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E. S. Panakhov (Firat Univ., Elazig, Turkey),
M. Sat (Erzincan Univ., Turkey)

INVERSE PROBLEM FOR INTERIOR SPECTRAL DATA OF THE HYDROGEN ATOM EQUATION

ОБЕРНЕНА ЗАДАЧА ДЛЯ ВНУТРІШНІХ СПЕКТРАЛЬНИХ ДАНИХ РІВНЯННЯ АТОМА ВОДНЮ

We consider the inverse problem for second-order differential operators with regular singularity and show that the potential function can be uniquely determined by the set of values of eigenfunctions at some interior point and parts of two spectra.

Розглянуто обернену задачу для диференціальних операторів другого порядку з регулярною сингулярністю та показано, що потенціальна функція однозначно визначається множиною значень власних функцій у деякій внутрішній точці та частинами двох спектрів.

1. Introduction. The inverse Sturm – Liouville problem is primarily a model problem. Typically, in an inverse eigenvalue problem, one measures the frequencies of a vibration system and tries to infer some physical properties of the system. Inverse problems of spectral analysis involve the reconstruction of a linear operator from its spectral characteristics [1, 2]. A problem of this kind was first investigated by Ambarzumyan in 1929 [7]. Later, inverse problems for a regular and singular Sturm – Liouville operator appeared in various versions [3 – 14].

The inverse problem for interior spectral data of the differential operator consists in reconstruction of this operator from the known eigenvalues and some information on eigenfunctions at some internal point. The technique employed is similar to those used in [9]. Similar problems for the Sturm – Liouville operator and Dirac operator were formulated and studied in [10].

The main goal of the present work is to study the inverse problem of reconstructing the singular Sturm – Liouville operator on the basis of spectral data of a kind: one spectrum and some information on eigenfunctions at the internal point.

Consider the following singular Sturm – Liouville operator L satisfying

$$Ly = -y'' + \left[\frac{\ell\,(\ell+1)}{x^2} - \frac{2}{x} + q(x)\right] y = \lambda y, \qquad 0 < x < \pi, \tag{1.1}$$

with boundary conditions,

$$y(0) = 0, \tag{1.2}$$

$$y'(\pi,\lambda) + Hy(\pi,\lambda) = 0, \tag{1.3}$$

where $q(x)$ is assumed to be real valued and square integrable, λ spectral parameter, $\ell \in \mathbb{N}_0$, and H finite real number. The operator L is self adjoint on the $L_2\,(0,\pi)$ and (1.2), (1.3) boundary conditions has a discrete spectrum $\{\lambda_n\}$.

Let us introduce the second singular Sturm – Liouville operator $\widetilde{L}$ satisfying

$$\widetilde{L}y = -y'' + \left[\frac{\ell\,(\ell+1)}{x^2} - \frac{2}{x} + \widetilde{q}(x)\right] y = \lambda y, \qquad 0 < x < \pi, \tag{1.4}$$

subject to the same boundary conditions (1.2), (1.3), where $\widetilde{q}(x)$ is assumed to be real valued and square integrable. The operator $\widetilde{L}$ is self adjoint on the $L_2(0,\pi)$ and (1.2), (1.3) boundary conditions has a discrete spectrum $\{\widetilde{\lambda}_n\}$.

2. Main results. Before giving main results of this article, we mention some known results. We will consider the equation

$$R'' + \frac{2}{x}R' - \frac{\ell(\ell+1)}{x^2}R + \left(E + \frac{2}{x}\right)R = 0, \qquad 0 < x < \infty. \tag{2.1}$$

In quantum mechanics, the study of the energy levels of the hydrogen atom leads to this equation [16]. The substitution $R = \dfrac{y}{x}$ reduces this equation to the form

$$y'' + \left[E + \frac{2}{x} - \frac{\ell(\ell+1)}{x^2}\right]y = 0. \tag{2.2}$$

As known [17 – 19] the solution of (2.2) is bounded at zero, one has the following asymptotic formula for $\lambda \to \infty$

$$y(x) = \frac{e^{\pi/2\sqrt{\lambda}}}{\left|\Gamma\left(\ell + 1 + \dfrac{i}{\sqrt{\lambda}}\right)\right|}\frac{1}{\sqrt{\lambda}}\cos\left[\sqrt{\lambda}x + \frac{1}{\sqrt{\lambda}}\ln\sqrt{\lambda}x - (\ell+1)\frac{\pi}{2} + \alpha\right] + o(1),$$

where

$$\alpha = \arg\Gamma\left(\ell + 1 + \frac{i}{\sqrt{\lambda}}\right).$$

Eigenvalues of the problem (1.1) – (1.3) are the roots of the (1.3). These spectral characteristics and eigenfunctions satisfy the following asymptotic expression, respectively [18]:

$$\rho_n = \sqrt{\lambda_n} = n + \frac{\ell}{2} + O\left(\frac{\ln n}{n}\right), \tag{2.3}$$

$$\varphi(x,\lambda_n) = \cos\left[\left(n + \frac{\ell}{2}\right)x - \frac{\ell\pi}{2}\right] + O\left(\frac{\ln n}{n}\right), \tag{2.4}$$

$$\varphi'(x,\lambda_n) = -\left(n + \frac{\ell}{2}\right)\sin\left[\left(n + \frac{\ell}{2}\right)x - \frac{\ell\pi}{2}\right] + O\left(\frac{\ln n}{n}\right). \tag{2.5}$$

Next, we present the main results in this article. When $b = \dfrac{\pi}{2}$, we get the following uniqueness theorem.

Theorem 2.1. *If for every $n \in \mathbb{N}$ we have*

$$\lambda_n = \widetilde{\lambda}_n, \qquad \frac{y_n'\left(\frac{\pi}{2}\right)}{y_n\left(\frac{\pi}{2}\right)} = \frac{\widetilde{y}_n'\left(\frac{\pi}{2}\right)}{\widetilde{y}_n\left(\frac{\pi}{2}\right)}, \tag{2.6}$$

then

$$q(x) = \widetilde{q}(x) \qquad \textit{a.e. on the interval} \quad (0,\pi).$$

In the case $b \neq \frac{\pi}{2}$, the uniqueness of $q(x)$ can be proved if we require the knowledge of a part of the second spectrum.

Let $m(n)$ be a sequence of natural numbers with a property

$$m(n) = \frac{n}{\sigma}(1 + \varepsilon_n), \qquad 0 < \sigma \leq 1, \quad \varepsilon_n \to 0. \tag{2.7}$$

Lemma 2.1. *Let $m(n)$ be a sequence of natural numbers satisfying* (2.7) *and $b \in \left(0, \frac{\pi}{2}\right)$ are so chosen that $\sigma > \frac{2b}{\pi}$. If for any $n \in \mathbb{N}$*

$$\lambda_{m(n)} = \widetilde{\lambda}_{m(n)}, \qquad \frac{y'_{m(n)}(b)}{y_{m(n)}(b)} = \frac{\widetilde{y}'_{m(n)}(b)}{\widetilde{y}_{m(n)}(b)}, \tag{2.8}$$

then

$$q(x) = \widetilde{q}(x) \qquad \textit{a.e. on} \quad (0, b].$$

Let $l(n)$ and $r(n)$ be a sequence of natural numbers such that

$$l(n) = \frac{n}{\sigma_1}(1 + \varepsilon_{1,n}), \qquad 0 < \sigma_1 \leq 1, \quad \varepsilon_{1,n} \to 0, \tag{2.9}$$

$$r(n) = \frac{n}{\sigma_2}(1 + \varepsilon_{2,n}), \qquad 0 < \sigma_2 \leq 1, \quad \varepsilon_{2,n} \to 0, \tag{2.10}$$

and let μ_n be the eigenvalues of the problem (1.1), (1.2) and (2.11) and $\widetilde{\mu}_n$ be the eigenvalues of the problem (1.4), (1.2) and (2.11)

$$y'(\pi, \lambda) + H_1 y(\pi, \lambda) = 0, \qquad H \neq H_1. \tag{2.11}$$

Using Mochizuki and Trooshin's method from Lemma 2.1 and Theorem 2.1, we will prove that the following theorem holds.

Theorem 2.2. *Let $l(n)$ and $r(n)$ be a sequence of natural numbers satisfying* (2.9) *and* (2.10), *and $\frac{\pi}{2} < b < \pi$ are so chosen that $\sigma_1 > \frac{2b}{\pi} - 1$, $\sigma_2 > 2 - \frac{2b}{\pi}$. If for any $n \in \mathbb{N}$ we have*

$$\lambda_n = \widetilde{\lambda}_n, \qquad \mu_{l(n)} = \widetilde{\mu}_{l(n)} \qquad \textit{and} \qquad \frac{y'_{r(n)}(b)}{y_{r(n)}(b)} = \frac{\widetilde{y}'_{r(n)}(b)}{\widetilde{y}_{r(n)}(b)}, \tag{2.12}$$

then

$$q(x) = \widetilde{q}(x) \qquad \textit{a.e. on} \quad (0, \pi).$$

3. Proof of the main results. In this section we present the proofs of main results in this paper.

***Proof of Theorem* 2.1.** Before proving the Theorem 2.1, we will mention some results, which will be needed later. We get the initial value problems

$$-y'' + \left[\frac{\ell\,(\ell + 1)}{x^2} - \frac{2}{x} + q(x)\right] y = \lambda y, \qquad 0 < x < \pi, \tag{3.1}$$

$$y(0) = 0, \tag{3.2}$$

and

$$-\widetilde{y}'' + \left[\frac{\ell(\ell+1)}{x^2} - \frac{2}{x} + \widetilde{q}(x)\right]\widetilde{y} = \lambda\widetilde{y}, \qquad 0 < x < \pi, \tag{3.3}$$

$$\widetilde{y}(0) = 0. \tag{3.4}$$

It can be shown [19] that there exists a kernel $K(x,t)$ $\left(\widetilde{K}(x,t)\right)$ continuous on $(0,\pi)\times(0,\pi)$ such that by using the transformation operator every solution of equations (3.1), (3.2) and (3.3), (3.4) can be expressed in the form

$$y(x,\lambda) = \cos\left[\left(n+\frac{\ell}{2}\right)x - \frac{\ell\pi}{2}\right] + \int\limits_0^x K(x,t)\cos\left[\left(n+\frac{\ell}{2}\right)t - \frac{\ell\pi}{2}\right]dt, \tag{3.5}$$

$$\widetilde{y}(x,\lambda) = \cos\left[\left(n+\frac{\ell}{2}\right)x - \frac{\ell\pi}{2}\right] + \int\limits_0^x \widetilde{K}(x,t)\cos\left[\left(n+\frac{\ell}{2}\right)t - \frac{\ell\pi}{2}\right]dt, \tag{3.6}$$

respectively, where the kernel $K(x,t)$ $\left(\widetilde{K}(x,t)\right)$ is the solution of the problem

$$\frac{\partial^2 K(x,t)}{\partial x^2} - \left(\frac{2}{x} - \frac{\ell(\ell+1)}{x^2} + \widetilde{q}(x)\right)K(x,t) = \frac{\partial^2 K(x,t)}{\partial t^2} - \left(\frac{2}{t} - \frac{\ell(\ell+1)}{t^2} + q(t)\right)K(x,t)$$

subject to the boundary conditions

$$K(x,x) = \frac{1}{2}\int\limits_0^x \left[\widetilde{q}(t) - q(t)\right],$$

$$K(x,0) = 0.$$

After the transformations

$$z = \frac{1}{4}(x+t)^2, \qquad w = \frac{1}{4}(x-t)^2, \qquad K(x,t) = (z-w)^{-v+(1/2)}\,u(z,w),$$

we obtain the following problem $\left(\alpha = -v + \dfrac{1}{2}\right)$:

$$\frac{\partial^2 u}{\partial z\partial w} - \frac{\alpha}{z-w}\frac{\partial u}{\partial z} + \frac{\alpha}{z-w}\frac{\partial u}{\partial w} = \frac{(\widetilde{q}-q)\,u}{4\sqrt{zw}} - \frac{u}{\sqrt{z}\,(z-w)},$$

$$u(z, z-\delta) = 0,$$

$$\frac{\partial u}{\partial z} + \frac{\alpha}{z}u = \frac{1}{4}\left[\widetilde{q}(\sqrt{z}\,) - q(\sqrt{z}\,)\right]z^{v-1},$$

for a constant δ. This problem can be solved by using the Riemann method [20, 21].

Multiplying (3.1) by $\widetilde{y}(x,\lambda)$ and (3.3) by $y(x,\lambda)$, subtracting and integrating from 0 to $\dfrac{\pi}{2}$, we obtain

$$\int_0^{\pi/2} (q(x) - \widetilde{q}(x))\, y(x,\lambda)\, \widetilde{y}(x,\lambda) dx = \left[\widetilde{y}(x,\lambda) y'(x,\lambda) - y(x,\lambda)\widetilde{y}'(x,\lambda)\right] \Big|_0^{\pi/2}. \tag{3.7}$$

The functions $y(x,\lambda)$ and $\widetilde{y}(x,\lambda)$ satisfy the same initial conditions (3.2) and (3.4), i.e.,

$$\widetilde{y}(0,\lambda) y'(0,\lambda) - y(0,\lambda)\widetilde{y}'(0,\lambda) = 0.$$

Let

$$Q(x) = q(x) - \widetilde{q}(x), \tag{3.8}$$

$$H(\lambda) = \int_0^{\pi/2} Q(x)\, y(x,\lambda)\, \widetilde{y}(x,\lambda) dx. \tag{3.9}$$

If the properties of $y(x,\lambda)$ and $\widetilde{y}(x,\lambda)$ are considered, the function $H(\lambda)$ is an entire function.

Therefore the condition of the Theorem 2.1 imply

$$\widetilde{y}\left(\frac{\pi}{2},\lambda_n\right) y'\left(\frac{\pi}{2},\lambda_n\right) - y\left(\frac{\pi}{2},\lambda_n\right) \widetilde{y}'\left(\frac{\pi}{2},\lambda_n\right) = 0,$$

and hence

$$H(\lambda_n) = 0, \qquad n \in \mathbb{N}.$$

In addition, using (3.5) and (3.9) for $0 < x \leq \pi$,

$$\left|H(\lambda)\right| \leq \frac{M}{\lambda}, \tag{3.10}$$

where M is constant.

Introduce the function

$$\omega(\lambda) = y'(\pi,\lambda) + Hy(\pi,\lambda). \tag{3.11}$$

By using the asymptotic forms of φ and φ', we obtain

$$\omega(\lambda) = -\left(n + \frac{\ell}{2}\right) \sin\left[\left(n + \frac{\ell}{2}\right) - \frac{\ell\pi}{2}\right] + O\left(\frac{\ln n}{n}\right). \tag{3.12}$$

The zeros of $\omega(\lambda)$ are the eigenvalues of L and hence it has only simple zeros λ_n because of the seperated boundary conditions. It is an entire function of order $\frac{1}{2}$ of λ. From this and from the asymptotics for $\omega(\lambda)$ and $H(\lambda)$, it follows that the function

$$\Psi(\lambda) = \frac{H(\lambda)}{\omega(\lambda)} \tag{3.13}$$

is an entire function. Asymptotic form of $\omega(\lambda)$ and with equation (3.13), we get

$$\left|\Psi(\lambda)\right| = O\left(\frac{\pi}{\sqrt{\lambda}}\right).$$

So, for all λ, from the Liouville theorem,

$$\Psi(\lambda) = 0,$$

or

$$H(\lambda) = 0.$$

It was proved in [19] that there exists absolutely continuous function $\widetilde{\widetilde{K}}(x,\tau)$ such that, we have

$$y(x,\lambda)\,\widetilde{y}(x,\lambda) =$$

$$= \frac{1}{2}\left\{1 + \cos 2\left[\left(n + \frac{\ell}{2}\right)x - \frac{\ell\pi}{2}\right] + \int_0^x \widetilde{\widetilde{K}}(x,\tau)\cos 2\left[\left(n + \frac{\ell}{2}\right)\tau - \frac{\ell\pi}{2}\right]d\tau\right\}, \qquad (3.14)$$

where

$$\widetilde{\widetilde{K}}(x,t) = 2\left[K(x, x - 2\tau) + \widetilde{K}(x, x - 2\tau)\right] +$$

$$+2\left[\int_{-x+2\tau}^{x} K(x,s)\,\widetilde{K}(x, s - 2\tau)\,ds + \int_{-x}^{x-2\tau} K(x,s)\,\widetilde{K}(x, s + 2\tau)\,ds\right].$$

We are now going to show that $Q(x) = 0$ a.e. on $\left(0, \frac{\pi}{2}\right]$. From (3.9), (3.14) we obtain

$$\frac{1}{2}\int_0^{\pi/2} Q(x)\left\{1 + \cos 2\left[\left(n + \frac{\ell}{2}\right)x - \frac{\ell\pi}{2}\right] + \int_0^x \widetilde{\widetilde{K}}(x,\tau)\cos 2\left[\left(n + \frac{\ell}{2}\right)\tau - \frac{\ell\pi}{2}\right]d\tau\right\}dx = 0.$$

This can be written as

$$\int_0^{\pi/2} Q(x)dx + \int_0^{\pi/2} \cos 2\left[\left(n + \frac{\ell}{2}\right)\tau - \frac{\ell\pi}{2}\right]\left[Q(\tau) + \int_\tau^{\pi/2} Q(x)\widetilde{\widetilde{K}}(x,\tau)\,dx\right]d\tau = 0.$$

Let $\lambda \to \infty$ along the real axis, by the Riemann – Lebesgue lemma, we should have

$$\int_0^{\pi/2} Q(x)dx = 0, \qquad (3.15)$$

and

$$\int_0^{\pi/2} \cos 2\left[\left(n + \frac{\ell}{2}\right)\tau - \frac{\ell\pi}{2}\right]\left[Q(\tau) + \int_\tau^{\pi/2} Q(x)\widetilde{\widetilde{K}}(x,\tau)\,dx\right]d\tau = 0. \qquad (3.16)$$

Thus from the completeness of the functions $\cos$, it follows that

$$Q(\tau)+\int\limits_{\tau}^{\pi/2} Q(x)\widetilde{\widetilde{K}}(x,\tau)\,dx=0, \qquad 0<x<\frac{\pi}{2}. \tag{3.17}$$

But this equation is a homogeneous Volterra integral equation and has only the zero solution. Thus we have obtained

$$Q(x)=q(x)-\widetilde{q}(x)=0,$$

or

$$\widetilde{q}(x)=q(x),$$

almost everywhere on $\left(0,\dfrac{\pi}{2}\right]$.

To prove that $q(x)=0$ on $\left[\dfrac{\pi}{2},\pi\right)$ almost everywhere, we should repeat the above arguments for the supplementary problem

$$Ly=-y''+\left[\frac{\ell(\ell+1)}{(\pi-x)x^2}-\frac{2}{\pi-x}+q(\pi-x)\right]y, \quad 0<x<\pi,$$

subject to the boundary conditions

$$y(\pi)=0,$$

$$y'(0,\lambda)+Hy(0,\lambda)=0,$$

where

$$q_0(x)=\frac{\ell(\ell+1)}{x^2}-\frac{2}{x}+q_1(x).$$

Consequently

$$q(x)=\widetilde{q}(x) \qquad \text{a.e. on the interval} \quad (0,\pi).$$

Therefore, Theorem 2.1 is proved.

Next, we show that Lemma 2.1 holds.

***Proof of Lemma* 2.1.** As in the proof of Theorem 2.1 we can show that

$$G(\rho)=\int\limits_0^b Q(x)y(x,\lambda)\,\widetilde{y}(x,\lambda)dx=\left[\widetilde{y}(x,\lambda)y'(x,\lambda)-y(x,\lambda)\widetilde{y}'(x,\lambda)\right]\Big|_{x=b}, \tag{3.18}$$

where $\rho=\sqrt{\lambda}=re^{i\theta}$ and $Q(x)=q(x)-\widetilde{q}(x)$. From the assumption

$$\frac{y'_{m(n)}(b)}{y_{m(n)}(b)}=\frac{\widetilde{y}'_{m(n)}(b)}{\widetilde{y}_{m(n)}(b)},$$

together with the initial condition at 0 it follows that,

$$G(\rho_{m(n)})=0, \qquad n\in\mathbb{N}.$$

Next, we will show that $G(\rho) = 0$ on the whole ρ plane. The asymptotics (2.4), (2.5) imply that the entire function $G(\rho)$ is a function of exponential type $\le 2b$.

Define the indicator of function $G(\rho)$ by;

$$h(\theta) = \lim_{r\to\infty} \sup \frac{\ln\left|G(re^{i\theta})\right|}{r}. \tag{3.19}$$

Since $\left|\mathrm{Im}\sqrt{\lambda}\,\right| = r\left|\sin\theta\right|$, $\theta = \arg\sqrt{\lambda}$ from (2.4) and (2.5) it follows that

$$h(\theta) \le 2b\left|\sin\theta\right|. \tag{3.20}$$

Let us denote by $n(r)$ the number of zeros of $G(\rho)$ in the disk $\{|\rho| \le r\}$. According to the [15] set of zeros of every entire function of the exponential type, not identically zero, satisfies the inequality

$$\lim_{r\to\infty} \inf \frac{n(r)}{r} \le \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta, \tag{3.21}$$

where $n(r)$ is the number of zeros of $G(\rho)$ in the disk $|\rho| \le r$. By (3.20),

$$\frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta \le \frac{b}{\pi}\int_0^{2\pi} \left|\sin\theta\right| d\theta = \frac{4b}{\pi}.$$

From the assumption and the known asymptotic expression (2.3) of the eigenvalues $\sqrt{\lambda_n}$ we obtain

$$n(r) \ge 2 \sum_{\frac{n}{\sigma}\left[1+O\left(\frac{\ln n}{n}\right)\right]<r} 1 = 2\sigma r\bigl(1 + o\,(1)\bigr), \qquad r \to \infty.$$

For the case $\sigma > \dfrac{2b}{\pi}$,

$$\lim_{r\to\infty} \frac{n(r)}{r} \ge 2\sigma > \frac{4b}{\pi} = 2b\int_0^{2\pi} \left|\sin\theta\right| d\theta \ge \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta. \tag{3.22}$$

The inequalities (3.21) and (3.22) imply that $G(\rho) = 0$ on the whole ρ plane.

Similar to the proof of the Theorem 2.1, we have

$$q(x) = \widetilde{q}(x) \qquad \text{a.e. on the interval} \quad (0, b].$$

Lemma 2.1 is proved.

Now we prove that Theorem 2.2 is valid.

***Proof of Theorem* 2.2.** From

$$\lambda_{r(n)} = \widetilde{\lambda}_{r(n)}, \qquad \frac{y'_{r(n)}(b)}{y_{r(n)}(b)} = \frac{\widetilde{y}'_{r(n)}(b)}{\widetilde{y}_{r(n)}(b)},$$

where $r(n)$ satisfies (2.10) and $\sigma_2 > 2 - \dfrac{2b}{\pi}$ according to Lemma 2.1, we get

$$q(x) = \widetilde{q}(x) \qquad \text{a.e. on} \quad [b, \pi). \tag{3.23}$$

Thus, it needs to be proved that $q(x) = \widetilde{q}(x)$ a.e. on $(0, b]$. The eigenfunctions $y_n(x, \lambda_n)$ and $\widetilde{y}_n(x, \lambda_n)$ satisfy the same boundary condition at π. It means that

$$y_n(x, \lambda_n) = \xi_n \widetilde{y}_n(x, \lambda_n) \tag{3.24}$$

on $[b, \pi)$ for any $n \in \mathbb{N}$ where ξ_n are constants.

From (3.18) and (3.24) we obtain that,

$$G(\rho) = 0, \qquad \text{for} \quad \rho^2 = \lambda_n, \qquad n \in \mathbb{N},$$

and

$$G(\rho) = 0, \qquad \text{for} \quad \rho^2 = \mu_{l(n)}, \qquad n \in \mathbb{N}.$$

W are going to show that inequality (3.21) fails and consequently, the entire function of exponential type $G(\rho)$ vanishes on the whole ρ-plane. Let $\rho_n = \sqrt{\lambda_n}$, $s_n = \sqrt{\mu_n}$. The ρ_n and s_n have the same asymptotics (2.3). Counting the number of ρ_n and s_n located inside the disc of radius r, we have

$$1 + 2r\left[1 + O\left(\frac{\ln n}{n}\right)\right]$$

of ρ_n's and

$$1 + 2r\sigma_1\left[1 + O\left(\frac{\ln n}{n}\right)\right]$$

of s_n's.

This means that

$$n(r) = 2 + 2\left[r(\sigma_1 + 1) + O\left(\frac{\ln n}{n}\right)\right]$$

and

$$\lim_{r \to \infty} \frac{n(r)}{r} = 2(\sigma_1 + 1).$$

Repeating the last part of the proof of Lemma 2.1, and considering the condition $\sigma_1 > \dfrac{2b}{\pi} - 1$, we can show that $G(\rho) = 0$ identically on the whole ρ-plane which implies that

$$q(x) = \widetilde{q}(x) \qquad \text{a.e. on} \quad (0, b]$$

and consequently

$$q(x) = \widetilde{q}(x) \qquad \text{a.e. on} \quad (0, \pi).$$

Theorem 2.2 is proved.

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