UDC 515.16

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GENERALIZED ELASTIC LINE DEFORMED ON NON-NULL SURFACE BY AN EXTERNAL FIELD IN THE 3-DIMENSIONAL SEMI-EUCLIDEAN SPACE $\mathbb{E}^3_1$

УЗАГАЛЬНЕНА ПРУЖНА ЛІНІЯ, ДЕФОРМОВАНА НА НЕНУЛЬОВІЙ ПОВЕРХНІ ЗОВНІШНІМ ПОЛЕМ У ТРИВИМІРНОМУ НАПІВЕВКІДОВОМУ ПРОСТОРІ $\mathbb{E}^3_1$

We deduce intrinsic equations for a generalized elastic line deformed on the non-null surface by an external field in semi-Euclidean space $\mathbb{E}^3_1$ and give some applications.

Виведено природні рівняння для узагальненої пружної лінії, деформованої на ненульовій поверхні зовнішнім полем у тривимірному напівевклідовому просторі $\mathbb{E}^3_1$, та наведено деякі застосування.

1. Introduction. In this section we give some definitions and theorems.

$\mathbb{E}^n$ with the metric, $\langle v, w\rangle = -\sum_{i=1}^{\nu} v_i w_i + \sum_{j=\nu+1}^{n} v_j w_j$, $v, w \in \mathbb{E}^n$, $0 \le \nu \le n$, is called semi-Euclidean space and is denoted by $\mathbb{E}^n_\nu$, where is called the index of the metric. For $n = 3$, $\mathbb{E}^3_1$ is called semi-Euclidean 3-space. Let $\mathbb{E}^n_\nu$ be a semi-Euclidean space furnished with a metric tensor $\langle\, ,\rangle$. A vector v to $\mathbb{E}^n_\nu$ is called, spacelike if $\langle v, v\rangle > 0$ or $v = 0$ and timelike if $\langle v, v\rangle < 0$. Spacelike and timelike vectors are non-null vectors [1].

Apart from the Frenet frame $\{E_1, n, b\}$, there also exist a second frame at every point of the curve γ. At the point $\gamma(s)$ of γ, let $E_1(s) = \gamma'(s)$ denote the unit tangent vector to γ, N denote the unit normal of non-null surface and $E_2(s) = \varepsilon_2 N \wedge E_1$. Then $\{E_1, E_2, N\}$ gives an orthonormal basis for all vectors at $\gamma(s)$ [2].

The analogue of the Frenet – Serret formulas is given

$$\begin{bmatrix} E_1' \\ E_2' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2 c_g & \varepsilon_3 c_n \\ -\varepsilon_1 c_g & 0 & -\varepsilon_3 \tau_g \\ -\varepsilon_1 c_n & \varepsilon_2 \tau_g & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ N \end{bmatrix}, \tag{1}$$

where c_g is the geodesic curvature, τ_g is the geodesic torsion, c_n is the normal curvature and $\langle E_1, E_1\rangle = \varepsilon_1$, $\langle E_2, E_2\rangle = \varepsilon_2$, $\langle N, N\rangle = \varepsilon_3$.

Let $x(u, v)$ be the timelike surface, having parameter curves which are perpendicular to each other passing through point $\gamma(s)$ of any curve γ. χ is angle between u = constant curve with tangent vector of timelike surface, $(c_g)_1$, $(c_g)_2$ are curvatures u = constant and v = constant curves. The geodesic curvature is [2]

$$c_g = (c_g)_1 \cosh\chi - (c_g)_2 \sinh\chi - \frac{d\chi}{ds}. \tag{2}$$

Here $(c_g)_1 = -\dfrac{1}{2}\dfrac{E_v}{|E|\,|G|^{1/2}}$, $(c_g)_2 = \dfrac{1}{2}\dfrac{G_u}{|E|^{1/2}\,|G|}$ and the normal curvature is

$$c_n = c_1 \cosh^2\chi - c_2 \sinh^2\chi, \tag{3}$$

where c_1 and c_2 are principal curvatures. The geodesic torsion is [2]

$$\tau_g = (c_2 - c_1)\cosh\chi\sinh\chi. \tag{4}$$

2. Equilibrium conditions for generalized elastic line deformed. γ is called as an elastic curve, if it is a critical point of total squared curvature energy functional

$$\int_\gamma \kappa^2 ds. \tag{5}$$

The study of elastic curves have long research history. In the 1730, Bernoulli and Euler studied the bending energy functional (5) for in R^2. In last years, elastic problem has been reconsidered many geometers [3 – 8]. Problem play important role in the connection between the motion of curves and integrable systems [9]: the equation describing the evolution of the torsion with respect to certain length preserving vector fields coincides with the nonlinear Schrödinger equation. Vortex filaments and patches in fluids [10 – 13], classical magnetic spin chains [14, 15], interface dynamic contexts have such curve motions [16].

Generalization of the (5) is given by

$$\mathcal{H} = \int_\gamma h(\kappa,\tau) ds. \tag{6}$$

Here, κ, τ denote curvature and torsion of non-null curve γ. The arc γ is called a generalized elastic line if it is extremal for variational problem of minimizing the value of (6) within the family of all arcs of length l on non-null surface having the same initial point and initial direction as γ in the semi-Euclidean 3-space $\mathbb{E}^3_1$ [17, 18].

In this paper, we study a special case of (6)

$$I_1 = \int_0^l \tau^2\kappa ds. \tag{7}$$

In [19], Manning study the intrinsic equations for elastic line deformed an external field in Euclidean 3-space. In this paper, we study the equilibrium conditions for generalized elastic line deformed on a non-null surface in semi-Euclidean space $\mathbb{E}^3_1$.

If generalized elastic line is deformed an external field, it minimizes the sum of its generalized elastic energy and its energy of interaction with the field.

The problem is to minimize the energy

$$J = \int_0^l (\tau^2\kappa - \zeta\phi) ds = I_1(t) - \zeta I_2(t), \tag{8}$$

$$I_1(t) = \int_0^l \tau^2\kappa ds, \tag{9}$$

$$I_2(t) = \int_0^l \phi ds. \tag{10}$$

Here ζ is a constant measuring the strength of the external field and $\phi(u, v)$ gives its shape. For γ regular curve in $\mathbb{E}_1^3$,

$$\kappa = \frac{|\langle \gamma' \wedge \gamma'', \gamma' \wedge \gamma'' \rangle|^{1/2}}{|\langle \gamma', \gamma' \rangle|^{3/2}} \qquad \text{and} \qquad \tau = -\varepsilon_3 \frac{\langle \gamma' \wedge \gamma'', \gamma''' \rangle}{|\langle \gamma' \wedge \gamma'', \gamma' \wedge \gamma'' \rangle|},$$

$E_1(s)$ and $E_2(s)$, respectively, are expressed with suitable scalar functions $f(s)$ and $g(s)$

$$E_1(s) = \gamma'(s) = \frac{\partial x}{\partial u}\frac{du}{ds} + \frac{\partial x}{\partial v}\frac{dv}{ds}, \qquad E_2(s) = f(s)x_u + g(s)x_v,$$

f and g are expressed for spacelike surfaces and timelike surfaces with timelike arc γ as following [8]:

$$f = \frac{u' F + v' G}{|EG - F^2|^{1/2}}, \qquad g = -\frac{u' E + v' F}{|EG - F^2|^{1/2}}.$$

Here $E = \langle x_u, x_u \rangle$, $G = \langle x_v, x_v \rangle$ and $F = \langle x_u, x_v \rangle$.

We define

$$\Psi(\rho; t) = x\left(u(\rho) + t\eta(\rho), v(\rho) + t\xi(\rho)\right), \tag{11}$$

for $0 \leq \rho \leq l^*$, $\Psi(\rho; t)$ gives an arc with the same initial point and initial direction as γ. For $t = 0$, $\Psi(\rho; 0)$ is the same as γ^* and ρ is arc length. For $t \neq 0$, the parameter ρ is not non-null arc length in general. For fixed t, $|t| < \varepsilon$, let $L^*(t)$ denote the length of the non-null arc $\Psi(\rho; t)$, $0 \leq \rho \leq l^*$. Then

$$L^*(t) = \int_0^{l^*} \left(\left| \left\langle \frac{\partial \Psi}{\partial \rho}(\rho; t), \frac{\partial \Psi}{\partial \rho}(\rho; t) \right\rangle \right| \right)^{1/2} d\rho$$

with $L^*(0) = l^* > l$. We can restrict $\Psi(\rho; t)$, to an non-null arc of length l by restricting the parameter ρ to an interval $0 \leq \rho \leq \omega(t) \leq l^*$, $\omega(0) = l$ by requiring

$$\int_0^{\omega(t)} \left(\left| \left\langle \frac{\partial \Psi}{\partial \rho}, \frac{\partial \Psi}{\partial \rho} \right\rangle \right| \right)^{1/2} d\rho = l \tag{12}$$

and

$$\left. \frac{d\omega}{dt} \right|_{t=0} = \varepsilon_1 \int_0^l \delta c_g ds. \tag{13}$$

The proof of (13) and of other results below will depend on calculations in (11) such as

$$\left. \frac{\partial \Psi}{\partial \rho} \right|_{t=0} = E_1, \quad 0 \leq \rho \leq l,$$

which gives

$$\left.\frac{\partial^2\Psi}{\partial\rho^2}\right|_{t=0} = E_1' = \varepsilon_2 c_g E_2 + \varepsilon_3 c_n N, \tag{14}$$

$$\left.\frac{\partial^3\Psi}{\partial\rho^3}\right|_{t=0} = E_1' = -\varepsilon_1(\varepsilon_2 c_g^2 + \varepsilon_3 c_n^2)E_1 + (\varepsilon_2 c_g' + \varepsilon_2\varepsilon_3 c_n\tau_g)E_2 - (\varepsilon_2\varepsilon_3 c_g\tau_g + \varepsilon_3 c_n')N, \tag{15}$$

$$\left.\frac{\partial\Psi}{\partial t}\right|_{t=0} = \delta E_2. \tag{16}$$

(17), (18) are obtained with aid (14)–(16)

$$\left.\frac{\partial^2\Psi}{\partial t\partial\rho}\right|_{t=0} = -\varepsilon_1\delta c_g E_1 + \delta' E_2 - \varepsilon_3\delta\tau_g N, \tag{17}$$

$$\left.\frac{\partial^3\Psi}{\partial t\partial\rho^2}\right|_{t=0} = \Big(-2\varepsilon_1\delta' c_g - \varepsilon_1\delta c_g' + \varepsilon_1\varepsilon_3\delta\tau_g c_n\Big)E_1 +$$

$$+\Big(\delta'' - \varepsilon_1\varepsilon_2\delta c_g^2 - \varepsilon_2\varepsilon_3\delta\tau_g^2\Big)E_2 - \Big(2\varepsilon_3\delta'\tau_g + \varepsilon_1\varepsilon_3\delta c_g c_n + \varepsilon_3\delta\tau_g'\Big)N. \tag{18}$$

To prove (13), differentiate (12) with respect to t and evaluate at $t = 0$,

$$\left.\frac{d\omega}{dt}\right|_{t=0}\left(\left|\left\langle \left.\frac{\partial\Psi}{\partial\rho}\right|_{t=0}, \left.\frac{\partial\Psi}{\partial\rho}\right|_{t=0}\right\rangle\right|\right)^{1/2} +$$

$$+\int_0^l \left\langle \left.\frac{\partial\Psi}{\partial\rho}\right|_{t=0}, \left.\frac{\partial^2\Psi}{\partial\rho\partial t}\right|_{t=0}\right\rangle \frac{\left(\left|\left\langle \left.\frac{\partial\Psi}{\partial\rho}\right|_{t=0}, \left.\frac{\partial\Psi}{\partial\rho}\right|_{t=0}\right\rangle\right|\right)^{1/2}}{\left\langle \left.\frac{\partial\Psi}{\partial\rho}\right|_{t=0}, \left.\frac{\partial\Psi}{\partial\rho}\right|_{t=0}\right\rangle} d\rho = 0.$$

$I_1(t)$ is given as following from (9):

$$I_1(t) = \int_0^{\omega(t)} \left\langle \frac{\partial\Psi}{\partial\rho}\wedge\frac{\partial^2\Psi}{\partial\rho^2}, \frac{\partial^3\Psi}{\partial\rho^3}\right\rangle^2 \left|\left\langle \frac{\partial\Psi}{\partial\rho}\wedge\frac{\partial^2\Psi}{\partial\rho^2}, \frac{\partial\Psi}{\partial\rho}\wedge\frac{\partial^2\Psi}{\partial\rho^2}\right\rangle\right|^{-3/2} \left|\left\langle \frac{\partial\Psi}{\partial\rho}, \frac{\partial\Psi}{\partial\rho}\right\rangle\right|^{-1} d\rho.$$

We have

$$I_1'(t) = \frac{d\omega}{dt}\left\{\left\langle \frac{\partial\Psi}{\partial\rho}\wedge\frac{\partial^2\Psi}{\partial\rho^2}, \frac{\partial^3\Psi}{\partial\rho^3}\right\rangle^2 \left|\left\langle \frac{\partial\Psi}{\partial\rho}\wedge\frac{\partial^2\Psi}{\partial\rho^2}, \frac{\partial\Psi}{\partial\rho}\wedge\frac{\partial^2\Psi}{\partial\rho^2}\right\rangle\right|^{-3/2} \left|\left\langle \frac{\partial\Psi}{\partial\rho}, \frac{\partial\Psi}{\partial\rho}\right\rangle\right|^{-1}\right\}_{\rho=\omega(t)} +$$

$$+\int_0^{\omega(t)} \frac{\partial}{\partial t}\left(\left\langle \frac{\partial\Psi}{\partial\rho}\wedge\frac{\partial^2\Psi}{\partial\rho^2}, \frac{\partial^3\Psi}{\partial\rho^3}\right\rangle^2\right) \left|\left\langle \frac{\partial\Psi}{\partial\rho}\wedge\frac{\partial^2\Psi}{\partial\rho^2}, \frac{\partial\Psi}{\partial\rho}\wedge\frac{\partial^2\Psi}{\partial\rho^2}\right\rangle\right|^{-3/2} \left|\left\langle \frac{\partial\Psi}{\partial\rho}, \frac{\partial\Psi}{\partial\rho}\right\rangle\right|^{-1} d\rho +$$

$$+\int\limits_0^{\omega(t)}\left\langle\frac{\partial\Psi}{\partial\rho}\wedge\frac{\partial^2\Psi}{\partial\rho^2},\frac{\partial^3\Psi}{\partial\rho^3}\right\rangle^2\frac{\partial}{\partial t}\left(\left|\left\langle\frac{\partial\Psi}{\partial\rho}\wedge\frac{\partial^2\Psi}{\partial\rho^2},\frac{\partial\Psi}{\partial\rho}\wedge\frac{\partial^2\Psi}{\partial\rho^2}\right\rangle\right|^{-3/2}\left|\left\langle\frac{\partial\Psi}{\partial\rho},\frac{\partial\Psi}{\partial\rho}\right\rangle\right|^{-1}\right)d\rho. \quad (19)$$

$\left\langle\frac{\partial^2\Psi}{\partial\rho^2},\frac{\partial\Psi}{\partial\rho}\right\rangle$ vanishes at $t=0$, since $\langle E_1', E_1\rangle = 0$. After complicated computations, with differentiating of (10) at $t=0$ [8],

$$I_2'(0)=\int\limits_0^l\left[\left(\frac{\partial\phi}{\partial t}\right)\Big|_{t=0}-\varepsilon_1\delta c_g(\phi-\phi(l)\right]ds. \quad (20)$$

As function of coordinates along Ψ

$$\phi=\phi\big[u(\rho)+t\delta(\rho)f(\rho),v(\rho)+t\delta(\rho)g(\rho)\big]$$

and

$$\left(\frac{\partial\phi}{\partial t}\right)\Big|_{t=0}=\delta\left[f\left(\frac{\partial\phi}{\partial u}\right)+g\left(\frac{\partial\phi}{\partial v}\right)\right].$$

From (19) for $t=0$ and (20), $J'(0)$ is obtained as following:

$$J'(0)=\int\limits_0^l\delta\Big[(U\Upsilon)''-2\varepsilon_2(c_nU)''+(U\Omega)'+\varepsilon_1U(l)-$$

$$-\zeta\left(f\left(\frac{\partial\phi}{\partial u}\right)+g\left(\frac{\partial\phi}{\partial v}\right)-\varepsilon_1c_g(\phi-\phi(l))\right)\Big]ds-$$

$$-2\varepsilon_2\delta''(l)c_n(l)U(l)+\delta'(l)\Big[(-2\varepsilon_2(c_nU)'(l)+U(l)\Upsilon(l)]+$$

$$+\delta(l)[(2\varepsilon_2(c_nU)''(l)-(U\Upsilon)'(l)+U(l)\Omega(l)\Big], \quad (21)$$

where

$$U=\frac{(c_gc_n)'+\varepsilon_3(c_n\tau_g)-\varepsilon_2(c_g\tau_g)}{\left|\varepsilon_2c_g^2+\varepsilon_3c_n^2\right|^{3/2}},\qquad V=\left|\varepsilon_2c_g^2+\varepsilon_3c_n^2\right|^{-1}((c_gc_n)'+\tau_g(\varepsilon_3c_n-\varepsilon_2c_g)),$$

$$\Omega=(2c_n^3+2(3-4\varepsilon_1)c_nc_g^2+4\varepsilon_1c_g\tau_g^2+4\varepsilon_3c_n'\tau_g-6\varepsilon_3c_n\tau_g^2-6c_g\tau_g'-3(\varepsilon_2c_n\tau_g'+c_g\tau_g^2)V), \quad (22)$$

$$\Upsilon=(2\varepsilon_1\varepsilon_3c_n'-4c_g\tau_g+3\varepsilon_1c_gV).$$

3. Intrinsic equations for generalized elastic line deformed on timelike surface with timelike arc. For E_1 is timelike, E_2 and N are spacelike, respectively,

$$\langle E_1,E_1\rangle=\varepsilon_1=-1,\qquad \langle E_2,E_2\rangle=\varepsilon_2=1\qquad\text{and}\qquad\langle N,N\rangle=\varepsilon_3=1. \quad (23)$$

We consider the case $c_g^2>c_n^2$.

Using (21), (22) and (23) for all choices of the function $\delta(s)$ with arbitrary values of $\delta(l)$, $\delta'(l), \delta''(l)$, and $J'(0) = 0$, the path of timelike arc $\gamma(s)$ must satisfy as following boundary conditions and differential equation

$$c_n(l)U(l) = 0, \tag{24}$$

$$2(c_nU)'(l) = -U(l)\Upsilon(l), \tag{25}$$

$$2(c_nU)''(l) = (U\Upsilon)'(l) - U(l)\Omega(l), \tag{26}$$

$$(U\Omega)' + (U\Upsilon)'' - 2(c_nU)'' - U(l) - \zeta\left(f\left(\frac{\partial\phi}{\partial u}\right) + g\left(\frac{\partial\phi}{\partial v}\right) + c_g\big(\phi - \phi(l)\big)\right) = 0, \tag{27}$$

where

$$U = \frac{(c_gc_n)' + \tau_g(c_n - c_g)}{(c_g^2 + c_n^2)^{3/2}}, \qquad V = (c_g^2 + c_n^2)^{-1}((c_gc_n)' + \tau_g(c_n - c_g)),$$

$$\Omega = (2c_n^3 + 14c_nc_g^2 - 4c_g\tau_g^2 + 4c_n'\tau_g - 6c_n\tau_g^2 - 6c_g\tau_g' - 3(c_n\tau_g' + 3c_g\tau_g^2)V),$$

$$\Upsilon = -(2c_n' + 4c_g\tau_g + 3c_gV).$$

4. Intrinsic equations for generalized elastic line deformed on spacelike surface. If E, E_2 are spacelike and N is timelike,

$$\langle E_1, E_1\rangle = \varepsilon_1 = 1, \qquad \langle E_2, E_2\rangle = \varepsilon_2 = 1 \qquad \text{and} \qquad \langle N, N\rangle = \varepsilon_3 = -1. \tag{28}$$

We consider the case $c_g^2 < c_n^2$. With aid (21), (22) and (28), for all choices of the function $\delta(s)$ with arbitrary values of $\delta(l)$, $\delta'(l)$, $\delta''(l)$, and $J'(0) = 0$, the path of spacelike arc $\gamma(s)$ must satisfy as following conditions and differential equation

$$\begin{gathered} c_n(l)U(l) = 0, \\ -2(c_nU)'(l) + U(l)\Upsilon(l) = 0, \\ 2(c_nU)''(l) - (U\Upsilon)'(l) + U(l)\Omega(l) = 0, \\ -(U\Upsilon)'' - 2(c_nU)'' - (U\Omega)' + U(l) - \zeta\left(f\left(\frac{\partial\phi}{\partial u}\right) + g\left(\frac{\partial\phi}{\partial v}\right) - c_g(\phi - \phi(l)\right) = 0, \end{gathered} \tag{29}$$

where

$$U = \frac{(c_gc_n)' - \tau_g(c_n - c_g)}{(c_n^2 - c_g^2)^{3/2}}, \qquad V = (c_n^2 - c_g^2)^{-1}((c_gc_n)' - \tau_g(c_n - c_g)),$$

$$\Upsilon = (-2c_n' - 4c_g\tau_g + 3c_gV),$$

$$\Omega = (2c_n^3 - 2c_nc_g^2 + 4c_g\tau_g^2 - 4c_n'\tau_g + 6c_n\tau_g^2 - 6c_g\tau_g' - 3c_n(-c_n\tau_g' + c_g\tau_g^2)V).$$

5. Applications.

Theorem 1. *On $S_1^1 \times R$ Lorentz cylinder in $\mathbb{E}_1^3$, there is not generalized elastic line deformed.*

Proof. Parametric equation for r radius Lorentz cylinder

$$x(u,v) = (r\sinh u, r\cosh u, v).$$

Using shape operator, principal curvatures are obtained as following:

$$c_1 = -\frac{1}{r} \qquad \text{and} \qquad c_2 = 0.$$

With aid (3) and (4), respectively,

$$\begin{aligned} c_n &= -\frac{1}{r}\cosh^2\chi, \\ c_g &= \frac{1}{r}\cosh\chi\sinh\chi, \end{aligned} \tag{30}$$

boundary conditions (24) are just satisfied at $\chi = 0$, using (30). Therefore, on $S_1^1 \times R$ Lorentz cylinder, there is not generalized elastic line deformed.

Theorem 2. *There is not generalized elastic line deformed on $H^2(r)$ hyperbolic 2-space in $\mathbb{E}_1^3$.*

Proof. $H^2(r)$ hyperbolic 2-space satisfy

$$-x^2 + y^2 + z^2 = -r^2$$

and matrix for shape operator

$$\begin{bmatrix} -\frac{1}{r} & 0 \\ 0 & -\frac{1}{r} \end{bmatrix},$$

$c_1 = -\frac{1}{r}$ and $c_2 = -\frac{1}{r}$, $c_n = c_1\cos^2\chi + c_2\sin^2\chi = -\frac{1}{r} \neq 0$. From (29), proof is clear.

Theorem 3. *If γ is generalized elastic line deformed for non-null surface which geodesic torsion and normal curvature is zero, γ must satisfy the following differential equation*:

$$f\left(\frac{\partial\phi}{\partial u}\right) + g\left(\frac{\partial\phi}{\partial v}\right) = \varepsilon_1 c_g\big(\phi - \phi(l)\big).$$

Proof. If $c_n = 0$ and $\tau_g = 0$, proof is trivial from (21).

Theorem 4. *Any arc on spacelike plane is generalized elastic line deformed.*

Proof. For spacelike plane, $c_n = 0$ and $\tau_g = 0$. Thus, (29) is satisfied.

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Received 04.12.12,
after revision — 17.12.14