. So, $h^{\lozenge}(F) \in$ $\in \left(L^2\right)^{-1}$.

The theorem is proved.

Let us define a space B (a characterization space of $\left(L^2\right)^{-1}$ in terms of an S-transform) putting $B := S\left(\left(L^2\right)^{-1}\right) \equiv \left\{J \in \text{Hol}_0 \mid \exists\, F \in \left(L^2\right)^{-1}:\ J = SF\right\} \subset \text{Hol}_0$.

***Corollary* 3.1.** *The space B is an algebra with respect to the usual (pointwise) multiplication of functions. Moreover, if $J \in B$, $F^{(0)} \in \mathbb{C}$ is the kernel from* *decomposition* (1.6) *for* $S^{-1}J$ *and* $h:\mathbb{C}\to\mathbb{C}$ *is a holomorphic at* $F^{(0)}$ *function, then* $\tilde{J}(\cdot):=h(J(\cdot))\in B$. *In particular, for each entire* $h:\mathbb{C}\to\mathbb{C}$ *and* $J\in B$ $h(J(\cdot))\in B$.

***Remark* 3.6.** As it follows from Corollary 3.1, the space B has properties similar to properties of Hol_0. A characterization of $(L^2)^{-1}$ "in terms of B" will be very useful for study of a stochastic derivative on $(L^2)^{-1}$, we'll discuss this derivative in a forthcoming paper.

There is a simple interconnection between the Wick calculus and a stochastic integration. More exactly, for $t\in[0,+\infty]$ and $F\in(L^2)^{-1}\otimes\mathcal{H}_{\mathbb{C}}^{+}$ we define $\int_0^t F_s\lozenge G_s'\sigma(ds)$ (where $G_\cdot'=\langle L_1,\delta_\cdot\rangle$ is the Gamma-white noise) as a unique element of $(\mathcal{S}')'$ such that

$$\left\langle\!\left\langle\int_0^t F_s\lozenge G_s'\sigma(ds),f\right\rangle\!\right\rangle \equiv \int_0^t\langle\langle F_s\lozenge G_s',f\rangle\rangle\sigma(ds)\quad\forall f\in(\mathcal{S})$$

(so, $\int_0^t F_s\lozenge G_s'\sigma(ds)$ is the integral defined in a Pettis sense).

Theorem 3.3. *For all* $t\in[0,+\infty]$ *and* $F\in(L^2)^{-1}\otimes\mathcal{H}_{\mathbb{C}}^{+}$ $\int_0^t F_s\lozenge G_s'\sigma(ds)$ *can be extended to a linear continuous functional on* $(L^2)^1$ *that coinsides with* $\int_0^t F_s\hat{d}G_s$, *i.e.,*

$$\int_0^t F_s\lozenge G_s'\sigma(ds) = \int_0^t F_s\hat{d}G_s\in(L^2)^{-1}. \tag{3.15}$$

Proof. We have to prove that

$$\left\langle\!\left\langle\int_0^t F_s\lozenge G_s'\sigma(ds),f\right\rangle\!\right\rangle = \left\langle\!\left\langle\int_0^t F_s\hat{d}G_s,f\right\rangle\!\right\rangle\quad\forall f\in(\mathcal{S}).$$

It is easy to calculate that

$$\left\langle\!\left\langle\int_0^t F_s\hat{d}G_s,f\right\rangle\!\right\rangle = \sum_{n=1}^{\infty}n!\left\langle\hat{F}_{[0,t)}^{(n-1)},f^{(n)}\right\rangle_{\mathrm{ext}},\quad f^{(n)}\in\mathcal{S}_{\mathbb{C}}^{\hat{\otimes}n}$$

(we use the notation of (1.5) and (2.5)).

On the other hand, $F_\cdot\lozenge G_\cdot'=\sum_{m=1}^{\infty}\langle L_m,F_\cdot^{(m-1)}\lozenge\delta_\cdot\rangle$ (we use the notation of (2.4), see also Remark 3.2), whence

$$\langle\langle F_\cdot\lozenge G_\cdot',f\rangle\rangle = \sum_{n=1}^{\infty}n!\left\langle F_\cdot^{(n-1)}\lozenge\delta_\cdot,f^{(n)}\right\rangle_{\mathrm{ext}}.$$

So, in order to finish the proof we have to prove that for all $n\in\mathbb{N}$

$$\int_0^t\left\langle F_s^{(n-1)}\lozenge\delta_s,f^{(n)}\right\rangle_{\mathrm{ext}}\sigma(ds) = \left\langle\hat{F}_{[0,t)}^{(n-1)},f^{(n)}\right\rangle_{\mathrm{ext}}. \tag{3.16}$$

First, let us consider $f^{(n)}=\lambda^{\otimes n}$, $\lambda\in\mathcal{S}_{\mathbb{C}}$. Now $\int_0^t\left\langle F_s^{(n-1)}\lozenge\delta_s,\lambda^{\otimes n}\right\rangle_{\mathrm{ext}}\sigma(ds)=$ $=\int_0^t\left\langle F_s^{(n-1)},\lambda^{\otimes(n-1)}\right\rangle_{\mathrm{ext}}\lambda(s)\sigma(ds)$. But it was established in the proof of Theorem 2.2

in [3] that $\int_0^t \left\langle F_s^{(n-1)}, \lambda^{\otimes(n-1)} \right\rangle_{\text{ext}} \lambda(s)\sigma(ds) = \left\langle \hat{F}_{[0,t)}^{(n-1)}, \lambda^{\otimes n} \right\rangle_{\text{ext}}$. Because the set $\left\{\lambda^{\otimes n} : \lambda \in \mathcal{S}_{\mathbb{C}}\right\}$ is total in $\mathcal{S}_{\mathbb{C}}^{\hat{\otimes} n}$ and $\int_0^t \left\langle F_s^{(n-1)} \lozenge \delta_s, \circ \right\rangle_{\text{ext}} \sigma(ds)$, $\left\langle \hat{F}_{[0,t)}^{(n-1)}, \circ \right\rangle_{\text{ext}}$ are continuous functionals on $\mathcal{S}_{\mathbb{C}}^{\hat{\otimes} n}$, we can conclude that (3.16) holds true.

Property (3.15) and Corollary 3.1 give us a possibility to consider so-called stochastic differential equations wit Wick-type nonlinearity and solve such equations using an S-transform. Let us consider corresponding examples.

***Example* 3.1** (a linear equation). Let us consider the stochastic equation

$$X_t = X_0 + \int_0^t X_s \lozenge F_1 \lozenge \ldots \lozenge F_n \sigma(ds) + \int_0^t X_s \lozenge H_1 \lozenge \ldots \lozenge H_m \hat{d} G_s, \qquad (3.17)$$

where $X_0 \in \left(L^2\right)^{-1}$; $n, m \in \mathbb{N}$; $F_k = \left\langle L_1, F_k^{(1)} \right\rangle$, $F_k^{(1)} \in \mathcal{H}_{\text{ext}}^{(1)} = \mathcal{H}_{\mathbb{C}}$, $k \in \{1, \ldots, n\}$; $H_k = \left\langle L_1, H_k^{(1)} \right\rangle$, $H_k^{(1)} \in \mathcal{H}_{\text{ext}}^{(1)}$, $k \in \{1, \ldots, m\}$. Applying to (3.17) the S-transform (with regard to (3.15)), solving the obtained *algebraic* equation and applying the inverse S-transform (see Corollary 3.1 and Remark 3.6) we obtain the solution

$$X_t = X_0 \lozenge \exp^{\lozenge}\{F_1 \lozenge \ldots \lozenge F_n \sigma([0,t)) + H_1 \lozenge \ldots \lozenge H_m \lozenge G_t\} \in \left(L^2\right)^{-1}.$$

By analogy one can solve the more general equation

$$X_t = X_0 + \int_0^t X_s \lozenge F \sigma(ds) + \int_0^t X_s \lozenge H \hat{d} G_s,$$

where X_0, F, $H \in \left(L^2\right)^{-1}$, the solution has the form

$$X_t = X_0 \lozenge \exp^{\lozenge}\{F \sigma([0,t)) + H \lozenge G_t\} \in \left(L^2\right)^{-1}.$$

***Example* 3.2** (the Verhulst-type equation). Let us consider integral stochastic equation

$$X_t = X_0 + r \int_0^t X_s \lozenge (N - X_s)\sigma(ds) + \alpha \int_0^t X_s \lozenge (N - X_s) \hat{d} G_s, \qquad (3.18)$$

where $X_0 \in \left(L^2\right)^{-1}$, $N, r, \alpha \in \mathbb{R}$, $N > 0$, $r > 0$, $(SX_0)(0) > 0$. Applying to (3.18) the S-transform (with regard to (3.15)), solving the obtained *algebraic* equation and applying the inverse S-transform, one can show by the full analogy with [19] that the solution of (3.18) has the form

$$X_t = N\left[1 + \left(N X_0^{\lozenge(-1)} - 1\right) \lozenge \exp^{\lozenge}\{-N(r\sigma([0,t)) + \alpha G_t)\}\right]^{\lozenge(-1)} \in \left(L^2\right)^{-1},$$

where $Y^{\lozenge(-1)} := S^{-1} \dfrac{1}{SY}$.

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