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LONG-RANGE ORDER IN QUANTUM LATTICE LINEAR OSCILLATOR SYSTEMS

ДАЛЕКИЙ ПОРЯДОК У КВАНТОВИХ ҐРАТКОВИХ СИСТЕМАХ ЛІНІЙНИХ ОСЦИЛЯТОРІВ

Existence of ferromagnetic long-range order (lro) is proved for equilibrium quantum lattice systems of linear oscillators whose potential energy contains a strong ferromagnetic nn (nearest neighbor) pair interaction term and a weak nonferromagnetic term under a special condition on a superstability bound. It is shown that lro is possible if a mass of a quantum oscillator and a strength of a ferromagnetic nn interaction exceed special values. A generalized Peierls argument and a contour bound, proved with the help of a new superstability bound for correlation functions, are our main tools.

Для рівноважних квантових граткових систем лінійних осциляторів, потенціальна енергія яких містить сильну феромагнітну частину парної взаємодії близьких сусідів і слабку неферомагнітну частину, доведено існування феромагнітного далекого порядку при певній умові на нерівність суперстійкості. Показано, що далекий порядок може мати місце, якщо маса квантового осцилятора та сила феромагнітної взаємодії близьких сусідів перевищують певні значення. При цьому використано узагальнений принцип Пайєрлса та контурну нерівність, доведену з допомогою нової нерівності суперстійкості для кореляційних функцій.

1. Introduction and main result. Let's consider Gibbs quantum systems of one-dimensional oscillators on the d-dimensional hypercubic lattice $\mathbb{Z}^d$, with the potential energy $U(q_\Lambda) = U(-q_\Lambda)$ on a set Λ with the finite cardinality $|\Lambda|$, where q_Λ is an array of $(q_x, x \in \Lambda)$, q_x is the oscillator coordinate taking value in $\mathbb{R}$. The potential energy is assumed to be a growing function at infinity. It is invariant under the simplest discrete symmetry, namely $\mathbb{Z}_2$, which is realized as a transformation of changing of all the signs of the oscillator variables.

The Hamiltonian of the considered quantum system is given by

$$H^\Lambda = -\frac{1}{2m}\sum_{x\in\Lambda}\partial_x^2 + U(q_\Lambda),$$

where ∂_x is the partial derivative in q_x and U is the potential energy

$$U(q_\Lambda) = \sum_{x\in\Lambda} u(q_x) + \sum_{\langle x,y\rangle\in\Lambda}\phi(q_x,q_y) + U'(q_\Lambda), \tag{1.1}$$

$\langle x, y\rangle$ means nearest neighbors, the external potential u is a bounded below polynomial of the $2n$-th degree, the pair potential ϕ is a polynomial equal to zero at coinciding arguments and a finite-range U satisfies a special superstability and regularity conditions [1 – 4], written down below, which allow to pass to the thermodynamic limit.

If $\hat{F}_X$ is the operator of multiplication by the function $F_X(q_X)$ then the quantum Gibbs (equilibrium) average is given by

$$\langle F_X\rangle_\Lambda = Z_\Lambda^{-1}\mathrm{Tr}\big(\hat{F}_X e^{-\beta H^\Lambda}\big) = Z_\Lambda^{-1}\int F_X(q_X)e^{-\beta H^\Lambda}(q_\Lambda; q_\Lambda)dq_\Lambda =$$

$$= \int F_X(q_X)\rho^\Lambda(q_X|q_X)dq_X, \qquad Z_\Lambda = \mathrm{Tr}\big(e^{-\beta H^\Lambda}\big),$$

where the second function under the sign of the first integral is the kernel of the semigroup generated by the Hamiltonian and the integration is performed over $\mathbb{R}^{|\Lambda|}$, $\mathbb{R}^{|X|}$, respectively.

Ferromagnetic long-range order (lro) occurs in the systems if for $a' > 0$ independent of Λ either

$$\langle\sigma_x\sigma_y\rangle_\Lambda \geq a', \qquad \sigma_x(q_\Lambda) = q_x, \qquad x, y \in \Lambda, \tag{1.2}$$

or

$$\langle s_x s_y\rangle_\Lambda \geq a', \qquad s_x(q_\Lambda) = \frac{q_x}{|q_x|}. \tag{1.3}$$

It is believed that lro occurs if minima of the potential u are sufficiently deep and rare. In this paper we prove (1.3) for $d \geq 2$ with the help of the generalized Peierls argument based on an application of a new version of the Ruelle superstability bound (ferromagnetic superstability bound) given below.

For pair short-range potentials and high temperatures there is a decrease of correlations [5]. This should mean existence of phase transitions for such the systems. For $d \geq 3$ and the case of the pair nn(nearest neighbor) quadratic interaction potential potentials was derived in [6] with the help of the swc(spin wave condensation) method. For translation invariant U' having a small nonferromagnetic term and the quadratic nn-pair interaction (1.3) was claimed to hold in [7]. The technique of this paper was based on the conventional superstability bound.

The proposed here superstability Peierls argument is very general (the potential energy is more general than in [7]) and was introduced for nonequilibrium systems of interacting Brownian oscillators close to or far from an equilibrium [8] whose time dependent correlation functions have the structure of reduced density matrices of quantum oscillator systems with a three-body and pair interaction potentials. In this paper we reformulated and generalized our arguments from [7] proving the abstract ferromagnetic superstability bound. Its formulation and proof for quantum oscillator systems does not differ essentially from the case of nonequlibrium systems of interacting Brownian oscillators since the Winer integrals over unclosed paths in the latter have to be changed by Wiener integrals over closed paths (loops) in the former. In both cases Feynman – Kac formula has to be employed for its proof.

For the quadratic nearest neighbor pair interaction it was proved in [9] that lro disappears at sufficiently small mass of the oscillators m and appropriate general external potential and this implies a convergence of the cluster expansion for the Gibbs state at an arbitrary temperature [10].

We assume that

$$u(q) = \eta q^{2n} - gq^{2n_0} + \sum_{j=1}^{n-1} \eta_j q^{2j} = u_0(q) - gq^{2n_0}, \qquad n > n_0,$$

$$\eta \geq 1, \quad \eta_j \in \mathbb{R}.$$

If the coefficients η, η_j depend on $g > 0$ then they should be bounded functions. For existence of lro in our approach the following conditions are needed

$$m \geq g^{\frac{n-2}{2(n-n_0)}} m'(g), \qquad \lim_{g\to\infty} m'(g) = \infty. \tag{1.4}$$

That is, the mass has to be sufficiently large. The pair nearest neighbor potential ϕ is chosen as follows

$$\phi(q_x, q_y) = g_0(q_x - q_y)^{2n_1} Q'(q_x, q_y), \qquad g_0 = g^{\frac{\xi}{2(n-n_0)}} = z^{-\xi}, \quad \xi \in \mathbb{R},$$

where Q' is an even positive function such that the following inequalities hold:

$$Q'(q_x, q_y) \leq \alpha_0(q_x^{2n_2} + q_y^{2n_2}), \qquad n_1 + n_2 < n,$$

$$Q'(q_x, q_y) \geq q_x^{2n_2'} q_y^{2(n_2 - n_2')} + q_y^{2n_2'} q_x^{2(n_2 - n_2')},$$

the positive constants α_0 does not depend on g and $n_2 - n_2' \leq n_2' \leq n_2$. For such the choice of Q' the interaction constant g_0 can be interpreted as the strength of the nn-pair interaction. It'll be seen that the condition $\xi > 0$ is needed for occurrence of lro only if n_2 is comparable with n_1. For translation invariant U' the parameter ξ may be negative.

The above expressions for u with $\eta_j = 0$ and ϕ can be obtained if one starts from the following expressions (see [11] where this fact is proved):

$$u(q) = u_0(q), \qquad \phi(q_x, q_y) = -g(q_x^{n_0+k} q_y^{n_0-k} + q_y^{n_0+k} q_x^{n_0-k}), \quad 0 \leq k \leq n_0.$$

Our method is based on a control of asymtotics of the reduced density matrices at large g and all the conditions in our main Theorem 1.1 are aimed at that. In order to do this we have to deal with bounded functions u_g, u_*, U_*, ϕ_* in g and re-scale and translate variables by z^{n-1} and e_0, respectively, where e_0 is the deepest minimum of the potential $u_g(q)$. We put

$$u_g(q) = u(z^{n-1}q), \quad \phi_g(q, q') = \phi(z^{n-1}q, z^{n-1}q'), \quad U_g(q_X) = U(z^{n-1}q_X),$$

$$u_*(q) = u_g(q + e_0) - u_g(e_0), \qquad \phi_*(q, q') = \phi_g(q + e_0, q' + e_0),$$

$$U_*(q_X) = U_g(q_X + e_0) - |X| u_g(e_0), \quad U'_*(q_X) = U'_g(q_X + e_0).$$

Let $W'_*(q_X; q_Y) = U'_*(q_\Lambda) - U'_*(q_Y) - U'_*(q_X)$, $\Lambda = X \cup Y$. The same relation will hold, also, between U_* and W_*. We require the superstability and regularity conditions to hold

$$U_*(q_\Lambda) \geq \sum_{x \in \Lambda} u_*^-(q_x), \quad u_*^-(q) = u_*(q) - \zeta v^0(z^n q) - \zeta_0, \tag{1.5}$$

$$\left|W'_*(q_X; q_Y)\right| \leq \sum_{x \in X, y \in Y} \Psi'\big(|x - y|\big)\big(v^0(q_x) + v^0(q_y)\big), \quad v^0(q) = \sum_{j=1}^{n-1} q^{2j}, \tag{1.6}$$

where the positive L^1-function Ψ', and numbers $\zeta, \zeta_0 \geq 0$ do not depend on g. The second and third terms in the expression for u_*^- is a contribution of the negative (nonferromagnetic) term in U' which shows that the latter is always small, that is, it depends on positive powers of g^{-1}. Formulae (1.5), (1.6) allow positive (ferromagnetic) interaction terms in U to be large. Interaction is stronger for translation invariant potentials. For translation invariant interaction two conditions for the functions U', W' may be postulated in such the way that they will imply (1.5), (1.6) (see [7]). The following condition

$$\xi \leq 2(n-1)n_1 - 2n_2 \tag{1.7}$$

guarantees, also, that the nn-pair potential part of U_* and U_* itself satisfy (1.6) with an appropriate Ψ (see the appendix A in [11]). Let $\rho_*^\Lambda(q_X|q_X)$ be the correlation functions corresponding to the potential energy U_*. Then the ferromagnetic superstability bound is given by

$$\rho_*^\Lambda(q_X|q_X) \le e^{-\beta(H^X_{*\varepsilon}-|X|c_*(\varepsilon))}(q_X;q_X),$$

$$H^X_{*\varepsilon} = (2m_g)^{-1}\Delta_X + U_*^+(q_X) + \sum_{x\in X} u_*^+(q_x), \qquad m_g = mg^{-\frac{n-1}{n-n_0}}, \tag{1.7a}$$

$$u_*^+(q) = u_*^-(q) - 3\epsilon v^0(q),$$

where U_*^+ is the part of U_*, which is absent in the ordinary superstability bound, generated by a positive pair interaction potential. Function $c_*(\varepsilon)$ does not depend of the oscillator variables and is a positive continuous function tending to infinity in the limit of vanishing ε. Its analytical structure can be found [8].

Let $\chi_x^+(q_\Lambda) = \chi_{(0,\infty)}(q_x)$, $\chi_x^-(q_\Lambda) = \chi_{(-\infty,0)}(q_x)$, where $\chi_{(a,b)}$ is the characteristic function of the open interval (a,b). The corner stone of the Peierls principle (argument) is the contour bound whose universal character was demonstrated in [12] for classical oscillator systems with pair nn interaction (see also [13]) and Heisenberg models

$$\left\langle \prod_{\langle x,y\rangle\in\Gamma} \chi_x^+\chi_y^- \right\rangle_\Lambda \le e^{-|\Gamma|E}, \tag{1.8}$$

where Γ is the set of pairs of nearest neighbors adjacent to a contour, i.e., an external boundary of the connected union of unit cubes centered at the sites of the bounded subset of the hyper-cubic lattice and E can be made arbitrary large either at low temperatures or by varying a parameter in an expression of a potential energy (see Remark 4.1). The generalized Peierls principle proves (1.3) if (1.8) holds. It is not obvious that (1.3) implies (1.2) (see [12]). Our main result is formulated in the following theorem.

Theorem 1.1. *Let* (1.4) – (1.7) *be satisfied and for the constant in the superstability bound, corresponding to the potential energy* U_*, *the following equality hold*: $\lim_{g\to\infty} z^n c_*(z^{2(n-1)n}) = 0$. *Let, also, the following inequality hold:*

$$\xi > n - 2(n_1+n_2) + 2(n_2 - n_2')n. \tag{1.9}$$

Then the contour bound (1.8) *holds with* E *being a function which is growing in* g *as* $\beta\left(\frac{g^n n_0}{n\eta}\right)^{\frac{1}{2(n-n_0)}}$ *at infinity and* (1.3) *holds if* g *is sufficiently large.*

The proof that the condition for $c_*(\varepsilon)$ in this theorem is true for finite range interaction, determined by U', can be given following arguments from [8] and bounds given in this paper. The growth at infinity of $\beta^{-1}E$ coincides with the minimum $z^{-n}\mu^0$ of the external potential u_g with $\eta_j = 0$. Remark that z^n and $z^{2(n-1)n}$ behave at large g like e_0^{-1} and $e_0^{-2(n-1)}$, respectively, Conditions (1.7), (1.9) mean that the nontranslation invariant part of the nn-pair potential is not arbitrary and that the strength g_0 of the nn-pair interaction has to be correlated with the depth of the deepest minimum e_0 of the external potential u.

Inequality (1.9) arises from an application of (2.1) for the product of the characteristic functions in (1.8) and the necessity of a suppression of the first unbounded term in the expression Q_g in it (Q_g in (2.10) does not allow the strength of the nn ferromagnetic interaction g_0 be arbitrary small). This suppression is more complicated for quantum than for classical systems (see [11]). In our method an important role is played by the integral

$$\int e^{-\beta(u_*(q) - av^0(z^n q) - bz^{2n(n-1)}v^0(q))}dq. \tag{1.10}$$

We show that its asymptotics are the same for $v^0 = 0$ and $\eta_j = 0$ (this fact is more profound than the statement of the one-dimensional Laplace method). The proof is based on application of the inequalities (4.2), (4.3) or (3.6), (3.7), easily verified for $\eta_j = 0$. Our result can be generalized for the potential u obeying these inequalities. The bound for the integral is given in the third section and it is used later in a proof of our main bounds.

The expression for E is given in Theorem 2.1. The main term in it is given by the integral

$$I_*(2;r) = \int \mu(dq_x dq'_x)\mu(dq_y dq'_y) \exp\{-\beta r \Phi_*(q_{x,y}, q'_{x,y})\},$$

where $\mu(dqdq') = \exp\big\{-\beta u^+_*(q') - (m_g/2\beta)(q-q')^2\big\}dqdq'$, $\Phi_*(q_{x,y}, q'_{x,y}) = -Q_*(q_x, q_y) + \phi_*(q'_x, q'_y)$,

$$Q_*(q_x, q_y) = e_0^{-1}\left[(q_x - q_y)^2 + \frac{4}{3}\big(|q_x(q_x + 2e_0)| + |q_y(q_y + 2e_0)|\big)\right].$$

The polynomial Q_* is determined in the inequality (2.1) for the product of the characteristic functions in the contour bound (1.8) and m_g is the rescaled mass. The last polynomial has the remarkable property: $Q_*(-q_x, -q_y) = Q_*(q_x - 2e_0, q_y - 2e_0)$, which will be employed by us in a special way in the proof of Theorem 4.1.

Estimates of the above integrals are based on the control of the behavior of all the functions in the neighborhod of the deepest minimum e_0 of u_* based on application of (4.2), (4.3).

Our paper is organized as follows. In the second section we derive the expression for E in the contour bound (1.8) in terms of c_* and $I_*(2;r)$ (Theorem 2.1). In Theorem 2.2 we establish the character of asymptotics of this integral proving Theorem 1.1 in this way. In the third section some properties of the minima of the re-scaled external potential u_g are established (its higher derivatives tend to zero in the limit of infinite g) and the estimate of the integral in (1.10) is obtained. In the fourth section Theorem 2.2 is proved. The proofs of the Peierls principle can be found in [8, 9].

2. Contour bound and superstability. Derivation of the contour bound (1.8) is based on an application of the ferromagnetic superstability bound and the bound for the product of the characteristic functions whose more complicated version appeared for the first time in [14] for a continuum (classical) system of oscillators, corresponding to an interacting two-dimensional Euclidean quantum boson field. This bound (its proof can be found in [8, 11]) is given by

$$\prod_{\langle x,x'\rangle \in \Gamma} \chi^+(q_x)\chi^-(q_{x'}) \le \exp\Big\{-\beta\big[e_0|\Gamma| - Q_g(q_\Gamma)\big]\Big\}, \tag{2.1}$$

where

$$Q_g(q_\Gamma) = \sum_{\langle x,x'\rangle \in \Gamma} Q_g(q_x, q_y),$$

$$Q_g(q_x, q_y) = e_0^{-1}\left[(q_x - q_y)^2 + \frac{4}{3}\big(|q_x^2 - e_0^2| + |q_y^2 - e_0^2|\big)\right].$$

The exponent with the ferromagnetic part of the potential energy in the ferromagnetic superstability bound has to compensate a contribution of the first translation-invariant term in the expression for Q_g. The mechanism of a compensation leads to (1.9) and in quantum oscillator systems is more complicated than in classical oscillator systems and nonequilibrium systems of interacting Brownian oscillators explained in [11] and [8], respectively. In order to realize it we have to generalize the Golden – Thompson inequality. The main result of this section is formulated as follows.

Theorem 2.1. *Let the conditions of Theorem* 1.1 *be satisfied and* $e_*(g) = c_*(z^{2n(n-1)})$ *be determined by the superstability bound* (1.7a). *Then* E *in* (1.8) *is represented by*

$$E = \beta e_0 - e_*(g) - (2d-1)^{-1} \ln I_*(2; 2d-1) + \ln \frac{2\pi\beta I_*(2;0)}{e^2 m_g}$$

if $I_*(2;0) \geq 1$. *If* $I_*(2;0) \leq 1$ *then the inequality holds without* $I_*(2;0)$.

Proof. Using the fact that the scaling does not change the product of the characteristic functions and that Laplacian is translation invariant we derive

$$\left\langle \prod_{\langle x,y\rangle \in \Gamma} \chi_x^+ \chi_y^- \right\rangle_\Lambda = \left\langle \prod_{\langle x,y\rangle \in \Gamma} \chi_x^+ \chi_y^- \right\rangle_{*\Lambda}, \tag{2.2}$$

where the Gibbs average in the right-hand side corresponds to U_*. Let's insert (2.1) into the right-hand side of (2.2). Then (2.1) and the ferromagnetic superstability bound yield

$$\left\langle \prod_{\langle x,y\rangle \in \Gamma} \chi_x^+ \chi_y^- \right\rangle_\Lambda \leq e^{-(\beta e_0 - e_*(g))|\Gamma|} \hat{I}_*(\Gamma). \tag{2.3}$$

It's clear that

$$\hat{I}_*(\Gamma) = \mathrm{Tr} \left(e^{\beta \hat{Q}_{*\Gamma}} \exp\left\{ -\beta \left[-(2m_g)^{-1} \Delta_\Gamma + \sum_{\langle x,y\rangle \in \Gamma} \left(\hat{u}_{*x}^+ + \hat{u}_{*y}^+ + \hat{\phi}_{*(x,y)} \right) \right] \right\} \right), \tag{2.4}$$

where the operators with the hats correspond to the operators of multiplication by the functions depending on the variables indexed by the indices of the operators and Δ_Γ is the Laplacian in $q_{x,y}$, $x, y \in \Gamma$. Now we have to apply the generlized Golden – Thompson inequality formulated and proved in the following proposition.

Proposition 2.1. *Let* Δ *be the* d*-dimensional Laplacian and* $\hat{F}$, $\hat{v}$ *the operators of multiplication by real valued continuous functions* $F \geq 0$, v *such that* v *is bounded form below,* $v \in L^2(\mathbb{R}^d, e^{-\|q\|^2} dq)$ *and* $\int F(q) e^{tv(q)} dq < \infty$. *Then*

$$\mathrm{Tr}\big(\hat{F} e^{t((2m)^{-1}\Delta + \hat{v})}\big) \leq \big(m(2\pi t)^{-1} e^2\big)^d \int dq F(q) \int dq_1 e^{-\frac{m}{2t}\|q-q_1\|^2} e^{tv(q_1)}, \tag{2.5}$$

where $\|q\|$ *is the Euclidean norm of the vector* q.

Proof. Let H denote the operator under the sign of the exponent in the left-hand side of (2.5). The first condition for v implies that the Trotter formula holds in $L^2(\mathbb{R}^d)$: $e^{tH} = \mathrm{st} \lim_{n\to\infty} \big(e^{\frac{t}{2mn}\Delta} e^{\frac{t}{n}\hat{v}}\big)^n$ (see Theorem X.51, Exercise 3.X in [15, 16]). This implies that the function $\big(e^{\frac{t}{2mn}\Delta} e^{\frac{t}{n}\hat{v}}\big)^n f$, $f \in L^2(\mathbb{R}^d)$, converges almost everywhere to the limit function on a subsequence of positive integers. This means that $I_t(F|v) = \lim_{n\to\infty} I_t(F|v, n)$, where the limit is achieved on some subsequence of positive integers and

$$I_t(F|v,n) = \int dqF(q) \int dq_1 \dots dq_n p_0^{tn^{-1}}(q-q_1)e^{tn^{-1}v(q_1)} \times$$
$$\times \prod_{j=2}^{n} p_0^{tn^{-1}}(q_j - q_{j-1})e^{tn^{-1}v(q_j)}\delta(q-q_n),$$

where $\delta(q)$ is the point measure concentrated at zero,

$$p_0^t(q) = \sqrt{m^d(2\pi t)^{-d}} e^{-m\frac{\|q\|^2}{2t}}.$$

Now, lets apply the generalized Hölder inequality for a measure μ on arbitrary measure space (the usual Hölder inequality and induction are used to prove it)

$$\int \prod_{j=1}^{n} f_j d\mu \leq \prod_{j=1}^{n} \left[\int |f_j|^n d\mu \right]^{\frac{1}{n}}, \tag{2.6}$$

taking $\mathbb{R}^{d(n+1)}$ as the measure space and putting $f_j(q, q_1, \dots, q_n) = e^{\frac{t}{n}v(q_j)}$,

$$d\mu(q, q_1, \dots, q_n) = F(q)dqdq_1 \dots dq_n p_0^{tn^{-1}}(q-q_1) \prod_{j=2}^{n} p_0^{tn^{-1}}(q_j - q_{j-1})\delta(q-q_n).$$

From the semigroup property of p_0^t it follows that ($p_0^0(q) = \delta(q)$)

$$I_t(F|v,n) \leq \prod_{j=1}^{n} \left[\int dqF(q) \int dq_1 p_0^{t\frac{j}{n}}(q-q_1)e^{tv(q_1)} p_0^{t\frac{n-j}{n}}(q_1-q) \right]^{\frac{1}{n}} =$$

$$= m^d(2\pi t)^{-d} \prod_{j=1}^{n-1} \left[\left(\frac{n^2}{j(n-j)} \right)^d \int dqF(q) \int dq_1 e^{-(\frac{nm}{2tj} + \frac{nm}{2t(n-j)})\|q-q_1\|^2} e^{tv(q_1)} \right]^{\frac{1}{n}} \times$$

$$\times \left[\int dqF(q)e^{tv(q)} p_0^t(q,q) \right]^{\frac{1}{n}} \leq$$

$$\leq \left[m(2\pi t)^{-1} \left(\frac{n^{2n}}{(n!)^2} \right)^{\frac{1}{n}} \right]^d \left[\int dqF(q) \int dq_1 e^{-\frac{m}{t}\|q-q_1\|^2} e^{tv(q_1)} \right]^{\frac{n-1}{n}} \times$$

$$\times \left[\int dqF(q)e^{tv(q)} p_0^t(q,q) \right]^{\frac{1}{n}}.$$

Hence

$$\lim_{n\to\infty} I_t(F \mid v,n) \leq (m(2\pi t)^{-1}e^2)^d \int dqF(q) \int dq_1 e^{-\frac{m}{t}||q-q_1\|^2} e^{tv(q_1)}.$$

Here we used the elementary inequalities $\dfrac{n}{j} \geq \dfrac{n}{n-1}$, $\dfrac{n}{n-j} \geq \dfrac{n}{n-1}$ $\dfrac{n}{n-1} \geq 1$ and the formula $\lim\limits_{n\to\infty} \left(\dfrac{n^{2n}}{(n!)^2} \right)^{\frac{1}{n}} \leq e^2$. This formula follows from the Stirling inequality $n! \geq n^n e^{-n}\sqrt{2\pi n}$.

The proposition is proved.

Remark that (2.5) is useful if its right-hand side is finite. The additional condition $\int e^{-\frac{m}{2t}\|q\|^2}F(q)dq < \infty$, $t \in \mathbb{R}^+$, guarantee this (see (4.7)). For $F = 1$ (2.5) results in

$$\mathrm{Tr}\big(e^{t((2m)^{-1}\Delta+\hat{v})}\big) \leq \big(m(2\pi t)^{-1}e^2\big)^d \int dq \int dq_1 e^{-\frac{m}{2t}\|q-q_1\|^2} e^{tv(q_1)} =$$

$$= \big(m(2\pi t)^{-1}e^4\big)^{\frac{d}{2}} \int dq e^{tv(q)} = e^{2d}\,\mathrm{Tr}(e^{t(2m)^{-1}\Delta}e^{t\hat{v}}).$$

This bound coincide with the Golden – Thompson inequality if e^{2d} is dropped in the right-hand side. We'll apply this proposition for the case of an unbounded F.

As a result (2.3) – (2.5) for $d = |\Gamma|$

$$\hat{I}_*(\Gamma) \leq \big(m_g(2\pi\beta)^{-1}e^2\big)^{|\Gamma|}I_*(\Gamma), \quad I_*(\Gamma) = I_*(\Gamma;1),$$

$$I_*(\Gamma;r) = \int \prod_{\langle x,y\rangle\in\Gamma} \mu(dq_xdq'_x)\mu(dq_ydq'_y)\exp\big\{-\beta r\Phi_*(q_{x,y},q'_{x,y})\big\}. \tag{2.7}$$

Now, we show how to bound this integral in terms of the integral $I_*(2;r)$ coinciding with $I_*(\Gamma;r)$ when Γ is one pair adjacent to a corresponding face. In other words we have to decouple the variables in the integral. The contour Γ, which the set of near neighbor (nn)-pairs adjacent to the contour, contains overlapping pairs. The corners of the contour creates the crossing. Let's recollect that the contour is the external boundary of the connected (by faces) union of unit cubes centered at the sites of the bounded subset of the hyper-cubic lattice. A set of the nn-pairs is a union of the nonoverlapping set Γ^- of nonintersecting nn-pairs and the set Γ^+ which is a disjoint union of connected sets Γ^+_l of intersecting nn-pairs. Its clear that

$$I_*(\Gamma^-) = \big(I_*(2;1)\big)^{|\Gamma^-|}, \tag{2.8}$$

$$I_*(\Gamma) = I_*(\Gamma^-)\prod_l I_*(\Gamma^+_l). \tag{2.9}$$

Our main tool of the decoupling is the generalized Helder inequality (2.6). In the case of the d-dimensional hypercubic lattice no more than $2d-1$ contour nn-pairs can intersect. Let's describe the decoupling inductive process for it. Let the nn-pair (1.2) be the boundary nn-pair and 1 is the site which does not belong to other nn-pairs associated to Γ^+_l. Let's apply (2.6) for $n = 2d-1$ in s variables: a variable, indexed by 2, and the variables $j = 1, 3, s$, $s \leq 2d$, which are linked with the site 2 in other nn-pairs. $2d-1-(s-1)$ functions coincide with 1 and the other with $\exp\{-\beta\Phi_{*(2,j)}\}$, $j = 1, 3, \ldots, s$. Hence, instead of the integral in $(q,q')_{1,\ldots,s}$, we obtain the product

$$\left(\int \mu\big(d(q,q')\big)\right)^{\frac{s(2d-s)+(s-2)(s-1)}{2d-1}} \big(I_*(2;2d-1)\big)^{\frac{s-1-|\Gamma'_2|}{2d-1}} \times$$

$$\times \prod_{j\in\Gamma'_2}\left[\int \mu(d(q,q')_2)e^{-(2d-1)\beta\Phi_*((q,q')_{2,j})}\right]^{\frac{1}{2d-1}},$$

where $\Gamma'_2 \subseteq (3,\ldots,s)$ is the set of nn-sites which are linked with other nearest neighbors. Then this operation one has to employ for any site from Γ'_2 once more. Among the functions f_j in (2.6) there will be functions represented by the integrals in the right-hand side of the last inequality. Thus moving from one site to the next we'll fulfil the procedure of the decoupling of all nn-pairs for Γ_l^+. After that we have to use (2.8), (2.9) and the Hölder inequality for $I_*(2;1)$ with $n = 2d-1$

$$\int |f(x)|\mu(dx) \le \left(\int |f|^n(x)\mu(dx)\right)^{\frac{1}{n}} \left(\int \mu(dx)\right)^{\frac{n-1}{n}}. \qquad (2.10)$$

Since

$$\frac{s(2d-s)+(s-2)(s-1)}{2d-1} = s - \frac{2(s-1)}{2d-1} \le 2(s-1)$$

we proved the following proposition.

Proposition 2.2. *The following formula is valid for the integral defined in* (2.7)

$$I_*(\Gamma) \le \big(I_*(2;0)\big)^{|\Gamma|}\big(I_*(2;2d-1)\big)^{\frac{|\Gamma|}{2d-1}} \qquad (2.11)$$

if $I_*(2;0) \ge 1$. *If* $I_*(2;0) \le 1$ *then*

$$I_*(\Gamma) \le \big(I_*(2;2d-1)\big)^{\frac{|\Gamma|}{2d-1}}.$$

Inequality (2.3) and (2.8), (2.9) and the last proposition complete the proof of the theorem. It is not difficult to see that

$$I_*(2;0) = \left(\int \mu(dqdq')\right)^2 = \frac{2\pi\beta}{m_g}\big\|e^{-\beta u_*^+}\big\|_1^2. \qquad (2.12)$$

In the next section we'll show that the norm in the right-hand side of the last inequality is bounded in g (see (3.12)). Then Theorem 2.1 and the following theorem together will prove Theorem 1.1 since $e_*(g)$ grows at infinity weaker than e_0.

Theorem 2.2. *Let the conditions of Theorem* 1.1. *Then for arbitrary positive* δ_* *there exists a positive constant* C_* *independent of* g *such that the following inequality holds for sufficiently large* g

$$I_*(2;r) \le C_* e^{\beta\delta_* e_0}.$$

3. Potential minima. The main result of this section are formulas (3.8), (3.12). In order to prove them we have to introduce the new potential h and describe properties of the minima of u_g and u in terms of its minima.

$$u(z^{-1}q) = z^{-2n}h(q), \qquad u_g(q) = u(z^{n-1}q) = z^{-2n}h(z^n q)$$

where

$$h(q) = \eta q^{2n} - q^{2n_0} + h^1(q), \qquad h^1(q) = \sum_{j=1}^{n-1}\eta_j z^{2(n-j)}q^{2j} = z^{2n}u^1(z^{-1}q).$$

Let $e'(j)$, $\mu(j)$, $e(j)$ be the minima of u, h, u_g, respectively. The deepest among the sequences will be denoted by e_+, μ, e_0, respectively. Then

$$e'(j) = z^{-1}\mu(j), \qquad e(j) = z^{-n}\mu(j). \qquad (3.1)$$

It is so since for the equations for the minima we have

$$\partial u(z^{-1}q) = \partial h(q) = 0, \qquad \partial u_g(q) = \partial h(z^n q) = 0, \quad \partial = \frac{\partial}{\partial q}.$$

It is not difficult to check that

$$\partial^s u_g(q) = z^{-2n+sn}\partial^s h(z^n q), \qquad \partial^2 u_g(q) = \partial^2 h(z^n q). \tag{3.2}$$

Equalities (3.1), (3.2) yield

$$\partial^s u_g(e(j)) = z^{-2n+sn}\partial^s h(\mu(j)), \qquad \partial^2 u_g(e(j)) = \partial^2 h(\mu(j)). \tag{3.3}$$

This implies that the derivatives of $\partial^s u_g$ at the minima of u_g tend to zero if g tends to infinity and $s > 2$. This fact will play a significant role in our further consideration.

There is only one root μ of ∂h which converges to the unique positive root μ^0 of ∂h^0

$$h^0(q) = \eta q^{2n} - q^{2n_0}, \qquad \partial h^0(q) = 2q^{2n_0-1}(\eta n q^{2(n-n_0)} - n_0)$$

when z tends to zero. It is simple and an analytical function of z in the neighborhod of zero (see the first section in [15]), that is

$$\mu = \mu^0 + \sum_{k\geq 1} z^k \mu_k, \qquad \mu^0 = \left(\frac{n_0}{n\eta}\right)^{\frac{1}{2(n-n_0)}}. \tag{3.4}$$

It is the deepest minimum of h since other roots μ_j of ∂h converge to zero for vanishing z. For these roots there is the following convergent expansion in the neighborhod of zero [15]

$$\mu(j) = \sum_{k\geq 1} z^{\frac{k}{l_j}}\mu_{k,j}, \qquad \sum_s l_s = 2n_0 - 1, \quad 1 \leq l_j \leq 2n_0 - 1. \tag{3.5}$$

We'll restrict z to the neighborhod of zero, i.e., the interval $[0, z^0]$, $z_0 \leq 1$, where $\frac{1}{2}\mu^0 \leq$ $\leq \mu \leq 2\mu^0$, $\theta = \frac{1}{4}\min_{z\in[0,z^0]} \partial^2 h(\mu) > 0$. The last inequality is possible since $\partial^2 h^0(\mu^0) >$ > 0 and h, μ are continuous in z.

The Taylor expansion for h is given by

$$h(q+\mu) = h(\mu) + \sum_{s=2}^{2n} \frac{q^s}{s!}\partial h^s(\mu).$$

Then one can choose $\delta < 1$ such that (h is symmetric function)

$$h(q \pm \mu) - h(\mu) \geq \theta q^2, \qquad |q| \leq \delta\mu. \tag{3.6}$$

Both numbers μ_j and $h(\mu_j)$ tend to zero for vanishing z, that is μ_j are shallow minima with respect to the deepest minimum μ, and for sufficiently small δ we have

$$h(q) - h(\mu) \geq \theta_*, \quad q \in \mathbb{R}\backslash\big[(1-\delta)\mu, (1+\delta)\mu\big]\backslash\big[-(1+\delta)\mu, -(1-\delta)\mu\big],$$
$$\theta_* = \min_{z\in[0,z^0]}\Big(h\big((1+\delta)\mu\big) - h(\mu), h\big((1-\delta)\mu\big) - h(\mu)\Big) > 0. \tag{3.7}$$

This is obvious for the case $h^1 = 0$. We end this section my deriving a bound for the integral which will be used in the next section.

Proposition 3.1. *Let $\epsilon = z^{2n(n-1)}$ and $z \in [0, z^0]$ then for arbitrary positive numbers a, b there exists a positive number R independent of z such that*

$$\int e^{-\beta(u_*(q)-av^0(z^n q)-b\epsilon v^0(q))}dq \leq$$

$$\leq \left[\sqrt{\pi}\beta^{-\frac{1}{2}} + (\theta_*\beta)^{-\frac{1}{2n}}R\kappa + \sqrt{\pi}(\beta\theta)^{-\frac{1}{2}}\right]e^{\beta(a+b)v^0(6\mu^0+R)}, \tag{3.8}$$

where $\kappa = \sup_{a\geq 0}\sqrt{a}e^{-a}$.

Proof. Let us rescale the variable in the integral by z^{-n}, use the inequality $\epsilon v^0(z^{-n}q) \leq v^0(q)$ and translate the variable by $-\mu$. As result the integral is less than

$$z^{-n}\int e^{-\beta z^{-2n}(h(q)-h(\mu)-(a+b)z^{2n}v^0(q-\mu))}dq. \tag{3.8a}$$

Let' put ε instead of δ in (3.6), (3.7) and let $R > 0$ be such that

$$h(q) - h(\mu) - (a+b)v^0(q-\mu) \geq q^2, \quad |q| \geq R \geq 1.$$

Let's decompose the positive half-line into three sets (the first set is the set in the round brackets)

$$\mathbb{R}^+ = ([0,(1-\varepsilon)\mu]\bigcup[(1+\varepsilon)\mu, R])\bigcup[(1-\varepsilon)\mu,(1+\varepsilon)\mu]\bigcup[R,\infty]$$

change sign of q in the integral in (3.8a) and make estimates of the integral over these sets. For the integral over the second set we have

$$\int\limits_{(1-\varepsilon)\mu}^{(1+\varepsilon)\mu} e^{-\beta z^{-2n}[h(q)-h(\mu)-z^{2n}(a+b)v^0(q-\mu)]}dq =$$

$$= \int\limits_{-\varepsilon\mu}^{\varepsilon\mu} e^{-\beta z^{-2n}[h(q+\mu)-h(\mu)-z^{2n}(a+b)v^0(q+2\mu)]}dq \leq$$

$$\leq e^{\beta(a+b)v^0(6\mu^0)}\int\limits_{-\varepsilon\mu}^{\varepsilon\mu} e^{-\beta z^{-2n}(h(q+\mu)-h(\mu))}dq,$$

where we used $\mu \leq 2\mu^0$. Inequality (3.6) implies that the last integral is less than

$$\int\limits_{-\varepsilon\mu}^{\varepsilon\mu} e^{-\beta\theta z^{-2n}q^2}dq \leq \int e^{-\beta\theta z^{-2n}q^2}dq = \sqrt{\pi}(\beta\theta)^{-\frac{1}{2}}z^n. \tag{3.9}$$

The integral in (3.8a) over the first set is less (due to (3.7)) than

$$Re^{\beta(a+b)v^0(R+2\mu^0)}e^{-\beta\theta_* z^{-2n}}. \tag{3.10}$$

The integral in (3.8a) over the third set is less than

$$z^{-n}\int e^{-\beta z^{-2n}q^2}dq = \beta^{-\frac{1}{2}}\sqrt{\pi}. \tag{3.11}$$

Combining (3.9) – (3.11) we obtain the needed bound. Making the above decomposition for the right half-line and repeating the same arguments we prove the needed bound.

For $a = \zeta$, $b = 3$ we have

$$\|e^{-\beta u_*^+}\|_1 \le e^{\beta\zeta_0} C e^{\beta(\zeta+3)v(6\mu^0+R)}. \qquad (3.12)$$

4. Proof of Theorem 2.2. In this section we derive a bound for the integral $I_*(2;r)$. The main result is formulated in the following theorem which proves Theorem 2.2.

Theorem 4.1. *Let all the conditions, except the condition for the mass m, of Theorem* 1.1 *be satisfied, g be such that $z \in [0, z^0]$, $R_1 = \beta\theta_\mu - 12re_0^{-1} > 0$, $R_2 = 1 - 24r\beta^2(m_g e_0)^{-1} > 0$, $\theta_\mu = \mu^{-2}\theta$ and θ determined in* (3.6). *Then there exists positive numbers C_s, $s = 1, 2, 3$, independent of g, m, such that the following inequality holds for an arbitrary positive number δ_**

$$I_*(2;r) \le \tilde{I}(m,g) e^{\beta\delta_* e_0} e^{R_0}, \qquad R_0 = 16 m_g^{-1} r^2 \beta^3 R_2^{-1},$$

$$\tilde{I}(m,g) = \Big\{ C_1 + m_g^{-1}\big[C_2(1 + R_1^{-1}) + C_3 R_2^{-1}\big]\Big\}.$$

From the equalities

$$m_g^{-1} \le g^{\frac{2-n}{2(n-n_0)}} g^{\frac{n-1}{n-n_0}} m'(g)^{-1} = g^{\frac{n}{2(n-n_0)}} m'(g)^{-1} = e_0(\mu m'(g))^{-1},$$

$$(e_0 m_g)^{-1} = (\mu m'(g))^{-1},$$

where μ is bounded in g, it is seen that R_2 is uniformly bounded for large g, $e_0^{-1}R_0$ tends to zero in the limit of infinite g, that is the conditions of Theorem 1.1 are satisfied for sufficiently large g (R_1 is also uniformly bounded at large g since e_0^{-1} is small). This concludes the proof of Theorem 2.2.

***Proof of Theorem* 4.1.** Let the integral in prime variables in in the expression for $I_*(2;r)$ in (2.7) be denoted by $I(q_1, q_2)$. The main idea of our bounds is to determine the behavior of the function $I(q_1, q_2)$ in a neighborhood of the critical point $(-2e_0, 0)$, $(0, -2e_0)$ of $u_*(q_1) + u_*(q_2)$ and outside. We decompose the range of integration in this integral into several subsets $S_{j,l}$ covering $\mathbb{R}^2$ which are related to the behavior of u_* near the critical points (see (4.2), (4.3)). The restriction of the integral to the sets is denoted by $I_{S_{j,l}}$. We'll show that these integrals are bounded by the products $G_l(q_1)G_k(q_2)$, $G_l(q_1 + 2e_0)G_k(q_2 + 2e_0)$, $G_0(q_1 + 2e_0)G'(q_2 + e_0)$, $k, l = 0, 1, 2, 3$, of exponentially decreasing at infinity functions G_k, G' with special coefficients (the coefficient before the last product decrease exponentially in g at infinity). Then after translating the variables by $-2e_0$ (in the last two cases) we apply the bounds

$$e^{\beta re_0^{-1}(q_1-q_2)^2} \le e^{2\beta re_0^{-1}(q_1^2+q_2^2)}, \qquad e^{\beta re_0^{-1}|q_j(q_j \pm 2e_0)|} \le e^{\beta re_0^{-1}(q_j^2 + 2e_0|q_j|)|}, \quad (4.1)$$

and estimate the integrals

$$I'_j = \int G_j(q) e^{\beta r(3e_0^{-1}q^2 + 2|q|)} dq, \qquad I' = \int G'(q) e^{\beta r(3e_0^{-1}q^2+2|q|)} dq.$$

In this way we derive the left-hand side of the condition for ξ in Theorem 1.1 which is needed to make a contribution of the set $S_{2,1}$ in the asymptotics of $I_*(2;r)$ in g moderate.

In order to simplify integrals we'll omit r in them keeping in mind that g_0 is proportional to $r \ge 1$. From (3.6), (3.7) (h is a symmetric function), the equalities

$$u_*(z^{-n}q) = z^{-2n}\big(h(q+\mu) - h(\mu)\big), \quad e_0 = \mu z^{-n}$$

we derive

$$u_*(q) \geq \theta_{*\mu} e_0^2, \qquad \big(|q+2e_0| \geq \delta e_0\big) \bigcap \big(|q| \geq \delta e_0\big), \quad \theta_{*\mu} = \mu^{-2}\theta_*, \tag{4.2}$$

$$u_*(q) \geq \theta_\mu q^2, \qquad |q| \leq \delta e_0. \tag{4.3}$$

For brevity we'll omit the index μ in θ_*, θ in the formulae of this section.

Let's decompose the space of integration in q_1', q_2' into four sets S_j, $j = 1, \ldots, 4$,

$$|q_1' + 2e_0| \leq \delta e_0, \qquad |q_2' + 2e_0| \leq \delta e_0; \qquad |q_1' + 2e_0| \geq \delta e_0, \qquad |q_2' + 2e_0| \leq \delta e_0,$$

$$|q_1' + 2e_0| \leq \delta e_0, \qquad |q_2' + 2e_0| \geq \delta e_0; \qquad |q_1' + 2e_0| \geq \delta e_0, \qquad |q_2' + 2e_0| \geq \delta e_0.$$

Every set S_j, $j \geq 2$, will be, also, decomposed into subsets $S_{j,l}$ with the help of the inequalities

$$j = 2, \quad l = 1, \quad |q_1' + e_0| \geq \delta; \qquad l = 2, |q_1' + e_0| \leq \delta;$$

$$j = 3, \quad l = 1, \quad |q_2' + e_0| \geq \delta; \qquad l = 2, |q_2' + e_0| \leq \delta;$$

$$j = 4, \quad l = 1, \quad |q_1'| \leq \delta e_0, \quad |q_2'| \leq \delta e_0; \qquad l = 2, \quad |q_1'| \geq \delta e_0, \quad |q_2'| \leq \delta e_0;$$

$$j = 4, \quad l = 3, \quad |q_1'| \leq \delta e_0, \qquad |q_2'| \geq \delta e_0; \qquad l = 4, \quad |q_1'| \geq \delta e_0, \quad |q_2'| \geq \delta e_0.$$

The integral $I_*(2;r)$ will be denoted by $I_{*S_{j,l}}(2;r)$ if the integral $I(q_1,q_2)$ is replaced by $I_{S_{j,l}}(q_1,q_2)$. The most simple estimate is obtained for S_1 neglecting a dependence on positive ϕ_*

$$I_{S_1} \leq \int\limits_{S_1} \prod_{j=1}^{2} e^{-(2\beta)^{-1} m_g (q_j' - q_j)^2} e^{-\beta u_*^+(q_j')} dq_j' \leq$$

$$\leq \prod_{j=1}^{2} G_0(q_j + 2e_0) \left(\int e^{-\beta u_{*g}^+(q)} dq \right)^2, \tag{4.4}$$

where $G_0(q_j) = e^{-(2\beta)^{-1} m_g (|q_j| - \delta e_0)^2} + \chi^+(\delta e_0 - |q_j|)$. Here we used the inequalities

$$(q_j - q_j')^2 = \big((q_j + 2e_0) - (q_j' + 2e_0)\big)^2 \geq (|q_j + 2e_0| - \delta e_0)^2,$$

$$|q_j' + 2e_0| \leq \delta e_0, \qquad |q_j + 2e_0| \geq \delta e_0.$$

Formulae (4.4) and (4.1) lead to after translating variables by $-2e_0$

$$I_{*S_1}(2;r) \leq \big\| e^{-\beta u_*^+} \big\|_1^2 {I_0'}^2, \tag{4.4a}$$

$$I_0' \leq \int e^{3r\beta e_0^{-1} q^2} e^{2r\beta |q|} e^{-(2\beta)^{-1} m_g (|q| - \delta e_0)^2} dq + \delta e_0 e^{\beta r e_0 (3\delta^2 + 2\delta)} \leq$$

$$\leq 2 e^{r\beta e_0 (3\delta^2 + 2\delta)} \int e^{3r\beta e_0^{-1} q^2} e^{2r\beta q} e^{-(2\beta)^{-1} m_g q^2} dq + \delta e_0.$$

Here we shifted the variables in the integrals over the right half-line by δe_0. Hence

$$I_{*S_1}(2;r) \leq 4\left\|e^{-\beta u_*^+}\right\|_1^2 e^{2r\beta e_0(3\delta^2+2\delta)}\left[2\pi\beta m_g^{-1}R_2^{-1}e^{R_0}+\delta^2e_0^2\right], \tag{4.5}$$

where we used the elementary inequality $(a+b)^2 \leq 2(a^2+b^2)$ and the formula

$$\int e^{-bq^2}e^{aq}dq = \sqrt{b^{-1}\pi}e^{\frac{a^2}{4b}}, \quad a,b>0. \tag{4.5a}$$

That is, $I_{*S_1}(2;r)$ satisfies the main theorem bound (see (3.12)). Inequality (4.3) and the definition of u_*^+, v^+ (see Theorem 2.1) for the case $j=4, l=1$ yield

$$I_{S_{4,1}} \leq \prod_{j=1}^{2} \int\limits_{|q'_j|\leq\delta e_0} e^{-(2\beta)^{-1}m_g(q'_j-q_j)^2}e^{-\beta u_*^+(q'_j)}dq'_j \leq$$

$$\leq e^{\beta v^+(e_0\delta)}\prod_{j=1}^{2}\int e^{-(2\beta)^{-1}m_g(q'_j-q_j)^2}e^{-\beta\theta q_j'^2}dq'_j \leq e^{\beta v^+(\delta e_0)}\prod_{j=1}^{2}G_1(q_j), \tag{4.6}$$

$$G_1(q) = e^{-(8\beta)^{-1}m_gq^2}\int e^{-\beta\theta q^2}dq + e^{-4^{-1}\beta\theta q^2}\int e^{-(2\beta)^{-1}m_gq^2}dq.$$

Here we used the inequality for two monotonic functions f, h

$$\int f\big(|q-q'|\big)h\big(|q'|\big)dq' \leq f\big(2^{-1}|q|\big)\|h\|_1 + h\big(2^{-1}|q|\big)\|f\|_1. \tag{4.7}$$

To derive it we decomposed the range of integration into two sets $|q'| \geq 2^{-1}|q|$, $|q'| \leq$ $\leq 2^{-1}|q|$ and took into consideration that the functions are monotone.

For $j=4$, $l=4$ let's add and subtract in the exponent the polynomial $16r\beta e_0^{-1}[\sigma^2+$ $+\sigma'^2]$, $\sigma(q)=q$, and then apply (4.2) together with the Schwartz inequality in primed variables. This leads to

$$I_{S_{4,4}} \leq e^{-\beta\theta_*e_0^2}\prod_{j=1}^{2}\int e^{-(2\beta)^{-1}m_g(q-q_j)^2}e^{-\beta(u_*^+(q)-\frac{1}{2}u_*(q))}dq \leq$$

$$\leq \|e^{-2\beta(u_*^+-\frac{1}{2}u_*-16re_0^{-1}\sigma^2)}\|_1 e^{-\beta\theta_*e_0^2}\left(\prod_{j=1}^{2}\int e^{-\beta^{-1}m_g(q-q_j)^2}e^{-32r\beta e_0^{-1}q^2}dq\right)^{\frac{1}{2}}.$$

From (4.7) we deduce that

$$I_{S_{4,4}} \leq \left\|e^{-2\beta(u_*^+-\frac{1}{2}u_*-16re_0^{-1}\sigma^2)}\right\|_1 e^{-\beta\theta_*e_0^2}\prod_{j=1}^{2}G_2(q_j), \tag{4.8}$$

$$G_2^2(q_j) = e^{-8r\beta e_0^{-1}q_j^2}\int e^{-\beta^{-1}m_gq^2}dq + e^{-(4\beta)^{-1}m_gq_j^2}\int e^{-32r\beta e_0^{-1}q^2}dq.$$

For $S_{4,l}$, $l=2,3$, we have to use both (4.2), (4.3)

$$I_{S_{4,2}}(q_1,q_2) \le e^{\beta v^+(e_0\delta)} e^{\beta\theta_* e_0^2} \int e^{-(2\beta)^{-1}m_g(q'_j-q_1)^2} e^{-\beta(u_*^+(q'_j)-\frac{1}{2}u_*(q))} dq'_1 \times$$

$$\times \int e^{-(2\beta)^{-1}m_g(q'_2-q_2)^2} e^{-\beta\theta q'^2_j} dq'_2 \le$$

$$\le e^{\beta v^+(e_0\delta)} e^{-\beta\theta_* e_0^2} \left\| e^{-2\beta(u_*^+ - \frac{1}{2}u_* - 16re_0^{-1}\sigma^2)} \right\|_1^{\frac{1}{2}} G_2(q_1)G_1(q_2). \tag{4.9}$$

Here we applied the same arguments as in the case $j=4$, $l=4$. The same inequality holds for $I_{S_{4,3}}(q_1,q_2)$ with the permuted variables. From the definition of ϕ_* we derive putting

$$\phi_*(q'_1,q'_2) = \phi(z^{n-1}(q'_1+e_0), z^{n-1}(q'_2+e_0)) \ge$$

$$\ge z^{2(n-1)(n_1+n_2)}(q'_1-q'_2)^{2n_1} \times$$

$$\times\left[(q'_1+e_0)^{2n'_2}(q'_2+e_0)^{2(n_2-n'_2)} + (q'_2+e_0)^{2n'_2}(q'_1+e_0)^{2(n_2-n'_2)}\right].$$

As a result for $q'_1, q'_2 \in S_{2,1}$ we have $(1-\delta > \delta)$

$$\phi_*(q'_1,q'_2) \ge$$

$$\ge (|q'_1+2e_0| - \delta e_0)^{2n_1}\left[\delta^{2(n_2-n'_2)}\big((1-\delta)e_0\big)^{2n'_2} + \delta^{2n'_2}\big((1-\delta)e_0\big)^{2(n_2-n'_2)}\right] \ge$$

$$\ge \big(|q'_1+2e_0| - \delta e_0\big)^{2n_1} z_1,$$

$$z_1 = z^{2(n-1)(n_1+n_2)}\delta^{2n_2} e_0^{2n'_2}.$$

This yields after applying the Schwartz inequality

$$I_{S_{2,1}}(q_1,q_2) \le$$

$$\le \left\|e^{-\beta u_*^+}\right\|_1 G_0(q_2+2e_0) \int e^{-(2\beta)^{-1}m_g(q'-q_1)^2} e^{-\beta r z_1(|q'+2e_0|-\delta e_0)^{2n_1}} e^{-\beta u_*^+(q')} dq' \le$$

$$\le \left\|e^{-\beta u_*^+}\right\|_1 \left\|e^{-\beta u_*^+}\right\|_2 G_0(q_2+2e_0) \times$$

$$\times \left(\int e^{-\beta^{-1}m_g(q'-q_1)^2} e^{-\beta r z_1(|q'+2e_0|-\delta)^{2n_1}} dq'\right)^{\frac{1}{2}} \le$$

$$\le \left\|e^{-\beta u_*^+}\right\|_1 \left\|e^{-\beta u_*^+}\right\|_2 G_0(q_2+2e_0) G_3(q_1+2e_0), \tag{4.10}$$

$$G_3^2(q) = \left(e^{-\beta r z_1(2^{-1}|q|-\delta e_0)^{2n_1}} + \chi^+(\delta e_0 - 2^{-1}|q|)\right) \int e^{-(2\beta)^{-1}m_g q'^2} dq' +$$

$$+ e^{-(8\beta)^{-1}m_g q^2}\left(\int e^{-\beta r z_1(|q'|-\delta e_0)^{2n_1}} dq' + \delta e_0\right).$$

Here we applied (4.7) once more and put

$$f(q) = e^{-(2\beta)^{-1}m_g(q'-q)^2},$$

$$h(q) = e^{-\beta r z_1(|q|-\delta e_0)^{2n_1}} \chi^+(|q|-\delta e_0) + \chi^+(\delta e_0 - |q|).$$

Here h is a monotonic function. For $S_{2,2}$ we derive

$$I_{S_{2,2}}(q_1,q_2) \le$$

$$\le \|e^{-\beta u_*^+}\|_1 G_0(q_2+2e_0) \int\limits_{|q'|\le\delta} e^{-(2\beta)^{-1}m_g(q'-q_1-e_0)^2} e^{-\beta u_*^+(q'-e_0)}dq' \le$$

$$\le \|e^{-\beta u_*^+}\|_1 e^{\beta(u_g(e_0)+v^+(e_0+\delta))}\left(\int\limits_{|q'|\le\delta} e^{-\beta u_g(q')}dq'\right) G_0(q_2+2e_0)G'(q_1+e_0), \tag{4.11}$$

$$G'(q) = (e^{-(2\beta)^{-1}m_g(|q|-\delta)^2} + \chi^+(-|q+e_0|+\delta)).$$

Function u_g is a bounded in g, hence the integral in the second line in (4.11) is bounded in $g \ge 1$. Here one has to check that u_g tends to a finite limit when g tends to infinity. This follows from the fact that the coefficient before q^{2n_0} in the expression for u_g is equal to $g^{1-\frac{n_0(n-1)}{n-n_0}} = g^{-\frac{(n_0-1)n}{n-n_0}}$, $n_0 \ge 1$.

The same bounds are obtained for $I_{S_{3,1}}, I_{S_{3,2}}$ by permuting variables in the bounds (4.10), (4.11) for $I_{S_{2.1}}, I_{S_{2,2}}$. Now we can estimate $I_{*S_{j,l}}(2;r)$. For $j=4, l=1$ from (4.6), (4.5a) and

$$-12re_0^{-1}+\beta\theta = R_1 > 0, \qquad -24\beta^2 r(e_0m_g)^{-1}+1 = R_2 > 0 \tag{4.12}$$

it follows that

$$I_{*S_{4,1}}(2;r) \le {I_1'}^2 \le 416\pi^2(\theta m_g)^{-1}R_2^{-1}e^{R_0} + 2\pi^2\beta m_g^{-1}R_1^{-1}e^{4r^2\beta^2R_1^{-1}}. \tag{4.13}$$

For $j=4$, $l=4$ from (4.8) and the first inequality in (4.12) it follows that

$$I_{*S_{4,4}}(2;r) \le \left\|e^{-2\beta(u_*^+-\frac{1}{2}u_*-16re_0^{-1}\sigma^2)}\right\|_1 e^{-\beta\theta_*e_0^2}{I_2'}^2 \le$$

$$\le 2\left\|e^{-2\beta(u_*^+-\frac{1}{2}u_*-16re_0^{-1}\sigma^2)}\right\|_1 e^{-\beta\theta_*e_0^2}\times$$

$$\times\left[\frac{4\pi\beta}{m_g}\left(\int e^{-r\beta(e_0^{-1}q^2-2|q|)}dq\right)^2 + \pi^2e_0m_g^{-1}(8rR_2)^{-1}e^{R_0}\right]. \tag{4.14}$$

Here we used the inequality $\sqrt{a+b} \le \sqrt{a}+\sqrt{b}$, $(a+b)^2 \le 2(a^2+b^2)$, $a, b \ge 0$ and (4.5a). By Proposition 3.1 the norm in this inequality behaves as $\exp\{ae_0\}, a>0$. The same is true for the integral inside the square brackets. This and the dependence of m_g on g mean that $I_{*S_{4,4}}(2;r)$ is bounded in g. The analog of (4.14) is easily obtained for $I_{*S_{4,l}}(2;r)$, $l=2,3$,

$$I_{*S_{4,l}}(2;r) \le 2e^{-\beta\theta_*e_0^2}\left[{I_1'}^2+{I_2'}^2\right].$$

From (4.13), (4.14) it follows that the integrals $I_{*S_{4,l}}(2;r)$, $l=2,3$, are bounded in g. For $j=2$, $l=1$ we have (see (4.10))

$$I_{*S_{2,1}}(2;r) \le 2\|e^{-\beta u_{*g}}\|_1\|e^{-\beta u_{*g}}\|_2\left[{I_0'}^2+{I_3'}^2\right],$$

$${I_3'}^2 \le \frac{2\beta\pi}{m_g}\left[\left(\int\limits_{|q|\le 2\delta e_0} e^{r\beta(3e_0^{-1}q^2+2|q|)}dq\right)^2+\right.$$

$$+e^{2\beta(3\delta^2+2\delta)e_0}\left(\left(\left(\int e^{-\beta rz_1(2^{-1}q)^{2n_1}}e^{\beta(3e_0^{-1}q^2+2|q|)}dq\right)^2+\right.\right.$$

$$\left.\left.+R_2^{-1}e^{R_0}2(\beta rz_1)^{-\frac{1}{n_1}}\int e^{-q'^{2n_1}}dq'\right)\right]. \qquad (4.15)$$

Here we performed the operations, given at the beginning of the proof, and applied the formulas

$$\sqrt{a_1+a_2+a_3}\le\sqrt{a_1}+\sqrt{a_2}+\sqrt{a_3},$$

$$(a_1+a_2+a_3)^2\le 2^3(a_1^2+a_2^2+a_3^2)$$

translated, rescaled and translated the variable by $\pm 2\delta e_0$ in the integrals (over $\mathbb{R}^+$) corresponding to the last and first terms, respectively, in the expression for G_3. The second integral in the right-hand side of (4.15) can be estimated by taking the product of maximums of two exponents, depending on the quadratic and linear terms, multiplied by the root of the third order of the first exponent. Hence, it is less than

$$2\left(\max_{q\ge 0}e^{-3^{-1}\beta rz_1(2^{-1}q)^{2n_1}}e^{2\beta q}\right)\times$$

$$\times\left(\max_{q\ge 0}e^{-3^{-1}\beta rz_1e_0^{n_1}(2^{-1}q)^{n_1}}e^{3\beta q}\right)\left(3^{-1}\beta rz_1\right)^{-\frac{1}{n_1}}\int e^{-q'^{2n_1}}dq'.$$

In the second round bracket we rescaled the variable by $\sqrt{e_0}$. The firsts and the second terms grow in g as the exponent of $z_1^{-\frac{1}{2n_1-1}}$, $(z_1e_0^{n_1})^{-\frac{1}{n_1-1}}$, respectively. This growth has to be weaker than the growth of e_0, i.e. z^{-n}. That is, $-\xi+2(n-1)(n_1+n_2)-2n_2'n<$ $<(2n_1-1)n$,

$$-\xi+2(n-1)(n_1+n_2)-2n_2'n-n_1n<(n_1-1)n.$$

Both conditions are identical to the condition of Theorem 4.1. The second and last terms in (4.15) grow as e_0 to some power. Thus $I_{*S_{2,1}}(2;r)$ satisfies the theorem main bound (provided (4.12) holds). For the case $j=2$, $l=2$ we obtain with the help of (4.1) and (4.11)

$$I_{*S_{2,2}}(2;r)\le\left\|e^{-\beta u_*^+}\right\|_1 e^{\beta(u_g(e_0)+v^+(e_0+\delta))}\left(\int\limits_{|q'|\le\delta}e^{-\beta u_g(q')}dq'\right)\times$$

$$\times 2\left[I_0'^2+\int e^{-(2\beta)^{-1}m_g(|q+e_0|-\delta)^2}e^{r\beta(3e_0^{-1}q^2+2|q|)}dq)^2\right]. \qquad (4.16)$$

After translations of the variables in two integrals the term in the square brackets is less than

$$e^{32r\beta e_0}\left\{I_0'^2+4e^{r\beta(6\delta^2+2\delta)}\left[\left(\int e^{-(2\beta)^{-1}m_gq^2}e^{r\beta(3e_0^{-1}q^2+2q)}dq\right)^2+\delta^2\right]\right\}. \quad (4.17)$$

Both integrals were estimated considering the case $j=1$. From (4.5) and (4.16), (4.17) it follows that $I_{*S_{2,2}}(2;r)$ is bounded in $g\ge 1$ since $u_g(e_0)=z^{-2n}h(\mu)=e_0^2\mu^{-2}h(\mu)$,

$h(\mu) < 0$ and $v^+(e_0 + \delta)$, $v^+(e_0\delta)$ are bounded in g. Moreover, the norms in (4.10) are bounded in $g >> 1$. It implies, also, that the norm in (4.14) grows at in infinity in g as $e^{\beta 32(6\mu^0)^2 e_0}$. Hence all $I_{*S_{j,l}}(2;r)$ satisfy the theorem main bound. This implies that the theorem is proven since $I_*(2;r) = \sum_{j,l} I_{*S_{j,l}}(2;r)$ and it is not difficult to calculate C_s from (4.5), (4.13), (4.15) – (4.17): (4.11) and the bound after (4.14) , i.e., bounds for $I_{*S_{j,2}}(2)$, $j = 2, 3$, and $I_{*S_{4,l}}(2;r)$, $l = 2, 3, 4$, generate the expression for C_1; (4.5), i.e., the bound for $I_{*S_1}(2)$ gives expressions for C_1, C_3 (e_0^2 is less than $e^{\delta_* e_0}$ multiplied by a coefficient depending on δ_*); (4.13) and (4.15), i.e., bounds for $I_{*S_{4,1}}(2;r)$ $I_{*S_{2,1}}(2;r)$, respectively, give expressions for C_2, C_3.

Remark **4.1.** In a general case a depth of the polynomial external potential is controlled by $n-1$ parameters. The different subsets of this space can be described with the help of one parameter g and the representation $u(q) = \eta q^{2n} - \sum_{s=1}^{n-1} \eta_s g^{n_s} q^{2s}$. To deal with this case one has to find an appropriate re-scaling reducing the polynomial to a sum of a polynomial with easily calculated minima and an additional polynomial depending on g^{-1}.

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