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ASYMPTOTIC BEHAVIOR OF HIGHER-ORDER NEUTRAL DIFFERENCE EQUATIONS WITH GENERAL ARGUMENTS

АСИМПТОТИЧНА ПОВЕДІНКА НЕЙТРАЛЬНИХ РІЗНИЦЕВИХ РІВНЯНЬ ВИЩОГО ПОРЯДКУ ІЗ ЗАГАЛЬНИМИ АРГУМЕНТАМИ

We study the asymptotic behavior of solutions of the higher-order neutral difference equation

$$\Delta^m\left[x(n)+cx(\tau(n))\right]+p(n)x(\sigma(n))=0,\quad \mathbb{N}\ni m\geq 2,\quad n\geq 0,$$

where $\tau(n)$ is a general retarded argument, $\sigma(n)$ is a general deviated argument, $c\in\mathbb{R}$, $(p(n))_{n\geq 0}$ is a sequence of real numbers, Δ denotes the forward difference operator $\Delta x(n)=x(n+1)-x(n)$, and Δ^j denotes the jth forward difference operator $\Delta^j\big(x(n)\big)=\Delta\left(\Delta^{j-1}\big(x(n)\big)\right)$ for $j=2,3,\ldots,m$. Examples illustrating the results are also given.

Вивчається асимптотична поведінка розв'язків нейтрального різницевого рівняння вищого порядку

$$\Delta^m\left[x(n)+cx(\tau(n))\right]+p(n)x(\sigma(n))=0,\quad \mathbb{N}\ni m\geq 2,\quad n\geq 0,$$

де $\tau(n)$— загальний аргумент із запізненням, $\sigma(n)$ — загальний аргумент із відхиленням, $c\in\mathbb{R}$, $(p(n))_{n\geq 0}$ — послідовність дійсних чисел, Δ — оператор правої різниці, $\Delta x(n)=x(n+1)-x(n)$, та Δ^j — j-й оператор правої різниці, $\Delta^j\big(x(n)\big)=\Delta\left(\Delta^{j-1}\big(x(n)\big)\right)$ при $j=2,3,\ldots,m$. Наведено також приклади, що ілюструють отримані результати.

1. Introduction. Consider the m^{th}-order neutral difference equation of the form

$$\Delta^m\left[x(n)+cx(\tau(n))\right]+p(n)x(\sigma(n))=0,\qquad \mathbb{N}\ni m\geq 2,\quad n\geq 0, \tag{E}$$

where $(p(n))_{n\geq 0}$ is a sequence of real numbers, $c\in\mathbb{R}$, $(\tau(n))_{n\geq 0}$ is an increasing sequence of integers which satisfies

$$\tau(n)\leq n-1\quad \forall n\geq 0\qquad\text{and}\qquad \lim_{n\to\infty}\tau(n)=+\infty, \tag{1.1}$$

$(\sigma(n))_{n\geq 0}$ is an increasing sequence of integers such that

$$\sigma(n)\leq n-1\quad \forall n\geq 0\qquad\text{and}\qquad \lim_{n\to\infty}\sigma(n)=+\infty, \tag{1.2a}$$

or

$$\sigma(n)\geq n+1\quad \forall n\geq 0, \tag{1.2b}$$

Δ denotes the forward difference operator $\Delta x(n)=x(n+1)-x(n)$, and Δ^j denotes the jth forward difference operator $\Delta^j\big(x(n)\big)=\Delta\left(\Delta^{j-1}\big(x(n)\big)\right)$ for $j=2,3,\ldots,m$.

Define

$$k=-\min_{n\geq 0}\{\tau(n),\sigma(n)\}\qquad\text{if}\quad \sigma(n)\quad\text{is a retarded argument.}$$

(Clearly, k is a positive integer.)

By a *solution* of (E), we mean a sequence of real numbers $(x(n))_{n\geq -k}$ which satisfies (E) for all $n \geq 0$. It is clear that, for each choice of real numbers $c_{-k}, c_{-k+1}, \ldots, c_{-1}, c_0$, there exists a unique solution $(x(n))_{n\geq -k}$ of (E) which satisfies the initial conditions $x(-k) = c_{-k}$, $x(-k+1) = c_{-k+1}, \ldots, x(-1) = c_{-1}$, $x(0) = c_0$.

If $\sigma(n)$ is an advanced argument, then:

By a *solution* of (E), we mean a sequence of real numbers $(x(n))_{n\geq 0}$ which satisfies (E) for all $n \geq 0$.

A solution $(x(n))_{n\geq -k}$ (or $(x(n))_{n\geq 0}$) of (E) is called *oscillatory,* if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory*.

In the last few decades, the asymptotic and oscillatory behavior of neutral difference equations has been extensively studied. See, for example, [2–8, 10, 11, 13–24] and the references cited therein. Most of these papers concern the special case where the delay $(n - \tau(n))_{n\geq 0}$ is constant, while a small number of these papers are dealing with the general case of Eq. (E), in which the delay $(n - \tau(n))_{n\geq 0}$ is variable. For the general theory of difference equations the reader is referred to the monographs [1, 9, 12].

The objective in this paper is to study the asymptotic behavior of the solutions of Eq. (E). We proceed to study the asymptotic behavior of the solutions of Eq. (E) by considering various cases on the sign of the coefficients $p(n)$. We examine two cases, according to whether the coefficients $p(n)$ are all non-negative (Case 1) or are all non-positive (Case 2). Examples illustrating the results are also given.

2. Some preliminaries. Throughout this paper, we are going to use the following notation:

$$\tau \circ \tau = \tau^2 \qquad \tau \circ \tau \circ \tau = \tau^3, \quad \text{and so on.} \tag{2.1}$$

Let the domain of τ be the set $D(\tau) = \mathbb{N}_{n_*} = \{n_*, n_* + 1, n_* + 2, \ldots\}$, where n_* is the smallest natural number that τ is defined with. Then for every $n > n_*$ there exists a natural number $m(n)$ such that

$$x(\tau^{m(n)}(n)) = x(\tau(n_*)) \qquad \text{and} \qquad \lim_{n\to\infty} m(n) = +\infty, \tag{2.2}$$

since $(m(n))$ is increasing and unbounded function of n. Clearly, $n_* = \tau^{m(n)-1}(n)$.

The following lemmas provide us with some useful tools, for establishing the main results:

Lemma 2.1. *Assume that* $\big(z(n)\big)$ *is a sequence of real numbers and* $m \in \mathbb{N}$. *Then the following statements hold*:

(i) *If*

$$\lim_{n\to\infty} \Delta^m z(n) > 0, \tag{2.3}$$

then $\big(z(n)\big)$ *tends to* $+\infty$.

(ii) *If*

$$\lim_{n\to\infty} \Delta^m z(n) < 0, \tag{2.4}$$

then $\big(z(n)\big)$ *tends to* $-\infty$.

(iii) *If*

$$\lim_{n\to\infty} \Delta^m z(n) = 0 \qquad \text{and} \qquad \Delta^{m+1} z(n) \geq 0 \quad \forall n \tag{2.5a}$$

or

$$\lim_{n\to\infty} \Delta^m z(n) = 0 \qquad \text{and} \qquad \Delta^{m+1} z(n) \leq 0 \quad \forall n, \tag{2.5b}$$

then the sequence $\big(z(n)\big)$ *is monotone and therefore its limit exists.*

Proof. (i) Suppose that $m = 1$. Then, if $\lim_{n\to\infty} \Delta z(n) > 0$, clearly $\big(z(n)\big)$ is eventually strictly increasing, and therefore $\lim_{n\to\infty} z(n) = +\infty$.

Assume that $m > 1$ and (2.3) holds. Then $\big(\Delta^{m-1} z(n)\big)$ is eventually strictly increasing. By previous case ($m = 1$), we have $\lim_{n\to\infty} \Delta^{m-1} z(n) = +\infty$.

Repeating this argument $m - 1$ times, we obtain $\lim_{n\to\infty} z(n) = +\infty$. The proof of part (i) of the lemma is complete.

(ii) An obvious consequence of part (i) by taking $-z(n)$ instead of $z(n)$.

(iii) Suppose that $m = 1$ and (2.5a) holds. Then $(\Delta z(n))$ is nondecreasing, and consequently $\Delta z(n) \leq 0$, since $\lim_{n\to\infty} \Delta z(n) = 0$. Therefore $\big(z(n)\big)$ is nonincreasing, which guarantees that its limit exists.

Assume that $m > 1$ and (2.5a) holds. Then $(\Delta^m z(n))$ is nondecreasing, and consequently $\Delta^m z(n) \leq 0$, since $\lim_{n\to\infty} \Delta^m z(n) = 0$. Therefore $\big(\Delta^{m-1} z(n)\big)$ is nonincreasing, which guarantees that its limit exists.

If $\lim_{n\to\infty} \Delta^{m-1} z(n) \neq 0$, by parts (i) and (ii) we have $\lim_{n\to\infty} z(n) = \pm\infty$.

If $\lim_{n\to\infty} \Delta^{m-1} z(n) = 0$, then $\Delta^{m-1} z(n) \geq 0$ since $\big(\Delta^{m-1} z(n)\big)$ is nonincreasing. Therefore $\big(\Delta^{m-2} z(n)\big)$ is nondecreasing, which guarantees that its limit exists.

Applying this procedure $m - 2$ times, we conclude that $\lim_{n\to\infty} z(n) = \pm\infty$ or $\big(z(n)\big)$ is monotone and therefore its limit exists.

In the case where (2.5b) holds, the proof is similar. The proof of part (iii) of the lemma is complete.

Lemma 2.1 is proved.

Lemma 2.2. *Assume that* $\big(z(n)\big)$ *is a sequence of real numbers and* $\mathbb{N} \ni m \geq 2$. *Then the following statements hold*:

(i) *If* m *is even and* $\Delta^m z(n) \leq 0$, *then* $\big(z(n)\big)$ *tends to* $\pm\infty$ *or it is nondecreasing.*

(ii) *If* m *is even and* $\Delta^m z(n) \geq 0$, *then* $\big(z(n)\big)$ *tends to* $\pm\infty$ *or it is nonincreasing.*

(iii) *If* m *is odd and* $\Delta^m z(n) \leq 0$, *then* $\big(z(n)\big)$ *tends to* $\pm\infty$ *or it is nonincreasing.*

(iv) *If* m *is odd and* $\Delta^m z(n) \geq 0$, *then* $\big(z(n)\big)$ *tends to* $\pm\infty$ *or it is nondecreasing.*

Proof. (i) First we will study the case where $m = 2$.

Assume that $\Delta^2 z(n) \leq 0$. Then $(\Delta z(n))$ is nonincreasing, and consequently $\lim_{n\to\infty} \Delta z(n) = -\infty$ or $\lim_{n\to\infty} \Delta z(n) = A \in \mathbb{R}$.

If $\lim_{n\to\infty} \Delta z(n) = -\infty$, then $(z(n))$ is eventually nonincreasing. By part (ii) of Lemma 2.1, we have $\lim_{n\to\infty} z(n) = -\infty$.

If $\lim_{n\to\infty} \Delta z(n) = A < 0$, then $\big(z(n)\big)$ is eventually nonincreasing. By part (ii) of Lemma 2.1, we have $\lim_{n\to\infty} z(n) = -\infty$.

If $\lim_{n\to\infty} \Delta z(n) = 0$, then $\Delta z(n) \geq 0$ since $(\Delta z(n))$ is nonincreasing. Therefore $(z(n))$ is nondecreasing.

If $\lim_{n\to\infty} \Delta z(n) = A > 0$, then $\big(z(n)\big)$ is eventually increasing. By part (i) of Lemma 2.1, we have $\lim_{n\to\infty} z(n) = +\infty$.

Now we will study the case where m is even and $m > 2$.

By parts (i) and (ii) of Lemma 2.1 we have that, if $\lim_{n\to\infty} \Delta^{m-1} z(n) \neq 0$, then $\lim_{n\to\infty} z(n) = = \pm\infty$.

Suppose that $\lim_{n\to\infty} \Delta^{m-1} z(n) = 0$. Since $\Delta^m z(n) \leq 0$, then $\Delta^{m-1} z(n) \geq 0$. This guarantees that $\big(\Delta^{m-2} z(n)\big)$ is nondecreasing.

If $\lim_{n\to\infty} \Delta^{m-2} z(n) \neq 0$, then in view of parts (i) and (ii) of Lemma 2.1, we have $\lim_{n\to\infty} z(n) = = \pm\infty$.

If $\lim_{n\to\infty} \Delta^{m-2} z(n) = 0$, then $\Delta^{m-2} z(n) \leq 0$ since $\big(\Delta^{m-2} z(n)\big)$ is nondecreasing.

Applying this procedure $\dfrac{m-2}{2}$-times, we conclude that $\lim_{n\to\infty} z(n) = \pm\infty$ or $\Delta^2 z(n) \leq 0$. This means that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nondecreasing. The proof of part (i) of the lemma is complete.

(ii) An obvious consequence of part (i) by taking $-z(n)$ instead of $z(n)$.

(iii) First we will study the case where $m = 3$.

Assume that $\Delta^3 z(n) \leq 0$. Then $\big(\Delta^2 z(n)\big)$ is nonincreasing, and consequently $\lim_{n\to\infty} \Delta^2 z(n) = = -\infty$ or $\lim_{n\to\infty} \Delta^2 z(n) = A \in \mathbb{R}$.

If $\lim_{n\to\infty} \Delta^2 z(n) = -\infty$, then by part (ii) of Lemma 2.1 we have $\lim_{n\to\infty} z(n) = -\infty$.

If $\lim_{n\to\infty} \Delta^2 z(n) = A < 0$, then by part (ii) of Lemma 2.1 we have $\lim_{n\to\infty} z(n) = -\infty$.

If $\lim_{n\to\infty} \Delta^2 z(n) = 0$, then $\Delta^2 z(n) \geq 0$ since $\big(\Delta^2 z(n)\big)$ is nonincreasing. By part (ii), we conclude that $\big(z(n)\big)$ tends to $+\infty$ or it is nonincreasing.

If $\lim_{n\to\infty} \Delta^2 z(n) = A > 0$, then by part (i) of Lemma 2.1 we have $\lim_{n\to\infty} z(n) = +\infty$.

Now we will study the case where m is odd and $m > 3$.

By parts (i) and (ii) of Lemma 2.1 we have that, if $\lim_{n\to\infty} \Delta^{m-1} z(n) \neq 0$, then $\lim_{n\to\infty} z(n) = = \pm\infty$.

Suppose that $\lim_{n\to\infty} \Delta^{m-1} z(n) = 0$. Since $\Delta^m z(n) \leq 0$, then $\Delta^{m-1} z(n) \geq 0$. This guarantees that $\big(\Delta^{m-2} z(n)\big)$ is nondecreasing.

If $\lim_{n\to\infty} \Delta^{m-2} z(n) \neq 0$, then in view of parts (i) and (ii) of Lemma 2.1 we have $\lim_{n\to\infty} z(n) = \pm\infty$.

If $\lim_{n\to\infty} \Delta^{m-2} z(n) = 0$, then $\Delta^{m-2} z(n) \leq 0$ since $\big(\Delta^{m-2} z(n)\big)$ is nondecreasing.

Applying this procedure $\dfrac{m-3}{2}$-times, we conlude that $\lim_{n\to\infty} z(n) = \pm\infty$ or $\Delta^3 z(n) \leq 0$. This means that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nondecreasing. The proof of part (iii) of the lemma is complete.

(iv) An obvious consequence of part (iii) by taking $-z(n)$ instead of $z(n)$.

Lemma 2.2 is proved.

Lemma 2.3. *Assume that $\big(x(n)\big)$ is a positive solution of* (E). *Set*

$$z(n) := x(n) + cx(\tau(n)), \tag{2.6}$$

where $(\tau(n))_{n\geq 0}$ *is an increasing sequence of integers such that* (1.1) *holds and* $c\in\mathbb{R}$. *Then the following statements hold*:

(i) *If* $\lim_{n\to\infty} z(n) = -\infty$ *and in addition*:

(ia) $c < -1$, *then* $\big(x(n)\big)$ *is unbounded*;

(ib) $c \geq -1$, *then (E) has no positive solution.*

(ii) *If* $\lim_{n\to\infty} z(n) = A \in \mathbb{R}_-$ *and in addition*:

(iia) $c < -1$, *then* $\liminf x(n) \geq \dfrac{A}{1+c}$ *and if* $\big(x(n)\big)$ *has a real accumulation point greater than* $\dfrac{A}{1+c}$, *it will have infinitely many real accumulation points including* $\dfrac{A}{1+c}$;

(iib) $c \geq -1$, *then* (E) *has no positive solution.*

(iii) *If* $\lim_{n\to\infty} z(n) = 0$ *and in addition*:

(iiia) $c < -1$, *then* $\big(x(n)\big)$ *tends to zero or tends to infinity or* $\big(x(n)\big)$ *has infenitely many accumulation points and* $\liminf x(n) = 0$; *furthermore, the condition* $z(n) > 0$ *guarantees that* $\big(x(n)\big)$ *tends to infinity*;

(iiib) $c = -1$, *then* $\big(x(n)\big)$ *and* $(x(\tau(n)))$ *have the same set of accumulation points*; *furthermore, if* $z(n) < 0$ *then* $\big(x(n)\big)$ *is bounded and, if* $z(n) > 0$ *then* $\liminf x(n) > 0$;

(iiic) $c > -1$, *then* $\big(x(n)\big)$ *tends to zero.*

(iv) *If* $\lim_{n\to\infty} z(n) = A \in \mathbb{R}_+$ *and in addition*:

(iva) $c \leq -1$, *then* $\big(x(n)\big)$ *tends to infinity*;

(ivb) $-1 < c \leq 0$, *then* $\lim_{n\to\infty} x(n) = \dfrac{A}{1+c}$;

(ivc) $c > 0$, *then* $\big(x(n)\big)$ *is bounded.*

(v) *If* $\lim_{n\to\infty} z(n) = +\infty$ *and in addition*:

(va) $c \leq 0$, *then* $\big(x(n)\big)$ *tends to infinity*;

(vb) $c > 0$, *then* $\big(x(n)\big)$ *is unbounded.*

Proof. (i) Assume that $\lim_{n\to\infty} z(n) = -\infty$ and $c < -1$. By (2.6) we have

$$\lim_{n\to\infty}\,[x(n) + cx(\tau(n))] = -\infty,$$

which guarantees that $(x(\tau(n)))$ tends to infinity, and therefore $\big(x(n)\big)$ is unbounded.

Assume that $\lim_{n\to\infty} z(n) = -\infty$ and $c \geq -1$.

If $c = -1$, we have $\lim_{n\to\infty}\,[x(n) - x(\tau(n))] = -\infty$, which guarantees that $\big(x(n)\big)$ is unbounded. On the other hand $x(n) - x(\tau(n)) < 0$ eventually, or

$$x(n) < x(\tau(n)) < x(\tau^2(n)) < \ldots < x(\tau^{m(n_\ell)}(n)) = x(\tau(n_\lambda)),$$

where, in view of (2.2), $n_\lambda = \tau^{m(n_\ell)-1}(n)$. This means that $\big(x(n)\big)$ has an upper bound, which contradicts $\big(x(n)\big)$ is unbounded. Therefore, Eq. (E) has no positive solution.

If $-1 < c < 0$, we have $\lim_{n\to\infty}\,[x(n) + cx(\tau(n))] = -\infty$, which guarantees that $\big(x(n)\big)$ is unbounded. On the other hand $x(n) + cx(\tau(n)) < 0$ eventually, or

$$x(n) < -cx(\tau(n)) < (-c)^2 x(\tau^2(n)) < \ldots < (-c)^{m(n_\mu)} x(\tau(n_s)) \to 0 \qquad \text{as} \quad n\to\infty,$$

where, in view of (2.2), $n_s = \tau^{m(n_\mu)-1}(n)$. Thus $\lim_{n\to\infty} x(n) = 0$ which contradicts $\big(x(n)\big)$ is unbounded. Therefore, Eq. (E) has no positive solution.

If $c \geq 0$, we have $\lim_{n\to\infty}[x(n)+cx(\tau(n))] = -\infty$, which contradicts $z(n) \geq x(n) > 0$. Therefore, Eq. (E) has no positive solution. The proof of part (i) of the lemma is complete.

(ii) Assume that $\lim_{n\to\infty} z(n) = A \in \mathbb{R}_-$ and $c < -1$. Then for every $\epsilon > 0$ with $0 < \epsilon < -A$, there exists $n_1 \geq n_*$ such that $z(n) < A + \epsilon \ \forall n \geq n_1$, or

$$x(n) < -cx(\tau(n)) + A + \epsilon <$$
$$< -c\left[-cx(\tau^2(n)) + A + \epsilon\right] + A + \epsilon =$$
$$= (-c)^2\, x(\tau^2(n)) - c\,(A+\epsilon) + A + \epsilon < \ldots$$
$$\ldots < (-c)^{m(n_\lambda)}\, x(\tau(n_t)) - \frac{A+\epsilon}{1+c}\left[(-c)^{m(n_\lambda)} - 1\right] =$$
$$= (-c)^{m(n_\lambda)}\left[x(\tau(n_t)) - \frac{A+\epsilon}{1+c}\right] + \frac{A+\epsilon}{1+c} \quad \forall n \geq n_1$$

where, in view of (2.2), $n_t = \tau^{m(n_\lambda)-1}(n)$. Since $x(n) > 0$, clearly $x(\tau(n_t)) \geq \dfrac{A+\epsilon}{1+c}$, or eventually $x(\tau(n)) \geq \dfrac{A+\epsilon}{1+c}$.

On the other hand, since $\lim_{n\to\infty} z(n) = A$, we have $z(n) > A - \epsilon$, or

$$x(n) > -cx(\tau(n)) + A - \epsilon > -c\frac{A+\epsilon}{1+c} + A - \epsilon = \frac{A-(1+2c)\epsilon}{1+c}.$$

The last inequality guarantees that $\liminf x(n) \geq \dfrac{A}{1+c} > 0$.

It is clear that $\dfrac{A}{1+c}$ could be an accumulation point of $\big(x(n)\big)$. Let $L > \dfrac{A}{1+c}$ be an accumulation point of $\big(x(n)\big)$. Then there exists a subsequence $(x(\theta(n)))$ of $\big(x(n)\big)$ such that $\lim_{n\to\infty} x(\theta(n)) = L$. Taking into account that $\lim_{n\to\infty} z(\theta(n)) = A$, we obtain $\lim_{n\to\infty} x(\tau(\theta(n))) = \dfrac{A}{c} + \dfrac{L}{-c}$.

In view of this, we have

$$\lim_{n\to\infty}\left[x(\tau(\theta(n))) + cx(\tau^2(\theta(n)))\right] = A,$$

or

$$\lim_{n\to\infty} x(\tau^2(\theta(n))) = \frac{A}{c} + \frac{A}{(-c)^2} + \frac{L}{(-c)^2}.$$

Following the above procedure, we can construct a sequence $(b_n)_{n\geq 1}$ of accumulation points with

$$b_n = \frac{A}{c} + \frac{A}{(-c)^2} + \frac{A}{(-c)^3} + \ldots + \frac{A}{(-c)^n} + \frac{L}{(-c)^n}, \quad n \geq 1.$$

Notice that this sequence of accumulation points converges to $\dfrac{A}{1+c}$.

Assume that $\lim_{n\to\infty} z(n) = A \in \mathbb{R}_-$ and $c \geq -1$.

If $c = -1$, we have $\lim_{n\to\infty}[x(n) - x(\tau(n))] = A$. Then for every $\varepsilon > 0$ with $0 < \varepsilon < -A$ there exists $n_2 > n_*$ such that $x(n) - x(\tau(n)) < A + \varepsilon \ \forall n \geq n_2$, or

$$x(n) < x(\tau(n)) + A + \varepsilon < x(\tau^2(n)) + 2\,(A+\varepsilon) < \ldots$$

$$\ldots < x(\tau^{m(n_\rho)}(n)) + m(n_\rho)\,(A+\varepsilon) =$$
$$= x(\tau(n_\eta)) + m(n_\rho)\,(A+\varepsilon) \quad \forall n \geq n_2,$$

where, in view of (2.2), $n_\eta = \tau^{m(n_\rho)-1}(n)$. This inequality, for sufficiently large n guarantees that $x(n) < 0$ which contradicts $x(n) > 0$. Therefore, Eq. (E) has no positive solution.

If $-1 < c < 0$, we have $\lim_{n\to\infty}[x(n) + cx(\tau(n))] = A < 0$, which, for sufficiently large n, means that $x(n) + cx(\tau(n)) < 0$, or

$$x(n) < -cx(\tau(n)) < (-c)^2\, x(\tau^2(n)) < \ldots$$
$$\ldots < (-c)^{m(n_\ell)}\, x(\tau^{m(n_\ell)}(n)) \to 0 \qquad \text{as} \quad n \to \infty.$$

Thus $\lim_{n\to\infty} x(n) = 0$, and consequently $\lim_{n\to\infty} z(n) = 0$ which contradicts $\lim_{n\to\infty} z(n) = A < 0$. Therefore, Eq. (E) has no positive solution.

If $c \geq 0$, clearly $z(n) > 0$ which contradicts $\lim_{n\to\infty} z(n) = A < 0$. Therefore, Eq. (E) has no positive solution. The proof of part (ii) of the lemma is complete.

(iii) Assume that $\lim_{n\to\infty} z(n) = 0$.

If $c < -1$, we have $\lim_{n\to\infty}[x(n) + cx(\tau(n)] = 0$, which means that $\lim_{n\to\infty} x(n) = 0$ or $\lim_{n\to\infty} x(n) = +\infty$ or $(x(n))$ has infinitely many accumulation points. Indeed, in the case where $(x(n))$ does not tend to zero or to infinity, let $L_0 > 0$ an accumulation point of $(x(n))$. Then there exists a subsequence $(x(\theta(n)))$ of $(x(n))$ such that $\lim_{n\to\infty} x(\theta(n)) = L_0$. Taking into account that $\lim_{n\to\infty} z(\theta(n)) = 0$, we obtain $\lim_{n\to\infty} x(\tau(\theta(n))) = \dfrac{L_0}{-c}$.

In view of this, we have

$$\lim_{n\to\infty}\left[x(\tau(\theta(n))) + cx(\tau^2(\theta(n)))\right] = 0,$$

or

$$\lim_{n\to\infty} x(\tau^2(\theta(n))) = \frac{L_0}{(-c)^2}.$$

Based on the above procedure, we can construct a sequence $(\lambda_n)_{n\geq 1}$ of accumulation points with

$$\lambda_n = \frac{L_0}{(-c)^n}, \quad n \geq 1.$$

Notice that this sequence of accumulation points converges to zero, and therefore $\liminf x(n) = 0$.

Furthermore, the condition $z(n) > 0$ guarantees that $\big(x(n)\big)$ tends to infinity. Indeed, $x(n) + cx(\tau(n)) > 0$, or

$$x(n) > -cx(\tau(n)) > \ldots > (-c)^{m(n)} x(\tau(n_*)) \to +\infty \qquad \text{as} \quad n \to \infty,$$

which means that $\big(x(n)\big)$ tends to infinity.

If $c = -1$ then $\lim_{n\to\infty}[x(n) - x(\tau(n))] = 0$, which means that $\big(x(n)\big)$ and $(x(\tau(n)))$ have the same set of accumulation points.

Furthermore, if $z(n) < 0$, then $x(n) - x(\tau(n)) < 0$, or

$$x(n) < x(\tau(n)) < \ldots < x(\tau(n_*))$$

which means that $\big(x(n)\big)$ is bounded.

If $z(n) > 0$, then $x(n) - x(\tau(n)) > 0$, or

$$x(n) > x(\tau(n)) > \ldots > x(\tau(n_*))$$

which means that $\liminf x(n) > 0$.

Suppose that $c > -1$.

If $-1 < c < 0$, then for every $\epsilon > 0$ there exists $n_3 > n_*$ such that $x(n) + cx(\tau(n)) < \epsilon$ $\forall n \geq n_3$. Thus

$$x(n) < -cx(\tau(n)) + \epsilon < -c\left[-cx(\tau^2(n)) + \epsilon\right] + \epsilon < \ldots$$

$$\ldots < (-c)^{m(n_\ell)}\, x(\tau^{m(n_\ell)}(n)) + \epsilon - c\epsilon + \ldots + (-c)^{m(n_\ell)-1}\,\epsilon \quad \forall n \geq n_3.$$

As $n \to \infty$, clearly $m(n_\ell) \to \infty$, and therefore

$$\lim_{n\to\infty} x(n) \leq \lim_{m(n_\ell)\to\infty}\left[\epsilon - c\epsilon + \ldots + (-c)^{m(n_\ell)-1}\,\epsilon\right] = \frac{\epsilon}{1+c}.$$

Since ϵ is an arbitrarily small, real positive number and as in addition, $x(n) > 0$, the last inequality guarantees that $\lim_{n\to\infty} x(n) = 0$.

If $c \geq 0$, clearly $z(n) > 0$. Taking into account that $\lim_{n\to\infty} z(n) = 0$, it is obvious that $\lim_{n\to\infty} x(n) = 0$. The proof of part (iii) of the lemma is complete.

(iv) Assume that $\lim_{n\to\infty} z(n) = A \in \mathbb{R}_+$.

If $c < -1$, then $\lim_{n\to\infty}[x(n) + cx(\tau(n))] = A > 0$, which means that eventually $x(n) +$ $+ cx(\tau(n)) > 0$. Thus

$$x(n) > -cx(\tau(n)) > (-c)^2\, x(\tau^2(n)) \ldots > (-c)^{m(n_\mu)}\, x(\tau(n_s)) \to +\infty \quad \text{as} \quad n \to \infty$$

which guarantees that $(x(n))$ tends to infinity.

If $c = -1$, then $\lim_{n\to\infty}[x(n) - x(\tau(n))] = A > 0$. Thus, for every $\varepsilon > 0$ with $0 < \varepsilon < A$, there exists $n_4 > n_*$ such that $x(n) - x(\tau(n)) > A - \varepsilon$ $\forall n \geq n_4$, or

$$x(n) > x(\tau(n)) + A - \varepsilon > x(\tau^2(n)) + 2\,(A - \varepsilon) > \ldots$$

$$\ldots > x(\tau^{m(n_\mu)}(n)) + m(n_\mu)\,(A - \varepsilon) =$$

$$= x(\tau(n_s)) + m(n_\mu)\,(A - \varepsilon) \to +\infty \quad \text{as} \quad n \to \infty,$$

which means that $(x(n))$ tends to infinity.

Suppose that $c > -1$.

If $-1 < c < 0$, then for every $\epsilon > 0$, there exists $n_5 > n_*$ such that $|z(n) - A| < \epsilon$ $\forall n \geq n_5$. Thus

$$-cx(\tau(n)) + A - \epsilon < x(n) < -cx(\tau(n)) + A + \varepsilon$$

or

$$-c\left[-cx(\tau^2(n)) + A - \varepsilon\right] + A - \varepsilon < x(n) < -c\left[-cx(\tau^2(n)) + A + \varepsilon\right] + A + \varepsilon.$$

Based on the above procedure we get

$$x(n) < (-c)^{m(n_\mu)}\, x(\tau(n_s)) - \frac{A+\varepsilon}{1+c}\left[(-c)^{m(n_\mu)} - 1\right]$$

and

$$x(n) > (-c)^{m(n_\mu)}\, x(\tau(n_s)) - \frac{A-\varepsilon}{1+c}\left[(-c)^{m(n_\mu)} - 1\right]$$

or

$$x(n) < (-c)^{m(n_\mu)}\left[x(\tau(n_s)) - \frac{A+\varepsilon}{1+c}\right] + \frac{A+\varepsilon}{1+c}$$

and

$$x(n) > (-c)^{m(n_\mu)}\left[x(\tau(n_s)) - \frac{A-\varepsilon}{1+c}\right] + \frac{A-\varepsilon}{1+c}.$$

Therefore as $n \to \infty$ (clearly $m(n_\mu) \to \infty$) we obtain

$$\frac{A-\varepsilon}{1+c} \le \liminf x(n) \le \limsup x(n) \le \frac{A+\varepsilon}{1+c}.$$

Since ϵ is an arbitrary real positive number, the last inequality gives $\lim_{n\to\infty} x(n) = \dfrac{A}{1+c}$.

If $c = 0$, then $z(n) = x(n)$ and therefore $\lim_{n\to\infty} x(n) = A$.

If $c > 0$, then $\lim_{n\to\infty}[x(n) + cx(\tau(n))] = A > 0$, which guarantees that $(x(n))$ is bounded. The proof of part (iv) of the lemma is complete.

(v) Assume that $\lim_{n\to\infty} z(n) = +\infty$.

If $c \le 0$, then $\lim_{n\to\infty}[x(n) + cx(\tau(n))] = +\infty$, which guarantees that $\big(x(n)\big)$ tends to infinity.

If $c > 0$, then $\lim_{n\to\infty}[x(n) + cx(\tau(n))] = +\infty$, which guarantees that $\big(x(n)\big)$ is unbounded. The proof of part (v) of the lemma is complete.

Lemma 2.3 is proved.

3. Main results. Throughout this section, we are going to use the following remarks:

Remark **3.1.** Assume that the coefficients $p(i)$ are always nonnegative or nonpositive, $\sum_{i=0}^{\infty} p(i) = \pm\infty$ and $\liminf x(n) > 0$. Then $\sum_{i=0}^{\infty} p(i)x(\sigma(i)) = \pm\infty$, respectively.

Remark **3.2.** Assume that the coefficients $p(i)$ are always nonnegative or nonpositive, $\sum_{i=0}^{\infty} p(i) = \pm\infty$ and $\sum_{i=0}^{\infty} p(i)x(\sigma(i)) \in \mathbb{R}$. Then $\liminf x(n) = 0$.

The asymptotic behavior of the solutions of the neutral type difference equation (E) is described in the following two cases:

Case 1. $p(n) \ge 0$.

Theorem 3.1. *Assume that* $p(n) \ge 0$ $\forall n \ge 0$ *and* $\sum_{i=0}^{\infty} p(i) = +\infty$. *Then for Eq.* (E) *the following statements hold*:

(i) *If* $c < -1$, *then every nonoscillatory solution* $\big(x(n)\big)$:

(ia) *has no real non-zero limit, if m is even*;

(ib) *is unbounded, if m is odd.*

(ii) *If $c = -1$, then*:

(iia) *every nonoscillatory solution $\big(x(n)\big)$ is bounded and* $\liminf x(n) = 0$, *if m is even*;

(iib) *every non-zero solution $\big(x(n)\big)$ oscillates, if m is odd.*

(iii) *If $-1 < c < 0$, then every nonoscillatory solution $\big(x(n)\big)$ tends to zero.*

(iv) *If $c = 0$ then*:

(iva) *every non-zero solution $\big(x(n)\big)$ oscillates, if m is even*;

(ivb) *every nonoscillatory solution $\big(x(n)\big)$ tends to zero, if m is odd.*

(v) *If $0 < c < 1$, then every nonoscillatory solution $\big(x(n)\big)$*:

(va) *is unbounded but* $\liminf x(n) = 0$, *if m is even*;

(vb) *tends to zero or it is unbounded but* $\liminf x(n) = 0$, *if m is odd.*

(vi) *If $c \geq 1$, then every nonoscillatory solution $\big(x(n)\big)$*:

(via) *cannot tend to zero but* $\liminf x(n) = 0$, *if m is even*;

(vib) *tends to zero or* $\liminf x(n) = 0$, *if m is odd.*

Proof. Assume that a solution $(x(n))_{n\geq -k}$ of (E) is nonoscillatory. Then it is either eventually positive or eventually negative. As $(-x(n))_{n\geq -k}$ is also a solution of (E), we may restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $n_0 \geq -k$ be an integer such that $x(n) > 0$ for all $n \geq n_0 \geq n_*$. Then, there exists $n_1 \geq n_0$ such that $x(\tau(n)) > 0$, $x(\sigma(n)) > 0$ $\forall n \geq n_1$.

In view of (2.6), Eq.(E) becomes $\Delta^m z(n) + p(n)x(\sigma(n)) = 0$, or

$$\Delta^m z(n) = -p(n)x(\sigma(n)). \tag{3.1}$$

Therefore, for sufficiently large n and since $p(n) \geq 0$, we have $\Delta^m z(n) \leq 0$.

Summing up (3.1) from n_1 to n, $n \geq n_1$ we obtain

$$\Delta^{m-1} z(n+1) = \Delta^{m-1} z(n_1) - \sum_{i=n_1}^{n} p(i)x(\sigma(i)). \tag{3.2}$$

(i) $c < -1$.

Assume that m is even. Since $\Delta^m z(n) \leq 0$, by part (i) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nondecreasing.

If $\lim_{n\to\infty} z(n) = -\infty$, then in view of part (ia) of Lemma 2.3 we have that $\big(x(n)\big)$ is unbounded.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of part (va) of Lemma 2.3 we have that $\big(x(n)\big)$ tends to infinity. Then $\sum_{i=n_1}^{\infty} p(i)x(\sigma(i)) = +\infty$. By (3.2), $\big(\Delta^{m-1} z(n+1)\big)$ tends to $-\infty$, which, in view of part (ii) of Lemma 2.1, guarantees that $\big(z(n)\big)$ tends to $-\infty$. This contradicts our assumption. Therefore $\lim_{n\to\infty} z(n) = +\infty$ is false.

If $\big(z(n)\big)$ is nondecreasing, clearly its limit exists.

Suppose that $\lim_{n\to\infty} z(n) = A < 0$. Then in view of part (iia) of Lemma 2.3 we have that $\liminf x(n) \geq \dfrac{A}{1+c}$. Therefore for every $\varepsilon > 0$ with $\varepsilon < \dfrac{A}{1+c}$, there exists n_2 such that

$$x(n) > \frac{A}{1+c} - \varepsilon \quad \forall n \geq n_2. \tag{3.3}$$

Thus, for every n_3 with $\sigma(n_3) \geq n_2$, by (3.2) and (3.3) we obtain

$$\Delta^{m-1}z(n+1) < \Delta^{m-1}z(n_3) - \left(\frac{A}{1+c} - \varepsilon\right)\sum_{i=n_3}^{n} p(i) \to -\infty \quad \text{as} \quad n \to \infty,$$

which guarantees that $\lim_{n\to\infty} \Delta^{m-1}z(n+1) = -\infty$. As in previous case, we conclude that $(z(n))$ tends to $-\infty$, which contradicts $\big(z(n)\big)$ is nondecreasing. Therefore $\lim_{n\to\infty} z(n) = A < 0$ is false.

Suppose that $\lim_{n\to\infty} z(n) = 0$. Then in view of part (iiia) of Lemma 2.3 we have that $\big(x(n)\big)$ either tends to zero or $\big(x(n)\big)$ has infenitely many accumulation points and $\liminf x(n) = 0$ or $\big(x(n)\big)$ tends to infinity. But, if $\lim_{n\to\infty} x(n) = +\infty$, then as in previous case we have that $\lim_{n\to\infty} z(n) = -\infty$ which contradicts our assumption.

Suppose that $\lim_{n\to\infty} z(n) = A > 0$. Since $(z(n))$ is nondecreasing, we have $z(n) > 0$ eventually. Therefore

$$x(n) > -cx(\tau(n)) > \ldots > (-c)^{m(n_\mu)}\, x(\tau(n_s)) \to \infty \quad \text{as} \quad n \to \infty, \tag{3.4}$$

which guarantees that $\big(x(n)\big)$ tends to infinity. As in previous case, we have $\lim_{n\to\infty} \Delta^{m-1}z(n + {} + 1) = -\infty$, which means that $\big(z(n)\big)$ tends to $-\infty$. This contradicts our assumption. Therefore $\lim_{n\to\infty} z(n) = A > 0$ is false.

In other words, $\big(x(n)\big)$ cannot have a non-zero real limit. Indeed, assume that $\lim_{n\to\infty} x(n) = {} = \ell > 0$. Then $\lim_{n\to\infty} z(n) = (1+c)\ell < 0$, which contradicts "$\lim_{n\to\infty} z(n) = A < 0$ is false".

Assume that m is odd. Since $\Delta^m z(n) \leq 0$, by part (iii) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nonincreasing.

If $\lim_{n\to\infty} z(n) = -\infty$, then in view of part (ia) of Lemma 2.3 we have that $\big(x(n)\big)$ is unbounded.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of part (va) of Lemma 2.3 we have that $\big(x(n)\big)$ tends to infinity. Then $\sum_{i=n_1}^{\infty} p(i)x(\sigma(i)) = +\infty$. By (3.2), $\big(\Delta^{m-1}z(n+1)\big)$ tends to $-\infty$, and therefore $(z(n))$ tends to $-\infty$, which contradicts our assumption. Therefore $\lim_{n\to\infty} z(n) = +\infty$ is false.

If $\big(z(n)\big)$ is nonincreasing, clearly its limit exists.

Suppose that $\lim_{n\to\infty} z(n) = A < 0$. As in previous case, this is false.

Suppose that $\lim_{n\to\infty} z(n) = A \geq 0$. Since $(z(n))$ is nonincreasing, we have $z(n) \geq 0$ eventually. Therefore (3.4) holds, and consequently $\big(x(n)\big)$ tends to infinity. Thus, as in previous case, we are led to a contradiction. Therefore $\lim_{n\to\infty} z(n) = A \geq 0$ is false. The proof of part (i) of the theorem is complete.

(ii) $c = -1$.

Assume that m is even. Since $\Delta^m z(n) \leq 0$, by part (i) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nondecreasing. By parts (ib) and (iib) of Lemma 2.3, the cases $\lim_{n\to\infty} z(n) = -\infty$ and $\lim_{n\to\infty} z(n) = A < 0$ are not valid.

Suppose that $\lim_{n\to\infty} z(n) = A > 0$. Since $(z(n))$ is nondecreasing, we have $z(n) > 0$ eventually. Therefore

$$x(n) > x(\tau(n)) > \ldots > x(\tau(n_s)), \tag{3.5}$$

which guarantees that $\sum_{i=n_1}^{n} p(i)x(\sigma(i)) = +\infty$.

Hence, by (3.2) we have $\lim_{n\to\infty}\Delta^{m-1}z(n+1)=-\infty$, which as in part (i) gives $\big(z(n)\big)$ tends to $-\infty$. This leads to a contradiction. Therefore $\lim_{n\to\infty}z(n)=A>0$ is false.

If $\lim_{n\to\infty}z(n)=0$, then $z(n)\le 0$ since $(z(n))$ is nondecreasing. Then in view of part (iiib) of Lemma 2.3 we have that $\big(x(n)\big)$ is bounded. Furthermore, in view of Remark 3.2 we have $\liminf x(n)=0$. Indeed, if $\liminf x(n)>0$ then, by Remark 3.1, we conclude that $\sum_{i=n_1}^{n}p(i)x(\sigma(i))=+\infty$. This guarantees that $\big(z(n)\big)$ tends to $-\infty$, which contradicts our assumption.

If $\lim_{n\to\infty}z(n)=+\infty$, then in view of part (va) of Lemma 2.3 we have that $\big(x(n)\big)$ tends to infinity. This guarantees that $\lim_{n\to\infty}\Delta^{m-1}z(n+1)=-\infty$, which as in part (i) gives that $\big(z(n)\big)$ tends to $-\infty$. This leads a contradiction. Therefore $\lim_{n\to\infty}z(n)=+\infty$ is false.

Assume that m is odd. Since $\Delta^m z(n)\le 0$, by part (iii) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nonincreasing. By parts (ib) and (iib) of Lemma 2.3, the cases $\lim_{n\to\infty}z(n)=-\infty$ and $\lim_{n\to\infty}z(n)=A<0$ are not valid.

If $\lim_{n\to\infty}z(n)=A\ge 0$, then $z(n)>0$ eventually since $\big(z(n)\big)$ is nonincreasing. Thus (3.5) holds and consequently $\lim_{n\to\infty}z(n)=-\infty$. This contradicts our assumption. Therefore $\lim_{n\to\infty}z(n)=A\ge 0$ is false.

If $\lim_{n\to\infty}z(n)=+\infty$, then in view of part (va) of Lemma 2.3 we have that $\big(x(n)\big)$ tends to infinity. This guarantees that $\lim_{n\to\infty}\Delta^{m-1}z(n+1)=-\infty$, which as in part (i) gives that $\big(z(n)\big)$ tends to $-\infty$. This leads to a contradiction. Therefore $\lim_{n\to\infty}z(n)=+\infty$ is false.

Consequently, $\big(x(n)\big)$ oscillates. The proof of part (ii) of the theorem is complete.

(iii) $-1<c<0$.

Assume that m is even. Since $\Delta^m z(n)\le 0$, by part (i) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nondecreasing. By parts (ib) and (iib) of Lemma 2.3, the cases $\lim_{n\to\infty}z(n)=-\infty$ and $\lim_{n\to\infty}z(n)=A<0$ are not valid.

Suppose that $\lim_{n\to\infty}z(n)=A>0$. By part (ivb) of Lemma 2.3 we have that $\lim_{n\to\infty}x(n)=\dfrac{A}{1+c}>0$. This guarantees that $\sum_{i=n_1}^{n}p(i)x(\sigma(i))=+\infty$. Thus, as in previous parts, we are led to a contradiction. Therefore $\lim_{n\to\infty}z(n)=A>0$ is false.

If $\lim_{n\to\infty}z(n)=0$, then by part (iiic) we have that $\big(x(n)\big)$ tends to zero.

If $\lim_{n\to\infty}z(n)=+\infty$, then in view of part (va) of Lemma 2.3 we have that $\big(x(n)\big)$ tends to infinity. Thus, as in previous parts, we are led to a contradiction. Therefore $\lim_{n\to\infty}z(n)=+\infty$ is false.

Assume that m is odd. Since $\Delta^m z(n)\le 0$, by part (iii) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nonincreasing. By parts (ib) and (iib) of Lemma 2.3, the cases $\lim_{n\to\infty}z(n)=-\infty$ and $\lim_{n\to\infty}z(n)=A<0$ are not valid.

If $\lim_{n\to\infty}z(n)=0$, then by part (iiic) of Lemma 2.3 we have that $\big(x(n)\big)$ tends to zero.

Suppose that $\lim_{n\to\infty}z(n)=A>0$. By part (ivb) of Lemma 2.3 we have that $\lim_{n\to\infty}x(n)=\dfrac{A}{1+c}>0$. As in case where m is even, we are led to a contradiction. Therefore $\lim_{n\to\infty}z(n)=A>0$ is false.

If $\lim_{n\to\infty}z(n)=+\infty$, then in view of part (va) of Lemma 2.3 we have that $\big(x(n)\big)$ tends to infinity. Thus, as in previous cases, we are led to a contradiction. Therefore $\lim_{n\to\infty}z(n)=+\infty$ is false. The proof of part (iii) of the theorem is complete.

(iv) $c = 0$.

Assume that m is even. Since $\Delta^m z(n) \le 0$, by part (i) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nondecreasing. By parts (ib) and (iib) of Lemma 2.3, the cases $\lim_{n\to\infty} z(n) = -\infty$ and $\lim_{n\to\infty} z(n) = A < 0$ are not valid.

Suppose that $\lim_{n\to\infty} z(n) = A > 0$. By part (ivb) of Lemma 2.3 we have that $\lim_{n\to\infty} x(n) = A > 0$. This guarantees that $\sum_{i=n_1}^{n} p(i)x(\sigma(i)) = +\infty$. Hence, as in previous cases, we are led to a contradiction. Therefore $\lim_{n\to\infty} z(n) = A > 0$ is false.

If $\lim_{n\to\infty} z(n) = 0$, then $\big(z(n)\big) \le 0$ since $\big(z(n)\big)$ is nondecreasing. This contradicts $z(n) = x(n) > 0$. Therefore $\lim_{n\to\infty} z(n) = 0$ is false.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of part (va) of Lemma 2.3 we have that $\big(x(n)\big)$ tends to infinity. This guarantees that $\sum_{i=n_1}^{n} p(i)x(\sigma(i)) = +\infty$. Hence, as in previous cases, we are led to a contradiction. Therefore $\lim_{n\to\infty} z(n) = +\infty$ is false. Consequently, $\big(x(n)\big)$ oscillates.

Assume that m is odd. Since $\Delta^m z(n) \le 0$, by part (iii) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nonincreasing. By parts (ib) and (iib) of Lemma 2.3, the cases $\lim_{n\to\infty} z(n) = -\infty$ and $\lim_{n\to\infty} z(n) = A < 0$ are not valid.

If $\lim_{n\to\infty} z(n) = 0$, then by part (iiic) of Lemma 2.3 we have that $\big(x(n)\big)$ tends to zero.

Suppose that $\lim_{n\to\infty} z(n) = A > 0$. By part (ivb) of Lemma 2.3 we have that $\lim_{n\to\infty} x(n) = A > 0$. As in case where m is even, we are led to a contradiction. Therefore $\lim_{n\to\infty} z(n) = A > 0$ is false.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of part (va) of Lemma 2.3 we have that $\big(x(n)\big)$ tends to infinity. This guarantees that $\sum_{i=n_1}^{+\infty} p(i)x(\sigma(i)) = +\infty$. Hence, as in previous cases, we are led to a contradiction. Therefore $\lim_{n\to\infty} z(n) = +\infty$ is false. The proof of part (iv) of the theorem is complete.

(v) $0 < c < 1$.

Assume that m is even. Since $\Delta^m z(n) \le 0$, by part (i) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nondecreasing. By parts (ib) and (iib) of Lemma 2.3, the cases $\lim_{n\to\infty} z(n) = -\infty$ and $\lim_{n\to\infty} z(n) = A < 0$ are not valid.

If $\lim_{n\to\infty} z(n) = 0$, then $\big(z(n)\big) \le 0$ since $\big(z(n)\big)$ is nondecreasing. This contradicts $z(n) = x(n) + cx(\tau(n)) > 0$. Therefore $\lim_{n\to\infty} z(n) = 0$ is false.

Suppose that $\lim_{n\to\infty} z(n) = A > 0$. By part (ivc) of Lemma 2.3 we have that $\big(x(n)\big)$ is bounded. This guarantees that $\sum_{i=n_1}^{+\infty} p(i)x(\sigma(i)) < +\infty$. Indeed, if $\sum_{i=n_1}^{+\infty} p(i)x(\sigma(i)) = +\infty$, then by (3.2) and part (ii) of Lemma 2.1 we have that $\lim_{n\to\infty} z(n) = -\infty$ which contradicts $\lim_{n\to\infty} z(n) = A > 0$. By Remark 3.2 we have that $\liminf x(n) = 0$. Then there exists a subsequence $(x(\theta(n)))$ of $\big(x(n)\big)$ such that $\lim_{n\to\infty} x(\theta(n)) = 0$. Taking into account that $\lim_{n\to\infty} z(\theta(n)) = A$, we obtain $\lim_{n\to\infty} x(\tau(\theta(n))) = \dfrac{A}{c}$.

In view of this, we have

$$\lim_{n\to\infty} \big[x(\tau(\theta(n))) + cx(\tau^2(\theta(n)))\big] = A,$$

or

$$\lim_{n\to\infty} x(\tau^2(\theta(n))) = \frac{A - \dfrac{A}{c}}{c} < 0,$$

which is impossible. Therefore $\lim_{n\to\infty} z(n) = A > 0$ is false.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of part (vb) of Lemma 2.3 we have that $\big(x(n)\big)$ is unbounded. If $\sum_{i=n_1}^{+\infty} p(i)x(\sigma(i)) = +\infty$, then by (3.2) $\big(\Delta^{m-1}z(n+1)\big)$ tends to $-\infty$. By part (ii) of Lemma 2.1 we conclude that $\big(z(n)\big)$ tends to $-\infty$, which contradicts $\lim_{n\to\infty} z(n) = +\infty$. Therefore $\sum_{i=n_1}^{+\infty} p(i)x(\sigma(i)) < +\infty$. By Remark 3.2, we conclude that $\liminf x(n) = 0$.

Assume that m is odd. Since $\Delta^m z(n) \leq 0$, by part (iii) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nonincreasing. By parts (ib) and (iib) of Lemma 2.3, the cases $\lim_{n\to\infty} z(n) = -\infty$ and $\lim_{n\to\infty} z(n) = A < 0$ are not valid.

If $\lim_{n\to\infty} z(n) = 0$, then by part (iiic) we have that $\big(x(n)\big)$ tends to zero.

Suppose that $\lim_{n\to\infty} z(n) = A > 0$. By part (ivc) of Lemma 2.3 we have that $\big(x(n)\big)$ is bounded. Thus, as in case where m is even, we are led to a contradiction. Therefore $\lim_{n\to\infty} z(n) = A > 0$ is false.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of part (vb) of Lemma 2.3 we have that $\big(x(n)\big)$ is unbounded. Thus, as in case where m is even, we have $\liminf x(n) = 0$. The proof of part (v) of the theorem is complete.

(vi) $c \geq 1$.

Assume that m is even. Since $\Delta^m z(n) \leq 0$, by part (i) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nondecreasing. By parts (ib) and (iib) of Lemma 2.3, the cases $\lim_{n\to\infty} z(n) = -\infty$ and $\lim_{n\to\infty} z(n) = A < 0$ are not valid.

If $\lim_{n\to\infty} z(n) = 0$, then $\big(z(n)\big) \leq 0$ since $\big(z(n)\big)$ is nondecreasing. This contradicts $z(n) = x(n) + cx(\tau(n)) > 0$. Therefore $\lim_{n\to\infty} z(n) = 0$ is false.

Suppose that $\lim_{n\to\infty} z(n) = A > 0$. By part (ivc) of Lemma 2.3 we have that $\big(x(n)\big)$ is bounded. Therefore, as in previous part, we have that $\sum_{i=n_1}^{+\infty} p(i)x(\sigma(i)) < +\infty$. By Remark 3.2 we have that $\liminf x(n) = 0$.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of part (vb) of Lemma 2.3 we have that $\big(x(n)\big)$ is unbounded. If $\sum_{i=n_1}^{+\infty} p(i)x(\sigma(i)) = +\infty$, then by (3.2) $\big(\Delta^{m-1}z(n+1)\big)$ tends to $-\infty$. By part (ii) of Lemma 2.1 we conclude that $\big(z(n)\big)$ tends to $-\infty$, which contradicts $\lim_{n\to\infty} z(n) = +\infty$. Therefore $\sum_{i=n_1}^{+\infty} p(i)x(\sigma(i)) < +\infty$. By Remark 3.2 we conclude that $\liminf x(n) = 0$. In other words, $\big(x(n)\big)$ cannot tend to zero but $\liminf x(n) = 0$.

Assume that m is odd. Since $\Delta^m z(n) \leq 0$, by part (iii) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nonincreasing. By parts (ib) and (iib) of Lemma 2.3, the cases $\lim_{n\to\infty} z(n) = -\infty$ and $\lim_{n\to\infty} z(n) = A < 0$ are not valid.

If $\lim_{n\to\infty} z(n) = 0$, then by part (iiic) we have that $\big(x(n)\big)$ tends to zero.

Suppose that $\lim_{n\to\infty} z(n) = A > 0$. Then by part (ivc) of Lemma 2.3 we have that $\big(x(n)\big)$ is bounded. Thus we have $\sum_{i=n_1}^{+\infty} p(i)x(\sigma(i)) < +\infty$. By Remark 3.2, $\liminf x(n) = 0$.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of part (vb) of Lemma 2.3 we have that $\big(x(n)\big)$ is unbounded. As in previous case, $\liminf x(n) = 0$. The proof of part (vi) of the theorem is complete.

Theorem 3.1 is proved.

Case 2. $p(n) \le 0$.

Theorem 3.2. *Assume that* $p(n) \le 0 \ \forall n \ge 0$ *and* $\sum_{i=0}^{\infty} p(i) = -\infty$. *Then for Eq.* (E) *the following statements hold*:

(i) *If* $c < -1$, *then every nonoscillatory solution* $\big(x(n)\big)$:

(ia) *is unbounded but* $\liminf x(n) = 0$ *or tends to infinity, if* m *is even*;

(ib) *has no a non-zero real limit but* $\liminf x(n) = 0$, *if* m *is odd.*

(ii) *If* $c = -1$, *then every nonoscillatory solution* $\big(x(n)\big)$:

(iia) *tends to infinity, if* m *is even*;

(iib) *tends to infinity or it is bounded and* $\liminf x(n) = 0$, *if* m *is odd.*

(iii) *If* $-1 < c < 0$, *then every nonoscillatory solution* $\big(x(n)\big)$ *tends to zero or tends to infinity.*

(iv) *If* $c = 0$, *then every nonoscillatory solution* $\big(x(n)\big)$:

(iva) *tends to zero or tends to infinity, if* m *is even*;

(ivb) *tends to infinity, if* m *is odd.*

(v) *If* $0 < c < 1$, *then every nonoscillatory solution* $\big(x(n)\big)$:

(va) *is unbounded or tends to zero, if* m *is even*;

(vb) *is unbounded, if* m *is odd.*

(vi) *If* $c \ge 1$, *then every nonoscillatory solution* $\big(x(n)\big)$:

(via) *is unbounded or is bounded and* $\liminf x(n) = 0$ *or tends to zero, if* m *is even*;

(vib) *cannot tend to zero and it is unbounded or it is bounded with* $\liminf x(n) = 0$, *if* m *is odd.*

Proof. Assume that the solution $(x(n))_{n\ge -k}$ of (E) is nonoscillatory. Then it is either eventually positive or eventually negative. As $(-x(n))_{n\ge -k}$ is also a solution of (E), we may restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $n_0 \ge -k$ be an integer such that $x(n) > 0$ for all $n \ge n_0 \ge n_*$. Then, there exists $n_1 \ge n_0$ such that $x(\tau(n)) > 0$, $x(\sigma(n)) > 0 \ \forall n \ge n_1$.

In view of (2.6), Eq.(E) becomes $\Delta^m z(n) + p(n)x(\sigma(n)) = 0$, or (3.1). Therefore, for sufficiently large n and since $p(n) \le 0$, we have $\Delta^m z(n) \ge 0$.

Summing up (3.1) from n_1 to n, $n \ge n_1$ we obtain (3.2).

(i) $c < -1$.

Assume that m is even. Since $\Delta^m z(n) \ge 0$, by part (ii) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nonincreasing.

If $\lim_{n\to\infty} z(n) = -\infty$, then in view of part (ia) of Lemma 2.3 we have that $\big(x(n)\big)$ is unbounded. If $\sum_{i=n_1}^{n} p(i)x(\sigma(i)) = -\infty$, then by (3.2) $\big(\Delta^{m-1}z(n+1)\big)$ tends to $+\infty$. By part (i) of Lemma 2.1 we conclude that $\big(z(n)\big)$ tends to $+\infty$, which contradicts $\lim_{n\to\infty} z(n) = -\infty$. Therefore $\sum_{i=n_1}^{n} p(i)x(\sigma(i)) > -\infty$. By Remark 3.2, we conclude that $\liminf x(n) = 0$.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of part (va) of Lemma 2.3 we have that $\big(x(n)\big)$ tends to infinity.

If $\big(z(n)\big)$ is nonincreasing, clearly its limit exists.

Suppose that $\lim_{n\to\infty} z(n) = A < 0$. Then in view of part (iia) of Lemma 2.3 we have that $\liminf x(n) \ge \dfrac{A}{1+c}$. Therefore for every $\varepsilon > 0$ with $\varepsilon < \dfrac{A}{1+c}$, there exists n_2 such that (3.3) holds. Thus, for every n_3 with $\sigma(n_3) \ge n_2$, by (3.2) and (3.3) we obtain

$$\Delta^{m-1}z(n+1) > \Delta^{m-1}z(n_3) - \left(\frac{A}{1+c} - \varepsilon\right)\sum\nolimits_{i=n_3}^{n} p(i) \to +\infty \quad \text{as} \quad n \to \infty,$$

which guarantees that $\lim_{n\to\infty}\Delta^{m-1}z(n+1) = +\infty$. By part (i) of Lemma 2.1 we conclude that $(z(n))$ tends to $+\infty$, which contradicts our assumption. Therefore $\lim_{n\to\infty} z(n) = A < 0$ is false.

Suppose that $\lim_{n\to\infty} z(n) = A \geq 0$. Then since $(z(n))$ is nonincreasing, we have $z(n) > 0$ eventually. Therefore (3.4) holds, which guarantees that $\big(x(n)\big)$ tends to infinity. Thus, as in previous case, we have that $\lim_{n\to\infty}\Delta^{m-1}z(n+1) = \infty$ which means that $\big(z(n)\big)$ tends to $+\infty$. This contradicts $\lim_{n\to\infty} z(n) = A \geq 0$. Therefore $\lim_{n\to\infty} z(n) = A \geq 0$ is false.

Assume that m is odd. Since $\Delta^m z(n) \geq 0$, by part (iv) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nondecreasing.

If $\lim_{n\to\infty} z(n) = -\infty$, then in view of part (ia) of Lemma 2.3 we have that $\big(x(n)\big)$ is unbounded. Also, as in case where m is even, $\liminf x(n) = 0$ is satisfied.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of part (va) of Lemma 2.3 we have that $\big(x(n)\big)$ tends to infinity.

If $\big(z(n)\big)$ is nondecreasing, clearly its limit exists.

Suppose that $\lim_{n\to\infty} z(n) = A < 0$. As in previous case, this is false.

Suppose that $\lim_{n\to\infty} z(n) = 0$. Then in view of part (iiia) of Lemma 2.3 we have that $\big(x(n)\big)$ either tends to zero or $\big(x(n)\big)$ has infenitely many accumulation points or $\big(x(n)\big)$ tends to infinity. But, if $\lim_{n\to\infty} x(n) = +\infty$, then as in previous case we have $\lim_{n\to\infty} z(n) = +\infty$, which contradicts our assumption.

Suppose that $\lim_{n\to\infty} z(n) = A > 0$. Thus (3.4) holds, and as in previous case we are led to a contradiction. Therefore $\lim_{n\to\infty} z(n) = A > 0$ is false.

In other words, $\big(x(n)\big)$ cannot have a non-zero real limit. Indeed, assume that $\lim_{n\to\infty} x(n) = \ell > 0$. Then $\lim_{n\to\infty} z(n) = (1+c)\ell < 0$, which contradicts "$\lim_{n\to\infty} z(n) = A < 0$ is false". The proof of part (i) of the theorem is complete.

(ii) $c = -1$.

Assume that m is even. Since $\Delta^m z(n) \geq 0$, by part (ii) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nonincreasing. By parts (ib) and (iib) of Lemma 2.3, the cases $\lim_{n\to\infty} z(n) = -\infty$ and $\lim_{n\to\infty} z(n) = A < 0$ are not valid.

If $\lim_{n\to\infty} z(n) = A \geq 0$, then $z(n) \geq 0$ eventually since $\big(z(n)\big)$ is nonincreasing. Thus (3.5) holds which means that $\big(x(n)\big)$ has a lower bound, i.e., $\liminf x(n) > 0$. By Remark 3.1 and relation (3.2) we conclude that $\lim_{n\to\infty}\Delta^{m-1}z(n) > 0$. Therefore, by part (i) of Lemma 2.1 we have that $\lim_{n\to\infty} z(n) = +\infty$, which contradicts our assumption. Therefore $\lim_{n\to\infty} z(n) = A \geq 0$ is false.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of part (va) of Lemma 2.3 we have that $\big(x(n)\big)$ tends to infinity.

Assume that m is odd. Since $\Delta^m z(n) \geq 0$, by part (iv) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nondecreasing. By parts (ib) and (iib) of Lemma 2.3, the cases $\lim_{n\to\infty} z(n) = -\infty$ and $\lim_{n\to\infty} z(n) = A < 0$ are not valid.

Suppose that $\lim_{n\to\infty} z(n) = A > 0$. Since $(z(n))$ is nondecreasing, we have $z(n) > 0$ eventually. Therefore (3.5) holds, and therefore as in case m is even, we are led to a contradiction. Therefore $\lim_{n\to\infty} z(n) = A > 0$ is false.

If $\lim_{n\to\infty} z(n) = 0$, then $z(n) \leq 0$ since $(z(n))$ is nondecreasing. Then in view of part (iiib) of Lemma 2.3 we have that $\big(x(n)\big)$ is bounded. Furthermore, in view of Remark 3.2 we have $\liminf x(n) = 0$.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of part (va) of Lemma 2.3 we have that $\big(x(n)\big)$ tends to infinity. The proof of part (ii) of the theorem is complete.

(iii) $-1 < c < 0$.

Assume that m is even. Since $\Delta^m z(n) \geq 0$, by part (iii) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nonincreasing. By parts (ib) and (iib) of Lemma 2.3, the cases $\lim_{n\to\infty} z(n) = -\infty$ and $\lim_{n\to\infty} z(n) = A < 0$ are not valid.

If $\lim_{n\to\infty} z(n) = 0$, then by part (iiic) we have that $\big(x(n)\big)$ tends to zero.

Suppose that $\lim_{n\to\infty} z(n) = A > 0$. By part (ivb) of Lemma 2.3 we have that $\lim_{n\to\infty} x(n) = \dfrac{A}{1+c} > 0$. This guarantees that $\sum_{i=n_1}^{n} p(i)x(\sigma(i)) = -\infty$. Thus, as in previous case, we are led to a contradiction. Therefore $\lim_{n\to\infty} z(n) = A > 0$ is false.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of part (va) of Lemma 2.3 we have that $\big(x(n)\big)$ tends to infinity.

Assume that m is odd. Since $\Delta^m z(n) \geq 0$, by part (iv) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nondecreasing. By parts (ib) and (iib) of Lemma 2.3, the cases $\lim_{n\to\infty} z(n) = -\infty$ and $\lim_{n\to\infty} z(n) = A < 0$ are not valid.

Suppose that $\lim_{n\to\infty} z(n) = A > 0$. By part (ivb) of Lemma 2.3 we have that $\lim_{n\to\infty} x(n) = \dfrac{A}{1+c} > 0$. This guarantees that $\sum_{i=n_1}^{n} p(i)x(\sigma(i)) = -\infty$. Thus, as in case where m is even, we are led to a contradiction. Therefore $\lim_{n\to\infty} z(n) = A > 0$ is false.

If $\lim_{n\to\infty} z(n) = 0$, then by part (iiic) we have that $\big(x(n)\big)$ tends to zero.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of part (va) of Lemma 2.3 we have that $\big(x(n)\big)$ tends to infinity. The proof of part (iii) of the theorem is complete.

(iv) $c = 0$.

Assume that m is even. Since $\Delta^m z(n) \geq 0$, by part (ii) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nonincreasing. By parts (ib) and (iib) of Lemma 2.3, the cases $\lim_{n\to\infty} z(n) = -\infty$ and $\lim_{n\to\infty} z(n) = A < 0$ are not valid.

If $\lim_{n\to\infty} z(n) = 0$, then by part (iiic) of Lemma 2.3 we have that $\big(x(n)\big)$ tends to zero.

Suppose that $\lim_{n\to\infty} z(n) = A > 0$. By part (ivb) of Lemma 2.3 we have that $\lim_{n\to\infty} x(n) = A > 0$. This guarantees that $\sum_{i=n_1}^{n} p(i)x(\sigma(i)) = -\infty$. Hence, as in previous cases, we are led to a contradiction. Therefore $\lim_{n\to\infty} z(n) = A > 0$ is false.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of part (va) of Lemma 2.3 we have that $\big(x(n)\big)$ tends to infinity.

Assume that m is odd. Since $\Delta^m z(n) \geq 0$, by part (iv) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nondecreasing. By parts (ib) and (iib) of Lemma 2.3, the cases $\lim_{n\to\infty} z(n) = -\infty$ and $\lim_{n\to\infty} z(n) = A < 0$ are not valid.

Suppose that $\lim_{n\to\infty} z(n) = A > 0$. By part (ivb) of Lemma 2.3 we have that $\lim_{n\to\infty} x(n) = A > 0$. This guarantees that $\sum_{i=n_1}^{n} p(i)x(\sigma(i)) = +\infty$. Hence, as in previous cases, we are led to a contradiction. Therefore $\lim_{n\to\infty} z(n) = A > 0$ is false.

If $\lim_{n\to\infty} z(n) = 0$, then $\big(z(n)\big) \le 0$ since $\big(z(n)\big)$ is nondecreasing. This contradicts $z(n) = x(n) > 0$. Therefore $\lim_{n\to\infty} z(n) = 0$ is false.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of part (va) of Lemma 2.3 we have that $\big(x(n)\big)$ tends to infinity. The proof of part (iv) of the theorem is complete.

(v) $0 < c < 1$.

Assume that m is even. Since $\Delta^m z(n) \ge 0$, by part (ii) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nonincreasing. By parts (ib) and (iib) of Lemma 2.3, the cases $\lim_{n\to\infty} z(n) = -\infty$ and $\lim_{n\to\infty} z(n) = A < 0$ are not valid.

If $\lim_{n\to\infty} z(n) = 0$, then by part (iiic) of Lemma 2.3 we have that $\big(x(n)\big)$ tends to zero.

If $\lim_{n\to\infty} z(n) = A > 0$, then $\sum_{i=n_1}^{n} p(i)x(\sigma(i)) > -\infty$. By Remark 3.2 we have that $\liminf x(n) = 0$. Then there exists a subsequence $(x(\theta(n)))$ of $(x(n))$ such that $\lim_{n\to\infty} x(\theta(n)) = 0$. By a similar procedure as in part (v) of Theorem 3.1 we are led to a contradiction. Therefore $\lim_{n\to\infty} z(n) = A > 0$ is false.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of part (vb) of Lemma 2.3 we have that $\big(x(n)\big)$ unbounded.

Assume that m is odd. Since $\Delta^m z(n) \ge 0$, by part (iv) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nondecreasing. By parts (ib) and (iib) of Lemma 2.3, the cases $\lim_{n\to\infty} z(n) = -\infty$ and $\lim_{n\to\infty} z(n) = A < 0$ are not valid.

If $\lim_{n\to\infty} z(n) = 0$, then $z(n) \le 0$ since $(z(n))$ is nondecreasing. This contradicts $z(n) = x(n) + cx(\tau(n)) > 0$. Therefore $\lim_{n\to\infty} z(n) = 0$ is false.

Suppose that $\lim_{n\to\infty} z(n) = A > 0$. By part (ivc) of Lemma 2.3 we have that $\big(x(n)\big)$ is bounded. This guarantees that $\sum_{i=n_1}^{n} p(i)x(\sigma(i)) > -\infty$. As in case where m is even, we are led to a contradiction. Therefore $\lim_{n\to\infty} z(n) = A > 0$ is false.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of part (vb) of Lemma 2.3 we have that $\big(x(n)\big)$ is unbounded. The proof of part (v) of the theorem is complete.

(vi) $c \ge 1$.

Assume that m is even. Since $\Delta^m z(n) \ge 0$, by part (ii) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nonincreasing. By parts (ib) and (iib) of Lemma 2.3, the cases $\lim_{n\to\infty} z(n) = -\infty$ and $\lim_{n\to\infty} z(n) = A < 0$ are not valid.

If $\lim_{n\to\infty} z(n) = 0$, then by part (iiic) we have that $\big(x(n)\big)$ tends to zero.

Suppose that $\lim_{n\to\infty} z(n) = A > 0$. By part (ivc) of Lemma 2.3 we have that $\big(x(n)\big)$ is bounded. This guarantees that $\sum_{i=n_1}^{n} p(i)x(\sigma(i)) > -\infty$. By Remark 3.2, $\liminf x(n) = 0$.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of part (vb) of Lemma 2.3 we have that $\big(x(n)\big)$ is unbounded.

Assume that m is odd. Since $\Delta^m z(n) \ge 0$, by part (iv) of Lemma 2.2 we have that $\big(z(n)\big)$ tends to $\pm\infty$ or it is nondecreasing. By parts (ib) and (iib) of Lemma 2.3, the cases $\lim_{n\to\infty} z(n) = -\infty$ and $\lim_{n\to\infty} z(n) = A < 0$ are not valid.

If $\lim_{n\to\infty} z(n) = 0$, then $\big(z(n)\big) \le 0$ since $\big(z(n)\big)$ is nondecreasing. This contradicts $z(n) = x(n) + cx(\tau(n)) > 0$. Therefore $\lim_{n\to\infty} z(n) = 0$ is false.

Suppose that $\lim_{n\to\infty} z(n) = A > 0$. By part (ivc) of Lemma 2.3 we have that $\big(x(n)\big)$ is bounded. This guarantees that $\sum_{i=n_1}^{n} p(i)x(\sigma(i)) > -\infty$. By Remark 3.2 we have that $\liminf x(n) = 0$.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of part (vb) of Lemma 2.3 we have that $\big(x(n)\big)$ is unbounded. The proof of part (vi) of the theorem is complete.

Theorem 3.2 is proved.

4. Examples. In this section we present some examples to illustrate the main results.

***Example* 4.1.** Consider the difference equation

$$\Delta^2(x(n) - 2x(n-1)) + p(n)x(n^2+1) = 0, \quad n \ge 2,$$

where $p(n) = \dfrac{(2n+10)\left(n^2+1\right)}{n\,(n+1)\,(n-1)\,(n+2)} > 0 \ \forall n \ge 2.$

Here m is even and $\sum_{i=2}^{\infty} p(i) = +\infty$. It is easy to see that all conditions of part (ia) of the Theorem 3.1 are satisfied, and hence every nonoscillatory solution $(x(n))$ of the above equation has no a real non-zero limit. In fact $(x(n)) = \left(\dfrac{1}{n}\right)$ is one such solution, since it satisfies the above equation for all $n \ge 2$ and $\lim_{n\to\infty} x(n) = 0$.

***Example* 4.2.** Consider the difference equation

$$\Delta^3(x(n) - x(n-1)) + 16x(n^2+2) = 0, \quad n \ge 1.$$

Here m is odd and $\sum_{i=1}^{\infty} p(i) = +\infty$. All conditions of part (iib) of the Theorem 3.1 are satisfied, and hence every non-zero solution $(x(n))$ oscillates. In fact $(x(n)) = ((-1)^n)$ is one such solution, since it satisfies the above equation for all $n \ge 1$ and oscillates.

***Example* 4.3.** Consider the difference equation

$$\Delta^2\left(x(n) - \frac{1}{4}x(n-1)\right) + p(n)x(n-2) = 0, \quad n \ge 41,$$

where

$$p(n) = \frac{5}{2} \cdot \frac{n^3 - 57n^2 + 722n - 2320}{n(n-1)(n+1)(n+2)} > 0 \quad \forall n \ge 41.$$

Clearly $\sum_{i=41}^{\infty} p(i) = +\infty$. All conditions of part (iii) of the Theorem 3.1 are satisfied, and hence every nonoscillatory solution $(x(n))$ tends to zero. In fact $(x(n)) = \left(\dfrac{10^n}{n!}\right)$ is one such solution, since it satisfies the above equation for all $n \ge 41$ and $\lim_{n\to\infty} x(n) = 0$.

***Example* 4.4.** Consider the difference equation

$$\Delta^3\big(x(n) + x(n-1)\big) + p(n)x(n+3) = 0, \quad n \ge 4,$$

where $p(n) = \dfrac{3n-11}{n+3} > 0 \ \forall n \ge 4.$

Here m is odd and $\sum_{i=4}^{\infty} p(i) = +\infty$. All conditions of part (vib) of the Theorem 3.1 are satisfied, and hence every nonoscillatory solution $(x(n))$ tends to zero or $\liminf x(n) = 0$. In fact $(x(n)) = \left(\dfrac{n}{2^n}\right)$ is one such solution, since it satisfies the above equation for all $n \ge 4$ and $\lim_{n\to\infty} x(n) = 0$.

***Example* 4.5.** Consider the difference equation

$$\Delta^2\left(x(n) - 2x(n-2)\right) + \left(-\frac{1}{16}\right)x(n+3) = 0, \quad n \ge 3.$$

Here m is even and $\sum_{i=3}^{\infty} p(i) = -\infty$. All conditions of part (ia) of the Theorem 3.2 are satisfied, and hence every nonoscillatory solution $(x(n))$ is unbounded but $\liminf x(n) = 0$ or tends to infinity. In fact $(x(n)) = (2^n)$ is one such solution, since it satisfies the above equation for all $n \geq 3$ and $\lim_{n\to\infty} x(n) = +\infty$.

***Example* 4.6.** Consider the difference equation

$$\Delta^2(x(n) - x(n-2)) + \left(-\frac{3}{8}\right) x(n+1) = 0, \quad n \geq 2.$$

Here m is even and $\sum_{i=2}^{\infty} p(i) = -\infty$. All conditions of part (iia) of the Theorem 3.2 are satisfied, and hence every nonoscillatory solution $(x(n))$ tends to infinity. In fact $(x(n)) = (2^n)$ is one such solution, since it satisfies the above equation for all $n \geq 2$ and $\lim_{n\to\infty} x(n) = +\infty$.

***Example* 4.7.** Consider the difference equation

$$\Delta^3(x(n) - \frac{1}{2}x(n-1)) + \left(-\frac{4}{3^7}\right) x(n-4) = 0, \quad n \geq 4.$$

Clearly $\sum_{i=4}^{\infty} p(i) = -\infty$. All conditions of part (iii) of the Theorem 3.2 are satisfied, and hence every nonoscillatory solution $(x(n))$ tends to zero or tends to infinity. In fact $(x(n)) = (3^{-n})$ is one such solution, since satisfies the above equation for all $n \geq 4$ and $\lim_{n\to\infty} x(n) = 0$.

***Example* 4.8.** Consider the difference equation

$$\Delta^2\big(x(n) + x(n-1)\big) + p(n)x(n^3 + 2) = 0, \quad n \geq 10,$$

where

$$p(n) = \frac{n^3 + 2}{\ln(n^3 + 2)} \ln \frac{\sqrt[n+1]{n+1}\,\sqrt[n]{n}}{\sqrt[n+2]{n+2}\,\sqrt[n-1]{n-1}} < 0 \quad \forall n \geq 10.$$

Here m is even and $\sum_{i=4}^{\infty} p(i) = -\infty$. All conditions of part (via) of the Theorem 3.2 are satisfied, and hence every nonoscillatory solution $(x(n))$ is unbounded or is bounded and $\liminf x(n) = 0$ or tends to zero. In fact $(x(n)) = \left(\frac{\ln n}{n}\right)$ is one such solution, since it satisfies the above equation for all $n \geq 10$ and $\lim_{n\to\infty} x(n) = 0$.

***Remark* 4.1.** Similarly, one can construct examples to illustrate the other parts of Theorems 3.1 and 3.2.

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