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G-SUPPLEMENTED MODULES
G-ДОПОВНЕНІ МОДУЛІ

Following the concept of generalized small submodule, we define g-supplemented modules and characterize some fundamental properties of these modules. Moreover, the generalized radical of a module is defined and the relationship between the generalized radical and radical of a module is investigated. Finally, the definition of amply g-supplemented modulec is given with its some basic properties.

Із застосуванням поняття узагальненого малого підмодуля визначено поняття g-доповнених модулів та охарактеризовано деякі фундаментальні властивості цих модулів. Крім того, визначено поняття узагальненого радикала модуля та вивчено співвідношення між узагальненим радикалом та радикалом модуля. Насамкінець наведено визначення поняття рясно g-доповнених модулів та вивчено основні властивості цих модулів.

1. Introduction. Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an R-module. We will denote a submodule N of M by $N \leq M$ and a proper submodule K of M by $K < M$. Let M be an R-module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a small submodule of M and denoted by $N << M$. Let M be an R-module and $N \leq M$. If there exists a submodule K of M such that $M = N + K$ and $N \cap K = 0$, then a submodule N of M is called a direct summand of M and it is denoted by $M = N \oplus K$. For any module M, we have $M = M \oplus 0$. $\operatorname{Rad} M$ indicates the radical of M. An R-module M is said to be simple if M have no proper submodules with distinct zero. A submodule N of an R-module M is called an essential submodule and denoted by $N \trianglelefteq M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$. Let M be an R-module and K be a submodule of M. K is called a generalized small submodule of M if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$, then we write $K <<_g M$. It is clear that every small submodule is a generalized small submodule but the converse is not true generally. Let M be an R-module. M is called a (generalized) hollow module if every proper submodule of M is (generalized) small in M. Here it is clear that every hollow module is generalized hollow module. The converse of this statement is not always true. M is called local module if M has a largest submodule, i.e., a proper submodule which contains all other proper submodules. Let U and V be submodules of M. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V << V$, then V is called a supplement of U in M. M is called a supplemented module if every submodule of M has a supplement.

Now we will give some important properties of generalized small submodules.

Lemma 1 [6]. *Let M be an R-module and K, $N \leq M$. Consider the following conditions*:

(1) *If $K \leq N$ and N is generalized small submodule of M, then K is a generalized small submodule of M.*

(2) *If K is contained in N and a generalized small submodule of N, then K is a generalized small submodule in submodules of M which contains submodule N.*

(3) *Let $f : M \to N$ be an R-module homomorphism. If $K <<_g M$, then $f(K) <<_g M$.*

(4) *If $K <<_g L$ and $N <<_g T$, then $K + N <<_g L + T$.*

***Corollary* 1.** *Let M be an R-module and $K \leq N \leq M$. If $N <<_g M$, then $N/K <<_g M/K$.*

***Corollary* 2.** *Let M be an R-module, $K <<_g M$ and $L \leq M$. Then $(K + L)/L <<_g M/L$.*

2. *G*-supplemented modules.

Definition 1. *Let M be an R-module and $U, V \leq M$. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, then V is called a g-supplement of U in M. If every submodule of M has a g-supplement in M, then M is called a g-supplemented module.*

Supplemented modules are g-supplemented.

Lemma 2. *Let M be an R-module, $U \leq M$ and $V \leq M$. Then V is a g-supplement of U in M if and only if $M = U + V$ and $U \cap V <<_g V$.*

Proof. $(\Rightarrow)$ Let $U \cap V + T = V$ and $T \trianglelefteq V$. Then $M = U + V = U + U \cap V + T = U + T$ and since V is a g-supplement of U in M and $T \trianglelefteq V$, $T = V$. Hence $U \cap V <<_g V$.

$(\Leftarrow)$ Let $M = U + V$ and $U \cap V <<_g V$. Let $M = U + T$ with $T \trianglelefteq V$. Since $M = U + T$ and $T \leq V$, by Modular Law $V = V \cap M = V \cap (U + T) = U \cap V + T$. Then by $U \cap V <<_g V$, $T = V$. Hence V is a g-supplement of U in M.

Lemma 3. *Let M be an R-module, $M_1 \leq M$, $U \leq M$ and M_1 be a g-supplemented module. If $M_1 + U$ has a g-supplement in M, then so does U.*

Proof. Let X be a g-supplement of $M_1 + U$ in M. Then $M_1 + U + X = M$ and $(M_1 + U) \cap X <<_g X$. Since M_1 is g-supplemented, $(U + X) \cap M_1$ has a g-supplement Y in M_1, i.e., $M_1 \cap (U + X) + Y = M_1$ and $M_1 \cap (U + X) \cap Y <<_g Y$. Following this, we have $M = M_1 \cap (U + X) + Y + U + X = U + X + Y$ and $U \cap (X + Y) \leq X \cap (U + Y) + Y \cap (U + X) \leq X \cap (M_1 + U) + Y \cap M_1 \cap (U + X) <<_g X + Y$. Hence $X + Y$ is a g-supplement of U in M.

Theorem 1. *Let $M = M_1 + M_2$. If M_1 and M_2 are g-supplemented modules, then M is a g-supplemented module.*

Proof. Clear from Lemma 3.

***Corollary* 3.** *Any finite sum of g-supplemented modules are g-supplemented.*

Lemma 4. *Let M be an R-module, $X \leq U \leq M$ and V be a g-supplement of U. Then $(V + X)/X$ is a g-supplement of U/X in M/X.*

Proof. Since V is a g-supplement of U in M, we have $M = U + V$ and $U \cap V <<_g V$. Thus $(U \cap V + X)/X <<_g (V + X)/X$ by Lemma 1. Since $M = U + V$, it is easy to see that $M/X = (U + V)/X = U/X + (V + X)/X$ and $U/X \cap (V + X)/X = (U \cap V + X)/X <<_g (V + X)/X$. Therefore $(V + X)/X$ is a g-supplement of U/X in M/X.

Theorem 2. *If M is a g-supplemented module, then every factor module of M is g-supplemented.*

Proof. Clear from Lemma 4.

***Corollary* 4.** *If M is a g-supplemented module, then the homomorphic image of M is g-supplemented.*

Theorem 3. *Let M be an R-module, K be a direct summand of M and $T \leq K$. Then $T <<_g K$ if and only if $T <<_g M$.*

Proof. ($\Rightarrow$) Clear from Lemma 1.

($\Leftarrow$) Let $T <<_g M$. Assume that $M = K \oplus Y$. If we consider the canonical map $\pi: M \to K$, then we get $T = \pi(T) <<_g \pi(M) = K$ by Lemma 1.

Definition 2. *Let M be an R-module and $T \leq M$. If T is both maximal and essential in M, then T is called a generalized maximal submodule of M. The intersection of all generalized maximal submodules of M is called the generalized radical of M denoted by $\operatorname{Rad}_g M$. If M has not a generalized maximal submodule, then we denote $\operatorname{Rad}_g M = M$.*

Lemma 5. *Let M be an R-module. If M has at least one generalized maximal submodule, then $\operatorname{Rad}_g M = \sum_{L<<_g M} L$.*

Proof. Let $L <<_g M$. If $L \nsubseteq T$ with T is a generalized maximal submodule of M, then we get $L + T = M$ since T is maximal. Thus $T = M$, which is a contradiction. Therefore L is contained in every generalized maximal submodule of M. Hence $\sum_{L<<_g M} L \subseteq \operatorname{Rad}_g M$.

Let $x \in \operatorname{Rad}_g M$. Suppose that Rx is not generalized small in M and $\Omega = \{T \leq M \mid x \notin T, T \trianglelefteq M \text{ and } Rx + T = M\}$. Since Rx is not generalized small in M, we get $\Omega \neq \varnothing$. It is clear that every chain has a upper bound by inclusion in Ω. Hence Ω contains a maximal element K by Zorn's lemma. We can easily show that K is a generalized maximal submodule of M. Since $K \in \Omega$, we have $x \notin K$. Since $\operatorname{Rad}_g M \subseteq K$, we get $x \notin \operatorname{Rad}_g M$. This is a contradiction. Therefore $Rx <<_g M$ and then $\operatorname{Rad}_g M \subseteq \sum_{L<<_g M} L$. So we get $\operatorname{Rad}_g M = \sum_{L<<_g M} L$.

***Corollary* 5.** *If M has no generalized maximal submodule, then $\operatorname{Rad}_g M = \sum_{L<<_g M} L$.*

Proof. Similar to the proof of Lemma 5.

***Corollary* 6.** *Let M be an R-module. Then $\operatorname{Rad} M \leq \operatorname{Rad}_g M$.*

***Example* 1.** For a non-zero simple R-module M, we have $\operatorname{Rad} M = 0 \neq M = \operatorname{Rad}_g M$.

Theorem 4. *Let M be an R-module with $\operatorname{Rad}_g M \neq M$. The following conditions are equivalent*:

(i) *M is a generalized hollow module,*

(ii) *M is a local module,*

(iii) *M is a hollow module.*

Proof. (i) $\Rightarrow$ (ii) Let M be a generalized hollow module and T be any proper submodule of M. Then $T <<_g M$ and we have $T \leq \operatorname{Rad}_g M$ by Lemma 5. Since $\operatorname{Rad}_g M \neq M$, M is local and so the proof is complete.

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (i) Clear.

Theorem 5. *If M is a finitely generated R-module and M has a proper essential submodule, then every proper essential submodule of M is contained in a generalized maximal submodule.*

Proof. Let K be any proper essential submodule of M. Since M is finitely generated, K is contained in a maximal submodule T and $T \trianglelefteq M$ due to $K \trianglelefteq M$.

Theorem 6. *Let M be an R-module and $\operatorname{Rad}_g M \neq M$. If every proper essential submodule of M is contained in a generalized maximal submodule, then $\operatorname{Rad}_g M <<_g M$.*

Proof. Clear.

3. Amply G-supplemented modules.

Definition 3. *Let M be an R-module and $U \leq M$. If, for every $V \leq M$ with $M = U + V$, U has a g-supplement T in M such that $T \leq V$, then we say that U has ample g-supplements in M. If every submodule of M has ample g-supplements in M, then M is called an amply g-supplemented module.*

Theorem 7. *Let M be an R-module, $U_1, U_2 \leq M$ and $M = U_1 + U_2$. If U_1 and U_2 have ample g-supplements in M, then $U_1 \cap U_2$ has also ample g-supplements in M.*

Proof. Let $U_1 \cap U_2 + T = M$. Then we have $M = U_1 + U_2 \cap T = U_2 + U_1 \cap T$. Since U_1 and U_2 have ample g-supplements in M, then U_1 has a g-supplement V_1 with $V_1 \leq U_2 \cap T$ and U_2 has a g-supplement V_2 with $V_2 \leq U_1 \cap T$. Since $M = U_1 + V_1$ and $V_1 \leq U_2$, by Modular Law $U_2 = U_2 \cap (U_1 + V_1) = U_1 \cap U_2 + V_1$. Similarly we have $U_1 = U_1 \cap U_2 + V_2$. Then $M = U_1 + U_2 = U_1 \cap U_2 + V_2 + U_1 \cap U_2 + V_1 = U_1 \cap U_2 + V_1 + V_2$ and $U_1 \cap U_2 \cap (V_1 + V_2) = U_1 \cap (V_1 + U_2 \cap V_2) = U_1 \cap V_1 + U_2 \cap V_2 <<_g M$. Hence $V_1 + V_2$ is a g-supplement of $U_1 \cap U_2$ and since $V_1 + V_2 \leq T$, $U_1 \cap U_2$ has ample g-supplements in M.

Theorem 8. *If M is an amply g-supplemented module, then every factor module of M is amply g-supplemented.*

Proof. Clear.

***Corollary* 7.** *If M is an amply g-supplemented module, then the homomorphic image of M is amply g-supplemented.*

Proof. Clear from Lemma 4.

Theorem 9. *Let M be an R-module. If every submodule of M is g-supplemented, then M is amply g-supplemented.*

Proof. Clear.

Lemma 6. *If M is a π-projective and g-supplemented module, then M is an amply g-supplemented module.*

Proof. Let $M = U + V$ and X be a g-supplement of U. Since M is π-projective and $M = U + V$, there exists an R-module homomorphism $f: M \to M$ such that $\mathrm{Im} f \subset V$ and $\mathrm{Im}(1-f) \subset U$. So, we have $M = f(M) + (1-f)(M) = f(U) + f(X) + U = U + f(X)$. Suppose that $a \in U \cap f(X)$. Since $a \in f(X)$, then there exists $x \in X$ such that $a = f(x)$. Since $a = f(x) = f(x) - x + x = x - (1-f)(x)$ and $(1-f)(x) \in U$ we obtain $x = a + (1-f)(x)$ and $x \in U$. Thus $x \in U \cap X$ and so $f(X) \in f(U \cap X)$. Therefore we get $U \cap f(X) \leq f(U \cap X) <<_g f(X)$. This means that $f(X)$ is a g-supplement of U in M. Moreover $f(X) \subset V$. Therefore M is amply g-supplemented.

Now the following corollary can be easily written as a consequence of Lemma 6.

***Corollary* 8.** *If M is a projective and g-supplemented module, then M is an amply supplemented module.*

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