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ANALOGUE OF LIOUVILLE EQUATION AND BBGKY HIERARCHY FOR A SYSTEM OF HARD SPHERES WITH INELASTIC COLLISIONS*

АНАЛОГ РІВНЯННЯ ЛІУВІЛЛЯ ТА ББГКІ ІЄРАРХІЇ ДЛЯ СИСТЕМИ ТВЕРДИХ СФЕР З НЕПРУЖНИМ РОЗСІЯННЯМ

Dynamics of a system of hard spheres with inelastic collisions is investigated. This system is a model for granular flow. The map induced by a shift along the trajectory does not preserve the volume of the phase space, and the corresponding Jacobian is different from one.

A special distribution function is defined as the product of the usual distribution function and the squared Jacobian. For this distribution function, the Liouville equation with boundary condition is derived. A sequence of correlation functions is defined for canonical and grand canonical ensemble. The generalized BBGKY hierarchy and boundary condition are derived for correlation functions.

Досліджується динаміка твердих сфер з непружним розсіянням. Така система є моделлю для гранульованих потоків. Відображення, індуковане зсувом уздовж траєкторій, не зберігає об'єм фазового простору, а відповідний якобіан є відмінним від одиниці.

Визначено спеціальну функцію розподілу як добуток звичайної функції розподілу та квадрата якобіана. Для цієї функції розподілу виведено рівняння Ліувілля з граничними умовами. Послідовність кореляційних функцій визначено для канонічного та великого канонічного ансамблів. Для кореляційних функцій виведено узагальнену ієрархію ББГКІ та відповідні граничні умови.

Introduction. It is commonly accepted that systems of hard spheres with inelastic collision are proper model of granular flow. Statistical mechanics of systems of hard spheres should be a theoretical basis of the theory of granular flow. In attempts to adapt classical statistical mechanics to systems of hard spheres with inelastic collisions, one is faced with new very difficult problems connected with inelasticity.

First of all it is necessary to define density of probability (distribution function) on phase space, because the Jacobian of the transformation induced by shift along trajectories of hard spheres with inelastic collision is different from one and is singular (its derivative with respect to time contains δ-functions). It is necessary to derive the Liouville equation for defined distribution function and correctly formulate boundary conditions associated with inelasticity. And at last one should derive the analogue of BBGKY hierarchy for corresponding correlation functions.

Above mentioned problems are solved in given paper. The distribution function is defined as follows:

$$D_N(t,(x)_N) = D_N(0, X(-t,(x)_N)) \left[\frac{\partial X(-t,(x)_N)}{\partial (x)_N} \right]^2 \tag{1}$$

where $D_N(0,(x)_N)$ is the initial distribution function, $X(-t,(x)_N)$ the trajectory of N hard spheres at time $-t$ with initial data $(x)_N$ at initial time $t=0$, $\dfrac{\partial X(-t,(x)_N)}{\partial (x)_N}$ the corresponding singular Jacobian different from one. Distribution function satisfies the law of conservation of full probability

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$$\int D_N(t,(x)_N)d(x)_N = \int D_N(0,(x)_N)d(x)_N \tag{2}$$

and the Liouville equation

$$\frac{\partial}{\partial t}D_N(t,(x)_N) = -\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} D_N(t,(x)_N), \qquad D_N(t,(x)_N)\,|_{t=0} = D_N(0,(x)_N) \tag{3}$$

with boundary condition according to which for $q_i - q_j - a\eta = 0$ (where $|\eta| = 1$ and a is the diameter of the sphere), and $\langle \eta, (p_i - p_j)\rangle > 0$, momenta p_i, p_j should be replaced by

$$p_i^* = p_i + \frac{\varepsilon}{1-2\varepsilon}\eta\langle\eta,(p_i - p_j)\rangle, \qquad p_j^* = p_j - \frac{\varepsilon}{1-2\varepsilon}\eta\langle\eta,(p_i - p_j)\rangle$$

in the operator $-\sum_{i=1}^{N} p_i \dfrac{\partial}{\partial q_i}$ and in $D_N(t,(x)_N)$; moreover the identity is valid

$$D_N(t, q_1, p_1, \ldots, q_i, p_i, \ldots, q_j, p_j, \ldots, q_N, p_N) =$$
$$= \frac{1}{(1-2\varepsilon)^2} D_N(t, q_1, p_1, \ldots, q_i, p_i^*, \ldots, q_j, p_j^*, \ldots, q_N, p_N)$$

for above mentioned phase points; ε is a parameter associated with inelastic collisions $\left(\dfrac{1}{2} < \varepsilon \leq 1\right)$. Momenta do not change if $\langle\eta,(p_i - p_j)\rangle < 0$, i.e., $p_i^* = p_i$, $p_j^* = p_j$.

The following BBGKY hierarchy is derived for a sequence of correlation functions

$$\rho_s^{(N)}(t,(x)_s) =$$
$$= N(N-1)\ldots(N-s+1)\int D_N(t, x_1, \ldots, x_s, x_{s+1}, \ldots, x_N)\, dx_{s+1}\ldots x_N : \tag{4}$$

$$\frac{\partial \rho_s^{(N)}(t,(x)_s)}{\partial t} =$$
$$= -\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i}\rho_s^{(N)}(t,(x)_s) + a^2 \sum_{i=1}^{s}\int dp_{s+1}\int_{S_+^2} d\eta \langle\eta,(p_i - p_{s+1})\rangle \times$$
$$\times \Bigg[\frac{1}{(1-2\varepsilon)^2}\rho_{s+1}^{(N)}\left(t, q_1, p_1, \ldots, q_i, p_i^*, \ldots, q_s, p_s, q_i - a\eta, p_{s+1}^*\right) -$$
$$-\rho_{s+1}^{(N)}\left(t, q_1, p_1, \ldots, q_i, p_i, \ldots, q_s, p_s, q_i + a\eta, p_{s+1}\right)\Bigg],$$

$$S_2^+\big(\eta \big| \langle\eta,(p_i - p_{s+1})\rangle \geq 0\big), \qquad \rho_s^{(N)}(t,(x)_s)\,\big|_{t=0} = \rho_s^{(N)}((x)_s), \quad 1 \leq s \leq N.$$

One should add the same boundary condition as for $D_N(t,(x)_N)$ in the Liouville equation.

In given paper we did not touch the problem of existence of solution of hierarchy (4) and existence of the thermodynamic limit.

1. Trajectories of system of hard spheres with inelastic collisions.

1.1. Dynamics. Consider in three-dimensional space $\mathcal{R}^3$ particles with mass m that are hard spheres with diameter a. Particles move freely until they touch each other and the distance between their centers is equal to a. Then they inelastically collide.

Denote position of the center of sphere by $q \in \mathcal{R}^3$ and its momentum by $p \in \mathcal{R}^3$. Let N be the number of particles of the considered system. Particles with numbers i

and j collide if $q_i - q_j = a\eta$. If their momenta before collisions are p_i and p_j and $\langle\eta, (p_i - p_j)\rangle < 0$, then after inelastic collision they become

$$\begin{aligned} p_i^* &= p_i - \varepsilon\eta\langle\eta, (p_i - p_j)\rangle, \\ p_j^* &= p_j + \varepsilon\eta\langle\eta, (p_i - p_j)\rangle \end{aligned} \tag{1.1}$$

where parameter $\frac{1}{2} < \varepsilon \leq 1$ characterizes inelastic collision, unit vector η is directed from the center of sphere with number i to the center of sphere with number j, $\langle\eta, (p_i - p_j)\rangle$ is scalar product of vectors η and $p_i - p_j$. Formulae (1.1) define a linear transformation of momenta p_i, p_j. If $\langle\eta, (p_i - p_j)\rangle > 0$ then after collision $p_i^* = p_i$, $p_j^* = p_j$.

Note that this law of inelastic collision (1.1) is true for real evolution of system with increasing time t (for dynamics forward in time). In statistical mechanics we also need an imaginary evolution of system with decreasing time (backward in time dynamics). We define the law of inelastic collision with decreasing time as inverse to (1.1) transformation. To obtain this desired transformation we consider (1.1) as an equation with respect to p_i, p_j for given p_i^*, p_j^*.

Calculating scalar product

$$\langle\eta, (p_i^* - p_j^*)\rangle = (1 - 2\varepsilon)\langle\eta, (p_i - p_j)\rangle$$

one obtains from (1.1) desired inverse to (1.1) linear transformation

$$\begin{aligned} p_i &= p_i^* + \frac{\varepsilon}{1 - 2\varepsilon}\eta\langle\eta, (p_i^* - p_j^*)\rangle, \\ p_j &= p_j^* - \frac{\varepsilon}{1 - 2\varepsilon}\eta\langle\eta, (p_i^* - p_j^*)\rangle, \langle\eta, (p_i^* - p_j^*)\rangle > 0. \end{aligned} \tag{1.2}$$

In what follows we will need only backward in time dynamics and it will be useful to change in (1.2) denotation and write instead of momenta (p_i, p_j) momenta (p_i^*, p_j^*) and vice versa. Then transformation (1.2) looks like the following one

$$\begin{aligned} p_i^* &= p_i + \frac{\varepsilon}{1 - 2\varepsilon}\eta\langle\eta, (p_i - p_j)\rangle, \\ p_j^* &= p_j - \frac{\varepsilon}{1 - 2\varepsilon}\eta\langle\eta, (p_i - p_j)\rangle, \langle\eta, (p_i - p_j)\rangle > 0. \end{aligned} \tag{1.3}$$

It follows from (1.3) that components of vectors p_i^*, p_j^* perpendicular to vector η do not change, and components parallel to vector η change according to (1.3). It is obvious that the Jacobian of transformation (1.3) J can be easily calculated:

$$J = \frac{1}{1 - 2\varepsilon}. \tag{1.4}$$

If $\langle\eta, (p_i - p_j)\rangle < 0$ momenta do not change, i.e., $p_i^* = p_i$, $p_j^* = p_j$. Let us calculate kinetic energy after collision in backward motion of particles with number i and j. We have according to (1.3)

$$p_i^{*2} + p_j^{*2} = p_i^2 + p_j^2 + 2\frac{\varepsilon - \varepsilon^2}{(1 - 2\varepsilon)^2}\left(\langle\eta, (p_i - p_j)\rangle\right)^2 \geq p_i^2 + p_j^2 \tag{1.5}$$

because $\varepsilon - \varepsilon^2 > 0$ for $\frac{1}{2} < \varepsilon < 1$.

Now calculate kinetic energy after collision in forward motion. According to (1.1) one obtains

$$p_i^{*2} + p_j^{*2} = p_i^2 + p_j^2 + 2(-\varepsilon + \varepsilon^2)\left(\langle \eta, (p_i - p_j) \rangle\right)^2 \leq p_i^2 + p_j^2 \tag{1.6}$$

because $-\varepsilon + \varepsilon^2 < 0$ for $\frac{1}{2} < \varepsilon < 1$. From (1.5), (1.6) one can see that $p_i^{*2} + p_j^{*2}$ is greater than $p_i^2 + p_j^2$ for backward motion $(t < 0)$, and $p_i^{*2} + p_j^{*2}$ is less than $p_i^2 + p_j^2$ for forward motion:

$$\begin{aligned} p_i^{*2} + p_j^{*2} &\geq p_i^2 + p_j^2, \qquad t < 0, \\ p_i^{*2} + p_j^{*2} &\leq p_i^2 + p_j^2, \qquad t > 0. \end{aligned} \tag{1.7}$$

Thus in defined above dynamics with inelastic collisions kinetic energy increases for $t < 0$ and decreases for $t > 0$. Only in the case $\langle \eta, (p_i - p_j) \rangle = 0$, one has $p_i^{*2} + p_j^{*2} = p_i^2 + p_j^2$ and kinetic energy is preserved, even for $\frac{1}{2} < \varepsilon < 1$.

1.2. Trajectory. Denote by $Q_1(-t), \ldots, Q_N(-t)$ positions of hard spheres at time $-t$, $t > 0$, by $P_1(-t), \ldots, P_N(-t)$ their momenta, and by $q_1, \ldots, q_N$ their initial positions, by $p_1, \ldots, p_N$ their initial momenta at time $t = 0$, by $(x)_N = (q_1, p_1, \ldots, q_N, p_N)$ the initial phase point. Obviously we will consider only admissible configurations, i.e., $|q_i - q_j| \geq a$ for all $(i, j) \subset (1, \ldots, N)$. As it was mentioned above, particles move freely until they touch each other and then collide and their momenta change according to (1.3).

We will neglect instantaneous collisions of three or more particles because the set of such initial positions and momenta has Lebesgue measure equal to zero. Denote by $t_{ij}((x)_N)$ the time of collision of particles with number i and j. Considered as a function of $(x)_N$, $t_{ij}((x)_N)$ is continuously differentiable outside of a certain set with Lebesgue measure equal to zero.

The trajectory $X(-t, (x)_N) = \Big(Q_1(-t, (x)_N), P_1(-t, (x)_N), \ldots, Q_N(-t, (x)_N), P_N(-t, (x)_N)\Big)$, $Q_i(-t) \equiv Q_i(-t, (x)_N)$, $P_i(-t) = P_i(-t, (x)_N)$, $i = 1, \ldots, N$, is constructed as follows. Until the first collision

$$X(-t, (x)_N) = (q_1 - p_1 t, p_1, \ldots, q_N - p_N t, p_N). \tag{1.8}$$

If at time $t_{ij}((x)_N)$ particles with numbers i and j collide, then for $t > t_{ij}(x)_N$ the trajectory $X(-t, (x)_N)$ is again given by formula (1.8), but positions and momenta of i-th and j-th particles are given by

$$q_i - p_i t_{ij}(x) - p_i^*(t - t_{ij}(x)), \quad p_i^*, \quad q_j - p_j t_{ij}(x) - p_j^*(t - t_{ij}(x)), \quad p_j^*, \tag{1.9}$$

where p_i^*, p_j^* are expressed in terms of p_i, p_j according to (1.3).

One can continue the trajectory according to (1.9) after all collisions if infinitely many collisions on finite time interval are absent. Then momenta of all particles involved in these infinite number of collisions coincide and their spheres touch each other. The corresponding set of initial phase points lie on the hyperplanes of lower dimension and has Lebesgue measure equal to zero.

It is obvious that the trajectory has the group property

$$X(-t_1 - t_2, (x)_N) = X(-t_1, X(-t_2, (x)_N)) = X(-t_2, X(-t_1, (x)_N))$$

and it satisfies the following boundary condition:

for $q_i - q_j = a\eta, \ \langle \eta, (p_i - p_j) \rangle > 0, \ (i, j) \subset (1, \dots, N)$,

$$X(-t, q_1, p_1, \dots, q_i, p_i, \dots, q_j, p_j, \dots, q_N, p_N) = \\ = X(-t, q_1, p_1, \dots, q_i, p_i^*, \dots, q_j, p_j^*, \dots, q_N, p_N); \tag{1.10}$$

if $q_i - q_j = a\eta$, but $\langle \eta, (p_i - p_j) \rangle < 0$ momenta p_i, p_j do not change. This boundary condition means that after collision particles depart and distance between them increases.

The trajectory $X(-t, (x)_N)$ is a continuously differentiable function almost everywhere with respect to its initial data $(x)_N$ and time on every time interval between collision. The detailed proof of above mentioned properties of the trajectory $X(-t, (x)_N)$ can be found in books [1, 2], after some modification connected with the inelastic character of collisions.

We summarize above formulated results in the following theorem.

Theorem I. *The trajectory $X(-t, (x)_N)$ of N hard spheres that inelastically collide exists for arbitrary time $t > 0$, is continuously differentiable with respect to initial phase points $(x)_N$ and time t on intervals between collisions, and has group property for almost all initial $(x)_N$, that belong to certain domain outside of hypersurface with Lebesgue measure equal to zero.*

Theorem I asserts that trajectories $X(-t, (x)_N)$ are well defined between times of collisions almost everywhere (a.e.) with respect to $(x)_N$. In many respects, trajectories of our system of hard spheres with inelastic collisions have the same properties as a system of hard spheres with elastic collisions. These properties were formulated in Theorem I. But trajectories of hard spheres with inelastic collisions have also certain specific properties different from those of the case of elastic collisions. One of these specific properties is that the map of the phase space induced by the shift along trajectories does not preserve the volume.

According to definition of trajectories (1.8), (1.9), the Jacobian

$$\frac{\partial(X_1(-t, (x)_N), \dots, X_N(-t, (x)_N))}{(\partial x_1, \dots, \partial x_N)} = \frac{\partial(X(-t, (x)_N))}{\partial(x)_N} \tag{1.11}$$

is equal to one if for initial point $(x)_N$ there are no collisions until time $-t$, and is equal to

$$\frac{\partial(P_1(-t, (x)_N), \dots, P_N(-t, (x)_N))}{(\partial p_1, \dots, \partial p_N)} = \Big(\frac{1}{1 - 2\varepsilon}\Big)^n \tag{1.12}$$

if there are n pair collisions for initial point $(x)_N$. The Jacobian of transformation (1.3) is equal to

$$\frac{\partial(p_i^*, p_j^*)}{\partial(p_i, p_j)} = \frac{1}{1 - 2\varepsilon}.$$

2. Evolution operator.

2.1. Definition of evolution operator. Let $f_N(x_1, \dots, x_N) = f_N((x)_N)$ be a continuous symmetric (permutation invariant) function defined on phase space $\mathcal{R}^{6N}$ of N particles and equal to zero on the set of forbidden configurations. Define at first formally operator $S_N(-t)$ as the operator of shift along the trajectory $X(-t, (x)_N)$ as follows:

$$(S_N(-t) f_N)(x_1, \dots, x_N) = \\ = f_N(X_1(-t, (x)_N), \dots, X_N(-t, (x)_N)) = f_N(X(-t, (x)_N)) \tag{2.1}$$

on admissible configurations, and

$$(S_N(-t)f_N)(x_1, \dots, x_N) = 0$$

on the set of forbidden configurations.

According to definition of the trajectory $X(-t,(x)_N)$, function $f_N(X(-t,(x)_N))$ has jumps of momenta at the time of collision $t_{ij}((x)_N)$, because momenta after collisions are different from momenta before collisions, and is again a symmetric function.

In classical statistical mechanics of systems of particles with elastic collisions function $f_N(X(-t,(x)_N))$ is proportional to the probability density of the considered system at time t in phase space. It should satisfy the law of conservation of full probability, i.e., full probability has to be independent of time. We also need to derive equation for $f_N(X(-t,(x)_N))$ and equation for the sequence of correlation functions. Therefore we impose some condition on function $f_N((x)_N)$.

We suppose that $f_N((x)_N)$ belongs to the Banach space L_N of functions equal to zero on the set of forbidden configurations, for which $|q_i - q_j| < a$ at least for one pair $(i,j) \subset (1, \dots, N)$, and Lebesgue integrable with norm

$$\|f_N\| = \int |f_N(x_1, \dots, x_N)| dx_1 \dots dx_N = \int |f_N((x)_N)| d(x)_N. \tag{2.2}$$

Denote by L_N^0 the subspace of L_N consisting of continuously differentiable functions with compact support and equal to zero in some neighbourhood of the forbidden configuration. Subspace L_N^0 is everywhere dense in L_N.

If $f_N \in L_N^0$, then $f_N(X(-t,(x)_N))$ *is a continuously differentiable function with respect to* t *and* $(x)_N$ *almost everywhere.* Indeed, the trajectory $X(-t,(x)_N)$ is a continuously differentiable function with respect to time t and initial points $(x)_N$ a.e. on time intervals between collisions. Collisions happen if $|Q_i(t,(x)_N) - Q_j(-t,(x)_N)| = a$ for some $(i,j) \subset (1, \dots, N)$, but function $f_N(X(-t,(x)_N))$ is equal to zero in some neighborhood of these hypersurfaces. Outside of these hypersurfaces trajectories are continuously differentiable with respect to time t and initial points $(x)_N$ a.e., and therefore functions $f_N(X(-t,(x)_N))$ have the same property because $f_N((x)_N) \in L_N^0$. According to definition (2.1), $f_N(X(-t,(x)_N))$ it is equal to zero on the forbidden configuration together with $f_N((x)_N) \in L_N^0$. (For more details see [1, 2].)

It is obvious that operator $S_N(-t)$ has the group property

$$S_N(-t_1 - t_2) = S_N(-t_1)S_N(-t_2) = S_N(-t_2)S_N(-t_1) \tag{2.3}$$

2.2. Properties of operator $S_N(-t)$. Consider again $f_N(X(-t,(x)_N))$ with $f_N((x)_N) \in L_N^0$ and show that it is Lebesgue integrable. Indeed it is continuous with respect to $(x)_N$ a.e., has compact support and therefore

$$\int |f_N(X(-t,(x)_N)| d(x)_N < \infty.$$

We need only to prove that $f_N(X(-t,(x)_N))$ has compact support with respect to $(x)_N$ if $f_N((x)_N)$ has compact support. If $f_N((x)_N)$ has compact support, say $\sum_{i=1}^N (q_i^2 + p_i^2) \le R$, $R > 0$, then $f_N(X(-t,(x)_N))$ has compact support $\sum_{i=1}^N \left[Q_i^2(-t,(x)_N) + P_i^2(-t,(x)_N\right] \le R$, with respect to Q_i, P_i, $i = 1, \dots, N$.

If $\sum_{i=1}^N P_i^2(-t,(x)_N) \le R$, then $\sum_{i=1}^N p_i^2 \le R$, because at each collision of i-th and j-th particles at time $0 \le \tau \le t$ one has $P_i^{*2}(-\tau,(x)_N) + P_j^{*2}(-\tau,(x)_N) \ge$

$\geq P_i^2(-\tau,(x)_N) + P_j^2(-\tau,(x)_N)$, and therefore $R \geq \sum_{i=1}^{N} P_i^2(-t,(x)_N) \geq \geq \sum_{i=1}^{N} p_i^2$.

One has $\sum_{i=1}^{N} Q_i^2(-t,(x)_N) \leq R^2$ and therefore $\sum_{i=1}^{N} q_i^2 < r$, where $r > 0$ is finite, because $Q_i(-t,(x)_N)$ is shifted from q_i at finite distance by finite $P_i(-\tau,(x)_N)$, $0 \leq \tau \leq t$.

Thus $f_N(X(-t,(x)_N))$ *has compact support with respect to* $(x)_N$, *together with* $f_N((x)_N)$.

It the case of elastic collision, operator $S_N(-t)$ is isometric, because the Jacobian (1.11) is equal to one. In our case of inelastic collision the Jacobian (1.11) is different from one for such initial $(x)_N$ that collisions occur.

If $\mathcal{D}$ is some domain in $\mathcal{R}^{6N}$ and $\mathcal{D}_{-t}$ is the image of $\mathcal{D}$ induced by shift along trajectories $X(-t,(x)_N)$, then

$$\int\limits_{\mathcal{D}} d(x)_N \neq \int\limits_{\mathcal{D}_{-t}} d(X(-t,(x)_N)) = \int\limits_{\mathcal{D}} \frac{\partial X(-t,(x)_N)}{\partial (x)_N} d(x)_N,$$

because $\dfrac{\partial(X(-\tau,(x)_N)}{\partial (x)_N} \neq 1$ and $\dfrac{\partial}{\partial \tau}\dfrac{\partial(X(-\tau,(x)_N))}{\partial (x)_N}$ is proportional to $\delta(\tau - \tau_l)$, for those $(x)_N$ that collisions occur at time $\tau = \tau_l$, $0 \leq \tau_l \leq t$, and the Jacobian has a jump at time τ_l.

Denote by $V(\mathcal{D})$ and $V(\mathcal{D}_{-t})$ the volumes of domains $\mathcal{D}$ and $\mathcal{D}_{-t}$ respectively. Then one has

$$V(\mathcal{D}_{-t}) = \int\limits_{\mathcal{D}} \frac{\partial X(-t,(x)_N)}{\partial (x)_N} d(x)_N =$$

$$= \int\limits_{\mathcal{D}} \frac{\partial X(0,(x)_N)}{\partial (x)_N} d(x)_N + \int\limits_{\mathcal{D}} \left[\int\limits_0^t \frac{\partial}{\partial \tau} \frac{\partial X(-\tau,(x)_N)}{\partial (x)_N} d\tau \right] d(x)_N =$$

$$= V(\mathcal{D}) + \int\limits_{\mathcal{D}} \left[\int\limits_0^t \frac{\partial}{\partial \tau} \frac{\partial X(-\tau,(x)_N)}{\partial (x)_N} d\tau \right] d(x)_N.$$

It follows from these formulae that contributions in $V(\mathcal{D}_{-t})$ from hypersurfaces

$$|Q_i(-\tau_l,(x)_N) - Q_j(-\tau_l,(x)_N)| = a, \quad (i,j) \subset (1,\ldots,N),$$

are finite (see for more details in Appendix A and B).

Nevertheless operator $S(-t)$ is "isometric" on L_N^0 in the following sense.

Consider function $f_N(X(-t,(x)_N))\left(\dfrac{\partial(X(-t,(x)_N))}{\partial (x)_N}\right)^2$. Later we will show that

$$\begin{gathered} \int f_N(X(-t,(x)_N))\Big(\frac{\partial(X(-t,(x)_N))}{\partial (x)_N}\Big)^2 d(x)_N = \int f_N((x)_N) d(x)_N, \\ \int |f_N(X(-t,(x)_N))|\Big(\frac{\partial(X(-t,(x)_N))}{\partial (x)_N}\Big)^2 d(x)_N = \int |f_N((x)_N)| d(x)_N. \end{gathered} \tag{2.4}$$

It is obvious that $\left.\dfrac{\partial(X(-t,(x)_N))}{\partial (x)_N}\right|_{t=0} = 1$ and it follows from (2.4) that function

$$D_N(t,(x)_N) = f_N(X(-t,(x)_N))\left(\frac{\partial(X(-t,(x)_N))}{\partial(x)_N}\right)^2, \tag{2.5}$$

that is equal to $f_N((x)_N)$ at $t = 0,$ may be considered as a probability density in the phase space of systems of hard spheres with inelastic collisions.

Function (2.5) has the following "group" property:

$$f_N(X(-t_1 - t_2,(x)_N))\left(\frac{\partial(X(-t_1 - t_2,(x)_N))}{\partial(x)_N}\right)^2 =$$

$$= f_N(X(-t_1, X(-t_2,(x)_N))\left(\frac{\partial(X(-t_1, X(-t_2,(x)_N)))}{\partial(x)_N}\right)^2 =$$

$$= f_N(X(-t_2, X(-t_1,(x)_N))\left(\frac{\partial(X(-t_2, X(-t_1,(x)_N)))}{\partial(x)_N}\right)^2, \tag{2.6}$$

$$\frac{\partial(X(-t_1, X(-t_2,(x)_N)))}{\partial(x)_N} = \frac{\partial(X(-t_1, X(-t_2,(x)_N)))}{\partial X(-t_2,(x)_N)}\frac{\partial(X(-t_2,(x)_N))}{\partial(x)_N},$$

$$\frac{\partial(X(-t_2, X(-t_1,(x)_N)))}{\partial(x)_N} = \frac{\partial(X(-t_2, X(-t_1,(x)_N)))}{\partial X(-t_1,(x)_N)}\frac{\partial(X(-t_1,(x)_N))}{\partial(x)_N}.$$

Note that function (2.5) is continuously differentiable together with functions $f_N(X(-t,(x)_N)),\ f_N((x)_N) \in L^0_N.$ Indeed function $f_N(X(-t,(x)_N))$ is continuously differentiable with respect to time t and initial data $(x)_N$ a.e. The Jacobian $\dfrac{\partial(X(-t,(x)_N))}{\partial(x)_N}$ is a constant function of time on time intervals between collisions and has a jump at time of collisions, but function $f_N(X(-t,(x)_N))$ is equal to zero in a neighbourhood of time of collisions and, therefore, *function* $D_N(t,(x)_N)$ *is continuously differentiable as well as* $f_N(X(-t,(x)_N))$. The proof is presented below.

2.3. Differential equation for $\boldsymbol{D_N(t,(x)_N)}$**.** Let us show that the function $D_N(t,(x)_N)$ is differentiable with respect to time. It is product of two functions $f_N(X(-t,(x)_N))$ and $\dfrac{\partial X(-t,(x)_N)}{\partial(x)_N}$. One obtains

$$\frac{\partial}{\partial t}D_N(t,(x)_N) =$$
$$= \left[\frac{\partial}{\partial t}f_N(X(-t,(x)_N))\right]\left(\frac{\partial X(-t,(x)_N)}{\partial(x)_N}\right)^2 +$$
$$+ f_N(X(-t,(x)_N))\left[\frac{\partial}{\partial t}\left(\frac{\partial X(-t,(x)_N)}{\partial(x)_N}\right)^2\right]. \tag{2.7}$$

Now calculate derivatives of $f_N(X(-t,(x)_N))$ for $f_N((x)_N) \in L^0_N$. Using group property of $S_N(-t)$ (2.3) one obtains (see details in [1, 2])

$$\frac{\partial}{\partial t}f_N(X(-t,(x)_N)) =$$
$$= \lim_{\Delta t \to 0}\left[S_N(-t)\frac{(S_N(-\Delta t) - I)f_N((x)_N)}{\Delta t}\right] =$$

$$= S_N(-t)\left[-\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} f_N((x)_N)\right] =$$

$$= -\sum_{i=1}^{N} P_i(-t,(x)_N)\frac{\partial}{\partial Q_i(-t,(x)_N)} f_N(X(-t,(x)_N)),$$

$$\frac{\partial}{\partial t} f_N(X(-t,(x)_N)) = \lim_{\Delta t\to 0}\frac{S_N(-\Delta t)-I}{\Delta t} S_N(-t) f_N((x)_N) = \tag{2.8}$$

$$= -\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} f_N(X(-t,(x)_N)).$$

Now explain derivation of formulas (2.8).

One has

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}(S_N(-\Delta t)-I) f_N((x)_N) = -\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} f_N((x)_N)$$

on set $|q_i - q_j| > a,\ (i,j) \subset (1,\ldots,N)$ because $f_N \in L_N^0$ and it is equal to zero on some neighbourhood of forbidden configuration.

The trajectory $X(-t,(x)_N)$ at $q_i - q_j = a\eta,\ \langle \eta, (p_i - p_j)\rangle > 0$ has the jump at $t = +0,\ X(-0,(x)_N) - X(0,(x)_N) = (x)_N^* - (x)_N,\ (x)_N^* = (q_1, p_1, \ldots, q_i, p_i^*, \ldots, q_j, p_j^*, \ldots, q_N, p_N)$, but function $f_N \in L_N^0$ is equal to zero in some neighbourhood of such points, therefore $f_N((x)_N^*) = f_N((x)_N) = 0$, i.e., function $f_N(X(-t,x))$ has not jump at $t = +0$. At time $t > 0\ X(-t,(x)_N) = X(-t,(x)_N^*)$ and $f_N(X(-t,(x)_N)) = f_N(X(-t,(x)_N^*))$ for $q_i - q_j = a\eta,\ \langle \eta, (p_i - p_j)\rangle > 0$.

Note that $f_N(X(-t,(x)_N))$ may be different from zero with respect to $(x)_N$ on this neighbourhood of forbidden configuration where $f_N((x)_N)$ is equal to zero. Therefore

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}(S_N(-\Delta t)-I) f_N(X(-t,(x)_N)) = -\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} f_N(X(-t,(x)_N))$$

with boundary condition according to which at $q_i - q_j = a\eta,\ \langle \eta, p_i - p_j\rangle > 0,\ (i,j) \subset (1,\ldots,N)$ in expression $-\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} f_N(X(-t,(x)_N))$ momenta $p_i,\ p_j$ should be replaced by $p_i^*,\ p_j^*$.

Thus we obtain two expressions for $\frac{\partial}{\partial t} f_N(X(-t,(x)_N))$, namely

$$\frac{\partial}{\partial t} f_N(X(-t,(x)_N)) = -\sum_{i=1}^{N} P_i(-t,(x)_N)\frac{\partial}{\partial Q_i(-t,(x)_N)} f_N(X(-t,(x)_N)), \tag{2.8a}$$

and

$$\frac{\partial}{\partial t} f_N(X(-t,(x)_N)) = -\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} f_N(X(-t,(x)_N)) =$$

$$= -\sum_{j=1}^{N}\left[\frac{\partial f_N(X(-t,(x)_N))}{\partial Q_j(-t,(x)_N)}\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} Q_j(-t,(x)_N) + \right.$$

$$\left. + \frac{\partial f_N(X(-t,(x)_N))}{\partial P_j(-t,(x)_N)}\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} P_j(-t,(x)_N)\right]. \tag{2.8b}$$

The right-hand side of (2.8a) is a continuous function with respect to $(x)_N$ a.e., because $f_N((x)_N) \in L^0_N$ and $P_i(-t,(x)_N)$, $i = 1,\ldots,N$, are continuous functions of time and $(x)_N$ a.e. on time intervals between collisions and have jumps only at time of collisions, but function $f_N(X(-t,(x)_N))$ is equal to zero in some neighbourhood of times of collisions. $Q_i(-t,(x)_N)$, $i = 1,\ldots,N$, are continuous functions of time and $(x)_N$ a.e. on time intervals between collisions.

The right-hand side of (2.8b) is also a continuous function with respect to $(x)_N$ a.e., because $(f_N(x)_N) \in L^0_N$, and $Q_j(-t,(x)_N)$, $P_j(-t,(x)_N)$, $1 \le j \le N$, are continuously differentiable functions with respect to $(x)_N$ a.e. on time intervals between collisions, but functions $\dfrac{\partial f_N(X(-t,(x)_N))}{\partial Q_j(-t,(x)_N)}$, $\dfrac{\partial f_N(X(-t,(x)_N))}{\partial P_i(-t,(x)_N)}$ are equal to zero in some neighbourhood of times of collisions.

Note that in the right-hand side of (2.8b) we have the *following boundary conditions*:

$$\text{at} \quad q_i - q_j = a\eta, \quad |\eta| = 1, \quad \langle \eta, (p_i - p_j)\rangle > 0, \quad (i,j) \subset (1,\ldots,N),$$

the expressions $-\sum_{i=1}^N p_i \dfrac{\partial}{\partial q_i} f_N(X(-t,(x)_N))$, $f_N(X(-t,(x)_N))$ should be replaced by:

$$-\sum_{i=1}^N p_i \frac{\partial}{\partial q_i} f_N(X(-t,(x)_N))|_{p_i=p_i^*, p_j=p_j^*}, \qquad f_N(X(-t,(x)_N))|_{p_i=p_i^*, p_j=p_j^*}. \quad (2.9)$$

At $\langle \eta, (p_i - p_j)\rangle < 0$, on the contrary, momenta do not change. These boundary conditions follow from the definition of the trajectory at $q_i - q_j = a\eta$, $\langle \eta, (p_i - p_j)\rangle > > 0$ (1.10), namely $X(-t,(x)_N) = X(-t,(x)_N)\ |_{p_i=p_i^*,\ p_j=p_j^*}$, and the fact that $f_N((x)_N) \in L^0_N$ (see [1, 2]).

In the right-hand side of (2.8a) the analogous boundary conditions are absent, because function $f_N(X(-t,(x)_N))$ is equal to zero when $|Q_i(-t,(x)_N) - Q_j(-t,(x)_N)| = = a$, $(i,j) \subset (1,\ldots,N)$, and the term $\dfrac{\partial}{\partial t} P_i(-t,(x)_N) \dfrac{\partial f_N(X(-t,(x)_N))}{\partial P_i(-t,(x)_N)}$ is equal to zero, because $f_N(X(-t,(x)_N))$ is equal to zero where $P_i(-t,(x)_N)$ have jumps, $i = 1,\ldots,N$.

At first sight, according to the boundary condition (2.9) at $q_i - q_j = a\eta$, $\langle \eta, (p_i - -p_j)\rangle > 0$, there are jumps in the right-hand side of (2.8b), because momenta (p_i, p_j) are replaced by (p_i^*, p_j^*). We show that it is not true.

As it was mentioned above, the right-hand side of (2.8a) is a continuous function of $(x)_N$ a.e. on the entire phase space of admissible configurations, i.e., $|q_i - q_j| \ge a$ for all pairs $(i,j) \subset (1,\ldots,N)$. The right-hand side of (2.8b) together with *boundary condition* (2.9) identically coincide with the right-hand side of (2.8a) and therefore *it is also a continuous function of* $(x)_N$ *a.e. on the entire phase space of admissible configurations.*

Note that only the sum $-\sum_{i=1}^N p_i \dfrac{\partial}{\partial q_i} f_N(X(-t,(x)_N))$ is continuous on admissible configuration. The each term $-p_i \dfrac{\partial}{\partial q_i} f_N(X(-t,(x)_N))$, $-p_j \dfrac{\partial}{\partial q_j} f_N(X(-t,(x)_N))$ has jumps for $q_i - q_j = a\eta$, $\langle \eta, (p_i - p_j)\rangle > 0$ because p_i, p_j should be replaced by p_i^*, p_j^* and $\dfrac{\partial}{\partial q_i} f_N(X(-t,(x)_N))$, $\dfrac{\partial}{\partial q_j} f_N(X(-t,(x)_N))$ are continuous with respect to $(x)_N$ on admissible configuration (see (2.8b)).

Now consider the second term in the right-hand side of (2.7) and show that it is equal to zero. Indeed, the Jacobian $\dfrac{\partial X(-t,(x)_N)}{\partial (x)_N}$ for given fixed $(x)_N$ is a constant function of t and at time of collisions has a jumps. Therefore $\dfrac{\partial}{\partial t}\dfrac{\partial X(-t,(x)_N)}{\partial (x)_N}$ is equal to zero on time intervals between collisions, but the function $f(X(-t,(x)_N))$ is equal to zero in neighbourhoods of times of collisions and, as result,

$$f_N(X(-t,(x)_N))\left[\frac{\partial}{\partial t}\Big(\frac{\partial X(-t,(x)_N)}{\partial (x)_N}\Big)^2\right] \equiv 0,$$

i.e., the second term in the right-hand side of (2.7) is equal to zero.

Taking into account above obtained results we have

$$\begin{aligned}
\frac{\partial}{\partial t} D_N(t,(x)_N) &= \left[\frac{\partial}{\partial t} f_N(X(-t,(x)_N))\right]\Big(\frac{\partial X(-t,(x)_N)}{\partial (x)_N}\Big)^2 = \\
= \left[-\sum_{i=1}^{N} P_i(-t,(x)_N)\frac{\partial}{\partial Q_i(-t,(x)_N)} f_N(X(-t,(x)_N))\right]\Big(\frac{\partial X(-t,(x)_N)}{\partial (x)_N}\Big)^2 &= \\
= \left[-\sum_{i=1}^{N} p_i\frac{\partial}{\partial q_i} f_N(X(-t,(x)_N))\right]\Big(\frac{\partial X(-t,(x)_N)}{\partial (x)_N}\Big)^2 &= \\
= -\sum_{i=1}^{N} p_i\frac{\partial}{\partial q_i}\left[f_N(X(-t,(x)_N))\Big(\frac{\partial X(-t,(x)_N)}{\partial (x)_N}\Big)^2\right] &= \\
= -\sum_{i=1}^{N} p_i\frac{\partial}{\partial q_i} D_N(t,(x)_N). \qquad (2.10)
\end{aligned}$$

The last equality in (2.10) follows from the fact that the Jacobian $\dfrac{\partial X(-t,(x)_N)}{\partial (x)_N}$ is constant (piece-wise constant) everywhere with respect to $(x)_N$ excluding points $(x)_N$ at which there are collisions at time t, but the function $f_N(X(-t,(x)_N))$ is equal to zero in neighbourhoods of such points and, therefore,

$$f_N(X(-t,(x)_N))\left(-\sum_{i=1}^{N} p_i\frac{\partial}{\partial q_i}\frac{\partial X(-t,(x)_N)}{\partial (x)_N}\right)$$

is equal to zero at such points. Remembering that

$$D_N(t,(x)_N) = f_N(X(-t,(x)_N))\Big(\frac{\partial X(-t,(x)_N)}{\partial (x)_N}\Big)^2$$

one obtains for $\eta \in S_+^2(\langle\eta,(p_i-p_j)\rangle > 0)$

$$\begin{aligned}
&D_N(-t,x_1,\ldots,q_i,p_i,\ldots,q_i-a\eta,p_j,\ldots,x_N) = \\
&= f_N(X(-t,x_1,\ldots,q_i,p_i,\ldots,q_i-a\eta,p_j,\ldots,x_N)) \times \\
&\times\Big(\frac{\partial X(-t,x_1,\ldots,q_i,p_i,\ldots,q_i-a\eta,p_j,\ldots,x_N)}{\partial (x)_N}\Big)^2 = \\
&= f_N(X(-t,x_1,\ldots,q_i,p_i^*,\ldots,q_i-a\eta,p_j^*,\ldots,x_N)) \times \\
&\times\Big(\frac{\partial X(-t-0,x_1,\ldots,q_i,p_i^*,\ldots,q_i-a\eta,p_j^*,\ldots,x_N)}{\partial (x)_N^*} \times
\end{aligned}$$

$$\times \frac{\partial X(-0, x_1, \ldots, q_i, p_i^*, \ldots, q_i - a\eta, p_j^*, \ldots, x_N)}{\partial (x)_N}\Bigg)^2 =$$

$$= f_N(X(-t-0, x_1, \ldots, q_i, p_i^*, \ldots, q_i - a\eta, p_j^*, \ldots, x_N)) \times$$

$$\times \left(\frac{\partial X(-t-0, x_1, \ldots, q_i, p_i^*, \ldots, q_i - a\eta, p_j^*, \ldots, x_N)}{\partial (x)_N^*}\right)^2 \frac{1}{(1-2\varepsilon)^2} =$$

$$= D_N(-t, x_1, \ldots, q_i, p_i^*, \ldots, q_i - a\eta, p_j^*, \ldots, x_N)\frac{1}{(1-2\varepsilon)^2}. \tag{2.11}$$

In deriving (2.11) we took into account that $f_N(X(-t,(x)_N)) = f_N(X(-t,(x)_N^*))$ for $q_i - q_j = a\eta$, $\langle \eta, (p_i - p_j)\rangle > 0$ and used that the Jacobian $\dfrac{\partial X(-t,(x)_N)}{\partial (x)_N}$ can be calculated as product of Jacobians on consecutive time intervals between collisions and extract Jacobian that corresponds to collisions of i-th and j-th particles at time $t = +0$. The last Jacobian is equal to $\dfrac{1}{1-2\varepsilon}$.

We must add to (2.10) the following *boundary condition*: at

$$q_i - q_j = a\eta, \quad |\, \eta \,| = 1, \quad \langle \eta, (p_i - p_j)\rangle > 0, \quad (i,j) \subset (1, \ldots, N),$$

expressions

$$-\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} D_N(t,(x)_N), \quad D_N(t,(x)_N)$$

should be replaced by

$$-\frac{1}{(1-2\varepsilon)^2}\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} D_N(t,(x)_N)\,|_{p_i = p_i^*, p_j = p_j^*},$$

$$\frac{1}{(1-2\varepsilon)^2} D_N(t,(x)_N)\,|_{p_i = p_i^*, p_j = p_j^*}, \tag{2.12}$$

and at $\langle \eta, (p_i - p_j)\rangle < 0$ momenta (p_i, p_j) do not change.

The boundary condition (2.12) for $D_N(t,(x)_N)$ follows directly from the boundary condition (2.9) for $f_N(X(-t,(x)_N))$, $f_N((x)_N) \in L_N^0$, and from equality (2.11).

Obtained results can be summarized in the following fundamental theorem.

Theorem II. *The probability density on phase space of system of hard spheres with inelastic collisions* $D_N(t,(x)_N)$ *is a differentiable function with respect to time t and* $(x)_N$ *a.e., and satisfies the Liouville equation*

$$\frac{\partial}{\partial t} D_N(t,(x)_N) = -\sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} D_N(t,(x)_N) \tag{2.13}$$

with boundary condition (2.12) *and initial condition* $D_N(t,(x)_N)|_{t=0} = D_N(0,(x)_N) = = f_N((x)_N)$ $\left(\dfrac{\partial X(-t,(x)_N)}{\partial (x)_N}\Bigg|_{t=0} = 1,\ \textit{because}\ X(-t,(x)_N)|_{t=0} = (x)_N\right)$.

Note that the right-hand side of (2.13) together with boundary conditions (2.12) is a continuous function on phase space of admissible configurations a.e., as it follows from the second expression in the right-hand side of (2.10). Indeed it was already shown that

$$-\sum_{i=1}^{N} P_i(-t,(x)_N) \frac{\partial}{\partial Q_i(-t,(x)_N)} f_N(X(-t,(x)_N))$$

is a continuous function of $(x)_N$ on admissible configuration of phase space a.e. The Jacobian $\dfrac{\partial X(-t,(x)_N)}{\partial (x)_N}$ has jumps only at such points $(x)_N$ that there are collisions at time $t,$ but above written multiplier is equal to zero in neighbourhoods of these points, and therefore expression

$$-\sum_{i=1}^{N} P_i(-t,(x)_N)\frac{\partial}{\partial Q_i(-t,(x)_N)} f_N(X(-t,(x)_N))\left(\frac{\partial X(-t,(x)_N)}{\partial (x)_N}\right)^2$$

has desired property of continuity. Expression $-\sum_{i=1}^{N} p_i \dfrac{\partial}{\partial q_i} D_N(t,(x)_N),$ together with boundary condition (2.12), coincide with this continuous function and has the same property of continuity a.e. on the entire phase space of admissible configurations.

Remark. One can impose some additional conditions on functions $f_N((x)_N)\in L_N^0$ in order to make function $D_N(t,(x)_N)$ continuous everywhere on admissible configurations in phase space.

Namely, we restrict ourselves with functions $f_N((x)_N)\in L_N^0$ also equal to zero in neighbourhoods of the hyperplanes where three or more particles collide instantaneously, times of collisions become infinite and the number of collisions on finite time interval is infinite. (The trajectories after instantaneous collisions of three or more particles are defined as the same as before collisions but in opposite directions.) Obviously this set of functions is again everywhere dense in L_N. We continue denoting it by L_N^0. Functions $D_N(t,(x)_N)$ that correspond to such $f_N((x)_N)$ are continuous with respect to $(x)_N$ everywhere on phase space and for each time t.

3. Equation for sequence of correlation functions.

3.1. Definition of correlation functions. We will use commonly accepted definition of correlation function. Namely, correlation functions $\rho_s^{(N)}(t,x_1,\ldots,x_s)$ in the framework of the canonical ensemble is defined through the probability density $D_N(t,x_1,\ldots$ $\ldots,x_N)$ as follows:

$$\rho_s^{(N)}(t,x_1,\ldots,x_s)=$$
$$= N(N-1)\ldots(N-s+1)\int D_N(t,x_1,\ldots,x_s,x_{s+1},\ldots,x_N)dx_{s+1}\ldots dx_N, \tag{3.1}$$
$$1\le s\le N.$$

We integrate in (3.1) over entire phase space of particles with numbers $s+1,\ldots,N,$ but function $D_N(t,(x)_N)$ is equal to zero on forbidden configurations, and actually integration in (3.1) is carried out over the admissible configuration $|q_i-q_j|\ge a,\ (i,j)\subset$ $\subset(1,\ldots,N)$. It is supposed that initial probability density $D_N(0,(x)_N)=f_N((x)_N)$ is normalized to unity:

$$\int D_N(0,(x)_N)d(x)_N=\int f_N((x)_N)d(x)_N=1.$$

3.2. Equation for correlation function. In order to derive equations for correlation functions we differentiate both sides of (3.1) with respect to time and use in the right-hand side of (3.1) the Liouville equation (2.12). One obtains

$$\frac{\partial}{\partial t}\rho_s^{(N)}(t,x_1,\dots,x_s) =$$
$$= N(N-1)\cdots(N-s+1)\times$$
$$\times\int\left(-\sum_{i=1}^{N}p_i\frac{\partial}{\partial q_i}D_N(t,x_1,\dots,x_s,x_{s+1},\dots,x_N)\right)dx_{s+1}\dots dx_N. \tag{3.2}$$

Now we are in the same situation as for system of N hard spheres with elastic collisions, and we obtain the following hierarchy of equations [1–3]:

$$\frac{\partial}{\partial t}\rho_s^{(N)}(t,x_1,\dots,x_s) = -\sum_{i=1}^{s}p_i\frac{\partial}{\partial q_i}\rho_s^{(N)}(t,x_1,\dots,x_s) +$$
$$+a^2\sum_{i=1}^{s}\int dp_{s+1}\int\limits_{S^2}d\eta\langle\eta,(p_i-p_{s+1})\rangle\rho_{s+1}^{(N)}(t,x_1,\dots,x_s,q_i-a\eta,p_{s+1}) +$$
$$+\frac{1}{2}a^2\int dx_{s+1}\int dp_{s+2}\int\limits_{S^2}d\eta\langle\eta,(p_{s+1}-p_{s+2})\rangle\times$$
$$\times\rho_{s+2}^{(N)}(t,x_1,\dots,x_{s+1},q_{s+1}-a\eta,p_{s+2}),\quad 1\le s\le N, \tag{3.3}$$

where η is unit vector and S^2 is unit sphere.

Now split spheres S^2 in the second and third terms in the right-hand side of (3.3) into two parts, $S^2 = S^2_+\cup S^2_-$, where

$$S^2_+(\eta|\langle\eta,(p_i-p_{s+1})\rangle>0),\qquad S^2_-(\eta|\langle\eta,(p_i-p_{s+1})\rangle<0),\quad i=1,\dots,s,$$

and

$$S^2_+(\eta|\langle\eta,(p_{s+1}-p_{s+2})\rangle>0),\qquad S^2_-(\eta|\langle\eta,(p_{s+1}-p_{s+2})\rangle<0).$$

It follows from (2.11) that correlation functions satisfy the following boundary conditions:

$$\begin{gathered}\rho_{s+1}^{(N)}(t,x_1,\dots,q_i,p_i,\dots,x_s,q_i-a\eta,p_{s+1}) =\\ = \rho_{s+1}^{(N)}(t,x_1,\dots,q_i,p_i^*,\dots,x_s,q_i-a\eta,p_{s+1}^*)\frac{1}{(1-2\varepsilon)^2},\\ \rho_{s+2}^{(N)}(t,x_1,\dots,x_s,q_{s+1},p_{s+1},q_{s+1}-a\eta,p_{s+2}) =\\ = \rho_{s+2}^{(N)}(t,x_1,\dots,x_s,q_{s+1},p_{s+1}^*,q_{s+1}-a\eta,p_{s+2}^*)\frac{1}{(1-2\varepsilon)^2}\end{gathered} \tag{3.4}$$

for $\langle\eta,(p_i-p_{s+1})\rangle>0$ and $\langle\eta,(p_{s+1}-p_{s+2})\rangle>0$ correspondingly.

Show that the third term in the right-hand side of (3.3) is equal to zero. To this aim, represent it as follows using (3.4):

$$\frac{a^2}{2}\int dq_{s+1}\int dp_{s+1}\int dp_{s+2}\Bigg[\int\limits_{S^2_+}d\eta\big|\langle\eta,(p_{s+1}-p_{s+2})\rangle\big|\times$$
$$\times\rho_{s+2}^{(N)}(t,x_1,\dots,q_{s+1},p_{s+1}^*,q_{s+1}-a\eta,p_{s+2}^*)\frac{1}{(1-2\varepsilon)^2} -$$
$$-\int\limits_{S^2_-}d\eta|\langle\eta,(p_{s+1}-p_{s+2})\rangle|\rho_{s+2}^{(N)}(t,x_1,\dots,q_{+1},p_{s+1},q_{s+1}-a\eta,p_{s+2})\Bigg]. \tag{3.5}$$

In the first term we use as new variables of integration momenta p^*_{s+1}, p^*_{s+2}. Taking into account that $\langle\eta, (p^*_{s+1} - p^*_{s+2})\rangle = \dfrac{1}{1-2\varepsilon}\langle\eta, (p_{s+1} - p_{s+2})\rangle < 0$ for $\langle\eta, (p_{s+1} - p_{s+2})\rangle > 0$, $\dfrac{1}{2} < \varepsilon < 1$, we have $\eta \in S^2_-$ with respect to the variables p^*_{s+1}, p^*_{s+2}. We have also $dp_{s+1}dp_{s+2}\left|\dfrac{1}{1-2\varepsilon}\right| = dp^*_{s+1}dp^*_{s+2}$. (Note that we used the constant Jacobian in momentum space equal to $\left|\dfrac{1}{1-2\varepsilon}\right|$ and take into account that linear transformation (1.3) maps domain $\langle\eta, (p_{s+1}-p_{s+2})\rangle > 0$ into domain $\langle\eta, (p^*_{s+1}-p^*_{s+2})\rangle < 0$.)

Therefore the first term is equal to

$$\frac{a^2}{2}\int dq_{s+1}\int dp^*_{s+1}\int dp^*_{s+2}\int_{S^2_-} d\eta|\langle\eta, (p^*_{s+1} - p^*_{s+2})\rangle| \times$$
$$\times\rho^{(N)}_{s+2}(t, x_1, \ldots, q_{s+1}, p^*_{s+1}, q_{s+1} - a\eta, p^*_{s+2})$$

and it cancels with the second term.

Now split spheres S_2 into two parts, $S^2_+(\eta\,|\,\langle\eta, (p_i - p_{s+1})\rangle > 0)$, $S^2_-(\eta\,|\,\langle\eta, (p_i - p_{s+1})\rangle < 0)$, change the vector $\eta \in S^2_-$ to vector $-\eta \in S^2_+$, and use for $\eta \in S^2_+$ the boundary conditions (3.4)

$$\rho^{(N)}_{s+1}(t, x_1, \ldots, q_i, p_i, \ldots, q_i - a\eta, p_{s+1}) =$$
$$= \rho^{(N)}_{s+1}(t, x_1, \ldots, q_i, p^*_i, \ldots, q_i - a\eta, p_{s+1})\frac{1}{(1-2\varepsilon)^2}$$

in the second term of the right-hand side of (3.3).

Finally, taking this into account, hierarchy (3.3) takes the following form:

$$\frac{\partial}{\partial t}\rho^{(N)}_s(t, x_1, \ldots, x_s) = -\sum_{i=1}^{s} p_i\frac{\partial}{\partial q_i}\rho^{(N)}_s(t, x_1, \ldots, x_s) +$$
$$+ a^2\sum_{i=1}^{s}\int dp_{s+1}\int_{S^2_+} d\eta\langle\eta, (p_i - p_{s+1})\rangle \times$$
$$\times\left[\frac{1}{(1-2\varepsilon)^2}\rho^{(N)}_{s+1}(t, x_1, \ldots, q_i, p^*_i, \ldots, q_i - a\eta, p^*_{s+1}) -\right.$$
$$\left.-\rho^{(N)}_{s+1}(t, x_1, \ldots, q_i, p_i, \ldots, q_i + a\eta, p_{s+1})\right], \quad N \geq s \geq 1, \tag{3.6}$$

with the same boundary condition at $q_i - q_j = a\eta$ for $-\sum_{i=1}^{s} p_i\dfrac{\partial}{\partial q_i}\rho^{(N)}_s(t, x_1, \ldots, x_s)$, $\rho^{(N)}_s(t, x_1, \ldots, x_s)$ as for the Liouville equation (2.12) for $D_s(t, x_1, \ldots, x_s)$.

(In the term $\rho_{s+1}(t, x_1, \ldots, q_i, p_i, \ldots, q_i - a\eta, p_{s+1})$, with $\eta \in S^2_-$, one uses a new $\eta' = -\eta$, $\eta' \in S^2_+$.) We have also initial condition $\rho^{(N)}_s(t, x_1, \ldots, x_s)|_{t=0} = \rho^{(N)}_s(0, x_1, \ldots, x_s)$, $s \geq 1$.

Consider equation for $\rho^{(N)}_1(t, x_1)$

$$\frac{\partial}{\partial t}\rho^{(N)}_1(t, x_1) = -p_1\frac{\partial}{\partial q_1}\rho^{(N)}_1(t, x_1) + a^2\int dp_2\int_{S^2_+} d\eta\langle\eta, (p_1 - p_2)\rangle \times$$

$$\times\left[\frac{1}{(1-2\varepsilon)^2}\rho_2^{(N)}(t,q_1,p_1^*,q_1-a\eta,p_2^*)-\rho_2^{(N)}(t,q_1,p_1,q_1+a\eta,p_2)\right]$$

and integrate it with respect to x_1 over the entire phase space. Using the same tricks as in proof that the term with ρ_{s+2} is zero and supposing that $\lim\limits_{|q_1|\to\infty}\rho_1^{(N)}(t,q_1,p_1)=0,$ one obtains

$$\frac{\partial}{\partial t}\int\rho_1^{(N)}(t,x_1)dx_1=0. \tag{3.7}$$

This means that $\int\rho_1^{(N)}(t,x_1)dx_1$ does not depend on $t,$ i.e.,

$$\int\rho_1^{(N)}(t,x_1)dx_1=\int\rho_1^{(N)}(0,x_1)dx_1.$$

Taking into account that, according to definition (3.1),

$$\rho_1^{(N)}(t,(x)_1)=N\int D_N(t,x_1,x_2,\ldots,x_N)dx_2\ldots dx_N$$

one obtains the law of conservation of full probability

$$\int D_N(t,x_1,\ldots,x_N)dx_1\ldots dx_N=\int D_N(0,x_1,\ldots,x_N)dx_1\ldots dx_N. \tag{3.8}$$

Summarize the obtained above results in the following theorem.

Theorem III. *The sequence of correlation functions* $\rho_s^{(N)}(t,x_1,\ldots,x_s)$ (3.1)*,* $1\leq$ $\leq s\leq N,$ *satisfies the hierarchy of equations* (3.6) *with boundary and initial condition and the probability density* $D_N(t,x_1,\ldots,x_N)$ (2.5) *satisfies the law of conservation of full probability* (3.8).

Remark. If one introduces the probability density by formula

$$D_N(t,(x)_N)=f_N(X(-t,(x)_N)\left|\left[\frac{\partial X(-t,(x)_N)}{\partial(x)_N}\right]\right|^n \tag{3.9}$$

for $n\geq1,$ $n\neq2,$ then the sequence of correlation functions (3.1) satisfies the following hierarchy:

$$\frac{\partial}{\partial t}\rho_s^{(N)}(t,x_1,\ldots,x_s)=-\sum_{i=1}^s p_i\frac{\partial}{\partial q_i}\rho_s^{(N)}(t,x_1,\ldots,x_s)+$$
$$+a^2\sum_{i=1}^s\int dp_{s+1}\int\limits_{S_+^2}d\eta\langle\eta,(p_i-p_{s+1})\rangle\times$$
$$\times\left[\left|\frac{1}{(1-2\varepsilon)^n}\right|\rho_{s+1}^{(N)}(t,x_1,\ldots,q_i,p_i^*,q_i-a\eta,p_{s+1}^*)-\right.$$
$$\left.-\rho_{s+1}^{(N)}(t,x_1,\ldots,q_i,p_i,\ldots,q_i+a\eta,p_{s+1})\right]+$$
$$+\frac{1}{2}a^2\int dx_{s+1}\int dp_{s+1}\int\limits_{S_+^2}d\eta\langle\eta,(p_{s+1}-p_{s+2})\rangle\times$$
$$\times\left[\left|\frac{1}{(1-2\varepsilon)^n}\right|\rho_{s+2}^{(N)}(t,x_1,\ldots,q_{s+1},p_{s+1}^*,q_{s+1}-a\eta,p_{s+2}^*)-\right.$$

$$-\rho_{s+2}^{(N)}(t, x_1, \ldots, q_{s+1}, p_{s+1}, q_{s+1} + a\eta, p_{s+2})\Bigg]. \tag{3.10}$$

The third term in the right-hand side of (3.10) is different from zero, because the calculation used in the case $n = 2$ is not true for $n \neq 2$. After the change of integration variables from (p_{s+1}, p_{s+2}) to (p^*_{s+1}, p^*_{s+2}) in the first term with ρ_{s+2} there will be left the multiplier $\left|\dfrac{1}{(1-2\varepsilon)^{n-2}}\right|$, and this term does not cancel with the second term.

For the probability density $D_N(t,(x)_N)$ (3.9) with $n \neq 2$ the law of conservation of full probability (3.8) is not true. *This means that for the system of hard spheres with inelastic collisions the unique "candidate" for the probability density is function* $D_N(t,(x)_N)$ (3.9) *with* $n = 2,$ *i.e.,* $D_N(t,(x)_N)$ *defined according to* (2.5).

3.3. Boundary conditions for correlation functions. According to boundary conditions for function $D_N(t,(x)_N)$ and for Liouville equation (2.11), we have boundary conditions (3.4) for correlation functions and the following boundary conditions for the BBGKY hierarchy (3.6): in expression

$$-\sum_{i=1}^{S} p_i \frac{\partial}{\partial q_i} \rho_s^{(N)}(t, x_1, \ldots, x_s)$$

for $q_i - q_j - a\eta = 0,\ \eta \in S^2_+,\ (i,j) \subset (1, \ldots, s),$ momenta p_i and p_j should be replaced by $p_i^*,\ p_j^*$ (1.3) and $\rho_s^{(N)}$ by $\dfrac{1}{(1-2\varepsilon)^2}\rho_s^{(N)}$.

At first sight momenta $p_i,\ p_{s+1},\ i = 1, \ldots, s,$ in $\langle \eta, (p_i - p_{s+1})\rangle$ in the first term of the integral in the right-hand side of equation (3.6) should be also replaced by p_i^*, p^*_{s+1}. It is not true. The reason is that under integral sign in (3.2) behavior of integrand on hypersurfaces of lower dimension can be neglected. But we prefer to explain this assertion on a very simple example of system of two spheres (rods) in one-dimensional case.

We have Liouville equation

$$\frac{\partial D_2(t, x_1, x_2)}{\partial t} = -\Big(p_1 \frac{\partial}{\partial q_1} + p_2 \frac{\partial}{\partial q_2}\Big) D_2(t, x_1, x_2)$$

with boundary condition: for $q_1 - q_2 - a\eta = 0,\ \langle \eta, (p_1 - p_2)\rangle > 0$

$$D_2(t, q_1, p_1, q_2, p_2) = \frac{1}{(1-2\varepsilon)^2} D_2(t, q_1, p_1^*, q_2, p_2^*),$$

$$\Big(-p_1 \frac{\partial}{\partial q_1} - p_2 \frac{\partial}{\partial q_2}\Big) D_2(t, q_1, p_1, q_2, p_2) =$$

$$= \frac{1}{(1-2\varepsilon)^2}\Big(-p_1{}^* \frac{\partial}{\partial q_1} - p_2^* \frac{\partial}{\partial q_2}\Big) D_2(t, q_1, p_1^*, q_2, p_2^*).$$

We have, following [3] and taking into account that $\dfrac{\partial}{\partial t} D_2(t, x_1, x_2)$ is different from zero on admissible configurations $|q_1 - q_2| \geq a$ and continuous with respect to (x_1, x_2),

$$\int \frac{\partial}{\partial t} D_2(t, x_1, x_2) dq_2 dp_2 =$$

$$= \int \Big(-p_1 \frac{\partial}{\partial q_1} - p_2 \frac{\partial}{\partial q_2}\Big) D_2(t, q_1, p_1, q_2, p_2) dq_2 dp_2 =$$

$$= \lim_{\varepsilon \to 0} \int \Bigg\{ \Bigg(\int\limits_{-\infty}^{q_1 - a - \varepsilon} dq_2 + \int\limits_{q_1 + a + \varepsilon}^{\infty} dq_2 \Bigg) \times$$

$$\times\left(-p_1\frac{\partial}{\partial q_1}-p_2\frac{\partial}{\partial q_2}\right)D_2(t,q_1,p_1,q_2,p_2)\Bigg\}dp_2. \tag{3.11}$$

Now calculate the following integrals:

$$\left(\int\limits_{-\infty}^{q_1-a-\varepsilon}dq_2+\int\limits_{q_1+a+\varepsilon}^{\infty}dq_2\right)\left(-p_2\frac{\partial}{\partial q_2}D_2(t,q_1,p_1,q_2,p_2)\right)=$$

$$=-p_2D_2(t,q_1,p_1,q_1-a-\varepsilon,p_2)+p_2D_2(t,q_1,p_1,q_1+a+\varepsilon,p_2),$$

$$\left(\int\limits_{-\infty}^{q_1-a-\varepsilon}dq_2+\int\limits_{q_1+a+\varepsilon}^{\infty}dq_2\right)\left(-p_1\frac{\partial}{\partial q_1}D_2(t,q_1,p_1,q_2,p_2)\right)=$$

$$=-p_1\frac{\partial}{\partial q_1}\left(\int\limits_{-\infty}^{q_1-a-\varepsilon}dq_2+\int\limits_{q_1+a+\varepsilon}^{\infty}dq_2\right)D_2(t,q_1,p_1,q_2,p_2)+$$

$$+p_1D_2(t,q_1,p_1,q_1-a-\varepsilon,p_2)-p_1D_2(t,q_1,p_1,q_1+a+\varepsilon,p_2).$$

Tend $\varepsilon\to 0$ in (3.11) using above obtained formulas, continuity of function $D_2(t,x_1,x_2)$ on admissible configurations and take into account that p_1,p_2 are fixed and independent on ε. One obtains

$$\frac{\partial F_1(t,q_1,p_1)}{\partial t}=$$

$$=\lim_{\varepsilon\to 0}\int\left\{\left(\int\limits_{-\infty}^{q_1-a-\varepsilon}dq_2+\int\limits_{q_1+a+\varepsilon}^{\infty}dq_2\right)\times\right.$$

$$\left.\times\left(-p_1\frac{\partial}{\partial q_1}-p_2\frac{\partial}{\partial q_2}\right)D_2(t,q_1,p_1,q_2,p_2)\right\}dp_2=$$

$$=-p_1\frac{\partial}{\partial q_1}F_1(t,q_1,p_1)+$$

$$+\int dp_2\Big\{(p_1-p_2)\Big[D_2(t,q_1,p_1,q_1-a,p_2)-D_2(t,q_1,p_1,q_1+a,p_2)\Big]\Big\}.$$

Consider the following two cases: 1) $p_1-p_2>0$; 2) $p_1-p_2<0$. In the first case one has, according to the boundary condition,

$$D_2(t,q_1,p_1,q_1-a,p_2)=\frac{1}{(1-2\varepsilon)^2}D_2(t,q_1,p_1^*,q_1-a,p_2^*),$$

$$D_2(t,q_1,p_1,q_1+a,p_2)=D_2(t,q_1,p_1,q_1+a,p_2).$$

In the second case one has

$$D_2(t,q_1,p_1,q_1-a,p_2)=D_2(t,q_1,p_1,q_1-a,p_2),$$

$$D_2(t,q_1,p_1,q_1+a,p_2)=\frac{1}{(1-2\varepsilon)^2}D_2(t,q_1,p_1^*,q_1+a,p_2^*).$$

Denote by η unit inner vector of sphere (rod) $|q_2-q_1|=a$ with center in q_1, $\eta=+1$ in point $q_2=q_1-a$, $\eta=-1$ in point $q_2=q_1+a$.

We have in the first case $(p_1-p_2)=\langle\eta,(p_1-p_2)\rangle>0$, $\eta=+1$

$$(p_1-p_2)\Big[D_2(t,q_1,p_1,q_1-a,p_2)-D_2(t,q_1,p_1,q_1+a,p_2)\Big]=$$

$$= \eta(p_1 - p_2)\left[\frac{1}{(1-2\varepsilon)^2} D_2(t, q_1, p_1^*, q_1 - a\eta, p_2^*) - D_2(t, q_1, p_1, q_1 + a\eta, p_2)\right]\Big|_{\eta=1}$$

and in the second case $-(p_1 - p_2) = (p_1 - p_2)\eta, \ \langle \eta, (p_1 - p_2)\rangle > 0, \ \eta = -1$

$$(p_1 - p_2)\Big[D_2(t, q_1, p_1, q_1 - a, p_2) - D_2(t, q_1, p_1, q_1 + a, p_2)\Big] =$$

$$= (p_1 - p_2)\left[D_2(t, q_1, p_1, q_1 + a\eta, p_2) - \frac{1}{(1-2\varepsilon)^2} D_2(t, q_1, p_1^*, q_1 - a\eta, p_2^*)\right] =$$

$$= \eta(p_1 - p_2)\left[\frac{1}{(1-2\varepsilon)^2} D_2(t, q_1, p_1^*, q_1 - a\eta, p_2^*) - D_2(t, q_1, p_1, q_1 + a\eta, p_2)\right]\Big|_{\eta=-1}.$$

Denote by S^1_+ vector η for which $\langle \eta, (p_1 - p_2)\rangle > 0$. For $(p_1 - p_2) > 0$, S^1_+ consists of vector $\eta = +1$, for $(p_1 - p_2) < 0$, S^1_+ consists of vector $\eta = -1$. Denote $D_2(t, q_1, p_1, q_2, p_2) = F_2(t, q_1, p_1, q_2, p_2)$. Finally one obtains equations

$$\frac{\partial F_1(t, q_1, p_1)}{\partial t} = -p_1 \frac{\partial}{\partial q_1} F_1(t, q_1, p_1) + \int dp_2 \sum_{\eta \subset S^1_+} \langle \eta, (p_1 - p_2)\rangle \times$$

$$\times\left[\frac{1}{(1-2\varepsilon)^2} F_2(t, q_1, p_1^*, q_1 - a\eta, p_2^*) - F_2(t, q_1, p_1, q_1 + a\eta, p_2)\right],$$

$$\frac{\partial F_2(t, q_1, p_1, q_2, p_2)}{\partial t} = \left(-p_1 \frac{\partial}{\partial q_1} - p_2 \frac{\partial}{\partial q_2}\right) F_2(t, q_1, p_1, q_2, p_2) \qquad (3.12)$$

and boundary condition for the second equation is the same as for $D_2(t, q_1, p_1, q_2, p_2)$.

Equations (3.12) is the hierarchy (3.6) for $N = 2$.

Analogous calculation has been performed for one-dimensional point-wise particles in [4] on formal level.

3.4. Grand canonical ensemble. As known [1, 2], in grand canonical ensemble one has a sequence of nonnormalized distribution functions $D_N(t, (x)_N)$, $N \geq 0$, $D_0 = 1$, that satisfy Liouville equation (2.13) with boundary condition (2.12). The sequence of correlation functions is defined as follows:

$$\rho_s(t, (x)_s) = \frac{1}{\Xi} \sum_{n=0}^{\infty} \int \frac{1}{n!} D_{s+n}(t, x_1, \ldots, x_s, x_{s+1}, \ldots, x_{s+n}) dx_{s+1} \ldots dx_{s+n}, \quad s \geq 1, \qquad (3.13)$$

where Ξ is the grand partition function

$$\Xi = 1 + \sum_{n=1}^{\infty} \int D_n(t, x_1, \ldots, x_n) dx_1 \ldots dx_n =$$

$$= 1 + \sum_{n=1}^{\infty} \int \frac{1}{n!} D_n(0, x_1, \ldots, x_n) dx_1 \ldots dx_n. \qquad (3.14)$$

In (3.14) we used the law of conservation of full probability (3.8).

By repeating the derivation of hierarchy (3.6) for canonical ensemble [1, 2], one obtains the hierarchy for grand canonical ensemble

$$\frac{\partial \rho_s(t, x_1, \ldots, x_s)}{\partial t} = -\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i} \rho_s(t, x_1, \ldots, x_s) +$$

$$+ a^2 \sum_{i=1}^{s} \int dp_{s+1} \int_{S^2_+} d\eta \langle \eta, (p_i - p_{s+1})\rangle \times$$

$$\times\left[\frac{1}{(1-2\varepsilon)^2}\rho_{s+1}(t,x_1,\ldots,q_i,p_i^*,\ldots,q_i-a\eta,p_{s+1}^*)-\right.$$
$$\left.-\rho_{s+1}(t,x_1,\ldots,q_i,p_i,\ldots,q_i-a\eta,p_{s+1})\right],\quad s\geq 1, \tag{3.15}$$

with the same boundary conditions as for canonical ensemble.

Appendix A. In this appendix we present two very simple examples that explains the reason why

$$\int f(X(-t,x))\frac{\partial(X(-t,x))}{\partial x}dx \neq \int f(x)dx.$$

Consider interval $[0,\infty)$ and on this interval define the following map T

$$\begin{gathered} T(x)=x,\quad \text{if}\quad 0\leq x\leq 1,\\ T(x)=2x,\quad \text{if}\quad x>1. \end{gathered} \tag{A.1}$$

Show that the Jacobian $\dfrac{dT(x)}{dx}$ is defined as follows:

$$\begin{gathered} \frac{dT(x)}{dx}=1,\quad \text{if}\quad 0\leq x<1,\\ \frac{dT(x)}{dx}=\delta(x-1),\quad \text{if}\quad x=1,\\ \frac{dT(x)}{dx}=2,\quad \text{if}\quad 1<x<\infty. \end{gathered} \tag{A.2}$$

Calculate $\dfrac{dT(x)}{dx}$ as distribution (generalized function). Let $\varphi(x)$ be a test function, then

$$\int_0^\infty \frac{dT(x)}{dx}\varphi(x)dx=-\int_0^\infty T(x)\varphi'(x)dx=-\int_0^1 x\varphi'(x)dx-\int_1^\infty 2x\varphi'(x)dx=$$
$$=-\varphi(1)+\int_0^1\varphi(x)dx+2\varphi(1)+2\int_1^\infty\varphi(x)dx=\varphi(1)+\int_0^1\varphi(x)dx+2\int_1^\infty\varphi(x)dx=$$
$$=\int_0^\infty\delta(x-1)\varphi(x)dx+\int_0^1 1\cdot\varphi(x)dx+\int_1^\infty 2\cdot\varphi(x)dx. \tag{A.3}$$

This formula gives us $\dfrac{dT(x)}{dx}$ as stated before in (A.2).

Now consider the following integral with arbitrary smooth function $f(x)$ defined on $[0,\infty)$:

$$\int\limits_0^\infty f(T(x))\frac{dT(x)}{dx}dx=\int\limits_0^1 1\cdot f(x)dx+f(1)+\int\limits_1^\infty 2\cdot f(2x)dx=$$

$$=f(1)+\int\limits_0^1 f(x)dx+\int\limits_2^\infty f(x)dx. \qquad (A.4)$$

From (A.4) one can see that there is finite contribution from "hypersurface" $x=1$ where the map $T(x)$ is discontinuous and the interval $1\le x\le 2$ is absent, i.e., is lost in the map $T(x)$. Let us suppose that $f(1)=0$. Then (A.4) is reduced to the following final formula:

$$\int\limits_0^\infty f(T(x))\frac{dT(x)}{dx}dx=\int\limits_0^1 f(x)dx+\int\limits_2^\infty f(x)dx=\int\limits_0^\infty f(x)dx-\int\limits_1^2 f(x)dx. \qquad (A.5)$$

It follows from (A.5) that

$$\int\limits_0^\infty f(T(x))\frac{dT(x)}{dx}dx<\int\limits_0^\infty f(x)dx \qquad (A.6)$$

for positive "distribution" $f(x)\ge 0$, different from zero on interval $(1,2]$.

Consider second example with map

$$T(x)=x,\ 0\le x\le 1,\quad T(x)=\frac{1}{2}x,\quad x>1.$$

For $T(x)$ one obtains $\dfrac{dT(x)}{dx}=1,\ 0\le x\le 1,\ \dfrac{dT(x)}{dx}=-\dfrac{1}{2}\delta(x-1),\ x=1,$ $\dfrac{dT(x)}{dx}=\dfrac{1}{2},\ x>1.$ If is easy to check that

$$\int\limits_0^\infty f(T(x))\frac{dT(x)}{dx}dx=-\frac{1}{2}f(1)+\int\limits_0^1 f(x)dx+\int\limits_{\frac{1}{2}}^\infty f(x)dx.$$

If $f(x)\ge 0$ and $f(1)=0$ then

$$\int\limits_1^\infty f(T(x))\frac{dT(x)}{dx}dx=\int\limits_0^\infty f(x)dx+\int\limits_{\frac{1}{2}}^1 f(x)dx>\int\limits_0^\infty f(x)dx. \qquad (A.7)$$

This two examples show that for discontinuous map there may be "loss" or "gain" of domains.

This simple examples can help us to understand why

$$\int f_N(X(-t,(x)_N))\frac{\partial(X(-t,(x)_N))}{\partial(x)_N}d(x)_N\ne\int f((x)_N)d(x)_N. \qquad (A.8)$$

It is because the map $X(-t,(x)_N)$ is discontinuous and after collisions $\dfrac{\partial(X(-t,(x)_N))}{\partial(x)_N}\ne 1,$ in the left hand side contributions from the hypersurfaces, where collisions occur, may be finite, and some domains in the phase space may be "lost" in the map induced by shift along trajectories $X(-t,(x)_N)$, or may be "gained" in the map induced by shift along trajectories $X(t,(x)_N)$.

We have shown in Section III that

$$\int f_N(X(-t,(x)_N))\left(\frac{\partial(X(-t,(x)_N))}{\partial(x)_N}\right)^2 = \int f_N((x)_N)d(x)_N$$

for $f_N((x)_N) \subset L_1^0$ and it means that additional multiplier $\dfrac{\partial(X(-t,(x)_N))}{\partial(x)_N}$, different from 1 after collisions compensates “loss” of domains in the phase space. Note that $f_N((x)_N) \subset L_1^0$ is equal to zero on hypersurfaces where collisions occur and, therefore, contributions from these hypersurfaces are equal to zero.

Appendix B. In deriving formulae (2.11) we did not take into account that for some p_i, p_j momenta after collisions p_i^*, p_j^* are equal to p_i, p_j and $\dfrac{\partial X(+0,(x)_N)}{\partial(x)_N} = 1$. For example, if $\langle\eta,(p_i - p_j)\rangle = 0$. These momenta belong to hypersurfaces of lower dimension and one can neglect them because $D_N(t,(x)_N) \subset L_N$ for $f_N((x)_N) \subset L_N^0$.

If one considers generalized functions $f_N((x)_N)$ concentrated, for example, on hypersurfaces $p_1 = \ldots = p_N = p$ and with compact support with respect to $(q)_N$ then $\dfrac{\partial X(-t,(x)_N)}{\partial(x)_N} = 1$ and in $D_N(t,x_1,\ldots,q_i,p_i^*,\ldots,q_i - a\eta,p_j^*,\ldots,x_N)$, $\langle\eta,(p_i - p_j)\rangle > 0$ momenta $p_i^* = p_i, p_j^* = p_j$.

For such initial distribution functions $f_N((x)_N)$ hierarchy for correlation functions (3.15) is reduced to the following one

$$\frac{\partial}{\partial t}\rho_s(t,x_1,\ldots,x_s) = -\sum_{i=1}^{s} p_i\frac{\partial}{\partial q_i}\rho_s(t,x_1,\ldots,x_s), \quad s \geq 1. \tag{B.1}$$

The second and third term in the right-hand side of (3.3) is equal to zero because $p_i - p_{s+1} = 0$, $p_{s+1} - p_{s+2} = 0$.

Hierarchy (B.1) has stationary solution

$$\rho_s(t,x_1,\ldots,x_s) = \prod_{i=1}^{s}(p_i - p)\prod_{i<j=1}^{s}\Theta(|q_i - q_j| = a).$$

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