UDC 512.5

S. K. Sekatskii (LPMV, Ecole Polytechnique Fédérale de Lausanne, Switzerland)

GENERALIZED BOMBIERI – LAGARIAS' THEOREM AND GENERALIZED LI'S CRITERION WITH ITS ARITHMETIC INTERPRETATION

УЗАГАЛЬНЕНА ТЕОРЕМА БОМБІЄРІ – ЛАГАРІАСА ТА УЗАГАЛЬНЕНИЙ КРИТЕРІЙ ЛІ ЗІ СВОЄЮ АРИФМЕТИЧНОЮ ІНТЕРПРЕТАЦІЄЮ

We show that Li's criterion equivalent to the Riemann hypothesis, viz. the statement that the sums $k_n = \Sigma_\rho \left(1-\left(1-\frac{1}{\rho}\right)^n\right)$ over Riemann xi-function zeroes and derivatives $\lambda_n \equiv \frac{1}{(n-1)!}\frac{d^n}{dz^n}\left(z^{n-1}\ln(\xi(z))\right)\Big|_{z=1}$, where $n = 1, 2, 3, \ldots$, are nonnegative if and only if the Riemann hypothesis is true, can be generalized and the nonnegativity of certain derivatives of the Riemann xi-function estimated at an *arbitrary* real point a, except $a = 1/2$, can be used as a criterion equivalent to the Riemann hypothesis. Namely, we demonstrate that the sums $k_{n,a} = \Sigma_\rho \left(1-\left(\frac{\rho-a}{\rho+a-1}\right)^n\right)$ for any real a and any $a < 1/2$ are nonnegative if and only if the Riemann hypothesis is true (correspondingly, the same derivatives with $a > 1/2$ should be nonpositive). The arithmetic interpretation of the generalized Li's criterion is given. Similarly to Li's criterion, the theorem of Bombieri and Lagarias applied to certain multisets of complex numbers is also generalized along the same lines.

Показано, що критерій Лі є еквівалентним гіпотезі Рімана, тобто твердження, що суми $k_n = \Sigma_\rho \left(1-\left(1-\frac{1}{\rho}\right)^n\right)$ по нулях рімановoї хі-функції та похідні $\lambda_n \equiv \frac{1}{(n-1)!}\frac{d^n}{dz^n}\left(z^{n-1}\ln(\xi(z))\right)\Big|_{z=1}$, де $n = 1, 2, 3, \ldots$, є невід'ємними тоді і тільки тоді, коли справедлива гіпотеза Рімана, може бути узагальнене, а невід'ємність деяких похідних рімановoї хі-функції, що оцінюються у *довільній* точці a, крім $a = 1/2$, може бути застосована, як критерій, еквівалентний гіпотезі Рімана. А саме, показано, що суми $k_{n,a} = \Sigma_\rho \left(1-\left(\frac{\rho-a}{\rho+a-1}\right)^n\right)$ для будь-яких дійсних a та будь-яких $a < 1/2$ є невід'ємними тоді і тільки тоді, коли справедлива гіпотеза Рімана (відповідно такі ж похідні з $a > 1/2$ повинні бути недодатніми). Наведено арифметичну інтерпретацію узагальненого критерію Лі. Подібно до критерію Лі теорема Бомбієрі та Лагаріаса, у застосуванні до деяких мультимножин комплексних чисел, також може бути узагальнена аналогічним чином.

1. Introduction. In 1997, Li has established the following criterion equivalent to the Riemann hypothesis concerning nontrivial zeroes of the Riemann ζ-function (see, e.g., [1] for standard definitions and discussion of the general properties of this function) and now bearing his name (Li's criterion) [2]:

Li's criterion. *Riemann hypothesis is equivalent to the nonnegativity of the following numbers*:

$$\lambda_n \equiv \frac{1}{(n-1)!}\frac{d^n}{dz^n}\left(z^{n-1}\ln(\xi(z))\Big|_{z=1}\right) \qquad (1)$$

for any nonnegative integer n.

Here $\xi(z)$ is the Riemann xi-function related with the Riemann ζ-function by the well-known relation [1]

$$\xi(z) = \frac{1}{2} z(z-1)\pi^{-z/2}\Gamma(z/2)\varsigma(z). \tag{2}$$

Two years later, Bombieri and Lagarias generalized Li's criterion [3]. If $\rho = 1/2 + iT$, T real and $i = \sqrt{-1}$, than $|(\rho-1)/\rho| = 1$ and hence can be written as $\exp(i\vartheta_i)$, where $\vartheta_i = \arctan \frac{T}{T^2 - 1/4}$. Let us introduce the sum $k_n = \Sigma_\rho \left(1 - \left(1 - \frac{1}{\rho}\right)^n\right) = \Sigma_\rho \left(1 - \left(\frac{\rho-1}{\rho}\right)^n\right)$ over nontrivial Riemann function zeroes (n is nonnegative integer, zeroes are counting taking into account their multiplicity, for $n = 1$ contributions of complex conjugate zeroes should be paired when summing). For two complex conjugate "correct" Riemann function zeroes $\rho = 1/2 \pm iT$ we easily see that their contribution to sum k_n is $2(1 - \cos(n\vartheta_i))$, and hence nonnegative; correspondingly, the sum k_n is also nonnegative. Quite the contrary, if some nontrivial Riemann function zero with $\operatorname{Re}\rho \neq 1/2$ exists, for large enough n we will have an arbitrary large (by module) negative contributions from these zeroes, and it is straightforward to show that for infinitely many n this contribution can not be compensated by all other "correct" $1 - \cos(n\vartheta_i)$ terms of the sum [3], whence infinitely many sums k_n are to be negative.

This consideration immediately shows that the nonnegativity of the sums k_n is equivalent to the Riemann hypothesis. Li also demonstrated that these sums are equal to derivatives presented in eq. (1) (certainly, this is the most technically difficult part of his work; another derivation of this relation will be given shortly below).

2. Generalized Li's and Bombieri – Lagarias' criteria. Now we note that for $\rho = 1/2 + iT$ and any real a $\left|\frac{\rho - a}{\rho + a - 1}\right| = \left|\frac{-a + 1/2 + iT}{a - 1/2 + iT}\right| = 1$ and introduce the sum

$$k_{n,a} = \sum_\rho \left(1 - \left(\frac{\rho - a}{\rho + a - 1}\right)^n\right) = \sum_\rho \left(1 - \left(1 - \frac{2a-1}{\rho + a - 1}\right)^n\right).$$

To demonstrate that on RH all these sums are nonnegative, just replace $\vartheta_i = \arctan \frac{T}{T^2 - 1/4}$ given above by $\vartheta_i = \arctan \frac{T(2a-1)}{T^2 - a^2 + a - 1/4}$ and repeat all the abovesaid. To demonstrate the inverse implication, let us briefly reproduce a slightly modified argument of Bomberi and Lagarias [3]; see their original paper for some more details.

Let $a < 1/2$. We observe that for any Riemann zero $\rho = \sigma + iT$, $\left|\frac{\rho - a}{\rho + a - 1}\right|^2 = 1 + \frac{(1-2a)(2\sigma - 1)}{|\rho + a - 1|^2}$, and thus if $\sigma > 1/2$ we may find at least one zero for which

$\left|\dfrac{\rho-a}{\rho+a-1}\right|>1$. Because $\dfrac{(1-2a)(2\sigma-1)}{|\rho+a-1|^2}$ tends to zero when $|\rho_k|$ tends to infinity, maximum of this expression over ρ is achieved and there are only finitely many, say K, zeroes ρ_k for which $\left|\dfrac{\rho-a}{\rho+a-1}\right|=1+t=\max$, for all others $\left|\dfrac{\rho-a}{\rho+a-1}\right|\le 1+t-\delta$ for some fixed positive δ. Clearly, taking n large enough, the term $1-\left(\dfrac{\rho_k-a}{\rho_k+a-1}\right)^n=1-(1+t)^n\exp(in\vartheta_k)$ $\left(\vartheta_k\right.$ is an argument of $\left.\left(\dfrac{\rho_k-a}{\rho_k+a-1}\right)\right)$ can be made very large by module and negative. Then, due to the Dirichlet's theorem on simultaneous Diophantine approximation, the sum of $1-\left(\dfrac{\rho_k-a}{\rho_k+a-1}\right)^n$ over all ρ_k can be made arbitrary close to $K(1-(1+t)^n)$ while the sum over all other zeroes is of the order of $O(n^2(1+t-\delta)^n)$, just due to their known density. The case $a>1/2$ is quite similar, so we have proven the following theorem.

Theorem 1. *Riemann hypothesis is equivalent to the nonnegativity of sums*

$$k_{n,a}=\sum_\rho\left(1-\left(\frac{\rho-a}{\rho+a-1}\right)^n\right)=\sum_\rho\left(1-\left(1-\frac{2a-1}{\rho+a-1}\right)^n\right)$$

taken over the Riemann xi-function zeroes for any real a, *except* $a=1/2$. *Here* n *is nonnegative integer, zeroes are counting taking into account their multiplicity, for* $n=1$ *contributions of complex conjugate zeroes should be paired when summing.*

Indeed, we proved this statement not only for the Riemann ξ-function zeroes but for certain multisets of complex numbers, see [3]. For completeness, here we formulate this result as a following theorem.

Theorem 2 (Generalized Bombieri – Lagarias' theorem). *Let* a *and* σ *are arbitrary real numbers,* $a<\sigma$, *and* R *be a multiset of complex numbers* ρ *such that*

(i) $2\sigma-a\notin R$;

(ii) $\sum_\rho\left(1+|\operatorname{Re}\rho|\right)/(1+|\rho+a-2\sigma|^2)<+\infty$.

Then the following conditions are equivalent:

(a) $\operatorname{Re}\rho\le\sigma$ *for every* ρ;

(b) $\sum_\rho\operatorname{Re}\left(1-\left(\dfrac{\rho-a}{\rho-2\sigma+a}\right)^n\right)\ge 0$ *for* $n=1,2,3,\ldots$;

(c) *for every fixed* $\varepsilon>0$ *there is a positive constant* $c(\varepsilon)$ *such that* $\sum_\rho\operatorname{Re}\left(1-\left(\dfrac{\rho-a}{\rho-2\sigma+a}\right)^n\right)\ge -c(\varepsilon)e^{\varepsilon n}$, $n=1,2,3,\ldots$.

If at the same conditions $a > \sigma$ i*s taken, the point* (a) *is to be changed to*

(a′) $\operatorname{Re}\rho \geq \sigma$ *for every* ρ,

points (b), (c) *remain unchanged*.

As you see, the statement of Theorem 2 is formulated for any σ, not only for $\sigma = 1/2$, provided $\sigma \neq a$. To demonstrate this, just note that for $\rho = \sigma + iT$, $\left|\dfrac{\rho - a}{\rho + a - 2\sigma}\right| = \left|\dfrac{\sigma - a + iT}{a - \sigma + iT}\right| = 1$ and for $\rho = q + iT$, $\left|\dfrac{\rho - a}{\rho + a - 2\sigma}\right|^2 = 1 + \dfrac{4(\sigma - a)(q - \sigma)}{|\rho + a - 2\sigma|^2}$, and then repeat all the abovesaid.

If, additionally to the aforementioned conditions of the generalized Bombieri – Lagarais' theorem, also the following takes place:

(iii) If $\rho \in R$, than $\bar{\rho} \in R$ with the same multiplicity as ρ, one can omit the operation of taking the real part in (b), (c), the expressions at question are real. (Here, as usual, $\bar{\rho}$ means a complex conjugate of ρ.)

Following again the paper of Bombieri and Lagarais [3], we conclude this section with the following theorem.

Theorem 3 (Generalized Li's criterion). *Let a is an arbitrary real number,* $a \neq \sigma$, *and* R *be a multiset of complex numbers* ρ *such that*

(i) $2\sigma - a \notin R$, $a \notin R$;

(ii) $\sum_\rho \left(1 + |\operatorname{Re}\rho|\right)/\left(1 + |\rho + a - 2\sigma|^2\right) < +\infty$, $\sum_\rho \left(1 + |\operatorname{Re}\rho|\right)/\left(1 + |\rho - a|^2\right) < +\infty$;

(iii) *if* $\rho \in R$, *than* $2\sigma - \rho \in R$.

Then the following conditions are equivalent:

(a) $\operatorname{Re}\rho = \sigma$ *for every* ρ;

(b) $\sum_\rho \operatorname{Re}\left(1 - \left(\dfrac{\rho - a}{\rho + a - 2\sigma}\right)^n\right) \geq 0$ *for any* a *and* $n = 1, 2, 3, \ldots$;

(c) *for every fixed* $\varepsilon > 0$ *and any* a *there is a positive constant* $c(\varepsilon, a)$ *such that* $\sum_\rho \operatorname{Re}\left(1 - \left(\dfrac{\rho - a}{\rho + a - 2\sigma}\right)^n\right) \geq -c(\varepsilon, a)e^{\varepsilon n}$, *for* $n = 1, 2, 3, \ldots$.

For clearly, in the conditions of the theorem we for all ρ have $\operatorname{Re}\rho \leq \sigma$ and $\operatorname{Re}(2\sigma - \rho) \leq \sigma$, whence $\operatorname{Re}\rho = \sigma$. If, additionally to the aforementioned conditions of the generalized Li's criterion, also the following takes place:

(iv) If $\rho \in R$, than complex conjugate $\bar{\rho} \in R$ with the same multiplicity as ρ,

one can omit the operation of taking the real part in (b), (c), the expressions at question are real.

***Remark* 1.** Similarly to the Li's criterion, generalized Li's criterion can be applied also to numerous other zeta-functions, as this was shown first by Li himself for Dedekind zeta-function [2], and afterwards was the subject of a number of sequel papers by other authors. We will not pursue this line of researches here.

3. Connection between generalized Li's sums and certain derivatives of Riemann xi-function. Our next aim is to establish relation "of the Li's type" similar to eq. (1), viz. the relation between sums $k_{n,a}$ and certain derivatives of the Riemann xi-function. For this we will use the generalized Littlewood theorem concerning contour integrals of logarithm of an analytical function, recently used by us to establish numerous equalities equivalent to the Riemann hypothesis [4], which for completeness we reproduce below. The proof [4] is a straightforward modification of familiar and well known corresponding Littlewood theorem (or lemma) proof (see, e.g., [5]). Actually, this theorem has been more or less explicitly used in Riemann researches already by Wang who in 1946 established the first integral equality equivalent to the Riemann hypothesis [6].

Theorem 4 (Generalized Littlewood theorem). *Let C denotes the rectangle bounded by the lines $x = X_1$, $x = X_2$, $y = Y_1$, $y = Y_2$, where $X_1 < X_2$, $Y_1 < Y_2$ and let $f(z)$ be analytic and nonzero on C and meromorphic inside it, let also $g(z)$ is analytic on C and meromorphic inside it. Let $F(z) = \ln(f(z))$, the logarithm being defined as follows: we start with a particular determination on $x = X_2$, and obtain the value at other points by continuous variation along $y = \text{const}$ from $\ln(X_2 + iy)$. If, however, this path would cross a zero or pole of $f(z)$, we take $F(z)$ to be $F(z \pm i0)$ according as we approach the path from above or below. Let also the poles and zeroes of the functions $f(z)$, $g(z)$ do not coincide.*

Then

$$\int_C F(z)g(z)\,dz = 2\pi i\left(\sum_{\rho_g}\operatorname{res}\left(g(\rho_g)\cdot F(\rho_g)\right) - \sum_{\rho_f^0}\int_{X_1+iY_\rho^0}^{X_\rho^0+iY_\rho^0} g(z)\,dz + \sum_{\rho_f^{\text{pol}}}\int_{X_1+iY_\rho^{\text{pol}}}^{X_\rho^{\text{pol}}+iY_\rho^{\text{pol}}} g(z)\,dz\right),$$

where the sum is over all ρ_g which are poles of the function $g(z)$ lying inside C, all $\rho_f^0 = X_\rho^0 + iY_\rho^0$ which are zeroes of the function $f(z)$ counted taking into account their multiplicities (that is the corresponding term is multiplied by m for a zero of the order m) and which lie inside C, and all $\rho_f^{\text{pol}} = X_\rho^{\text{pol}} + iY_\rho^{\text{pol}}$ which are poles of the function $f(z)$ counted taking into account their multiplicities and which lie inside C. For this is true all relevant integrals in the right-hand side of the equality should exist.

***Remark* 2.** Actually, the case of the coincidence of poles and zeroes of the functions $f(z)$, $g(z)$ often does not pose real problems and can be easily considered. We have dealt with a few such cases before [4].

The subtle moment related with this generalized Littlewood theorem is the circumstance that the function $\arg(F(z))$ (imaginary part of the $\ln(f(z))$ is not continuous on the left border of the contour (segment $X_1 + iY_1$, $X_1 + iY_2$) if there are zeroes or poles of the function $f(z)$ inside the contour. This is explicitly stated in the theorem condition: *If, however, this path would cross a zero or pole of $f(z)$, we take $F(z)$ to be $F(z \pm i0)$ according as we approach the path from above or below*. In practice, this means that when calculating the corresponding part of the contour integral, viz. the integral $-\int_{X_1+iY_1}^{X_1+iY_2} \arg(F(z))g(z)\,dz$ (minus sign comes from the necessity to round the contour counterclockwise), $\pm 2\pi il$ jumps should be added to an argument function at a point

$X_1 + iY_{z,p}$ whenever a zero or a pole of an order l of the function $f(z)$ occurs somewhere at a point $X + iY_{z,p}$ inside the contour. Corresponding integral should be properly modified if the use of a continuous argument branch is desirable. See our paper [7] for details, we also would like to note that the appropriateness of the necessary modification of an argument has been numerically tested (and confirmed) by us for a number of integrals, e.g., for the integral

$$\int_0^\infty \frac{t\arg(\varsigma(1/4+it))}{(1/16+t^2)^2}dt = \pi\frac{\varsigma'(1/2)}{\varsigma(1/2)} - 9\pi - \pi\sum_{\rho,\sigma_k>1/4,t_k>0}\left(\frac{1}{t_k^2+1/4}\right)$$

(similar equality in the form

$$\int_0^\infty \frac{t\arg(\varsigma(1/2+it))}{(1/16+t^2)^2}dt = \pi\frac{\varsigma'(3/4)}{\varsigma(3/4)} - \frac{32\pi}{3}$$

is equivalent to the Riemann hypothesis, our Theorem 5 from [4]). However, for what follows the asymptotic of the function $g(z)$ for large values of X_1 tending to minus infinity makes this modification irrelevant, the value of the integral $-\int_{X_1+iY_1}^{X_1+iY_2}\arg(F(z))g(z)\,dz$ tends to zero anyway.

First, as an exercise, we use this theorem to establish the Li's relation (1). For this, let us consider the rectangular contour C with vertices at $\pm X \pm iX$ with real $X \to +\infty$, if some Riemann zero occurs on the contour just shift it a bit to avoid this, and consider a contour integral $\int_C g(z)\ln(\xi(z))\,dz$ where

$$g(z) = \frac{n}{(z-1)^2}\left(\frac{z}{z-1}\right)^{n-1} - \frac{n}{(z-1)^2}. \tag{3}$$

Known asymptotic of the logarithm of the xi-function for large $|z|$, $\cong O(z\ln z)$ guaranties the "disappearance" of the contour integral value (it tends to zero when $X \to \infty$ due to the asymptotic $g(z) \cong O(1/z^3)$) thus we get, after division by $2\pi i$,

$$n\frac{1}{(n-1)!}\frac{d^n}{dz^n}\left(z^{n-1}\ln(\xi(z))\right)\Big|_{z=1} - n\frac{\xi'}{\xi}(1) - \sum_\rho\left(1-\left(1-\frac{1}{1-\rho}\right)^n\right) - n\sum_\rho\frac{1}{\rho} = 0. \tag{4}$$

(Complex conjugate zeroes are to be paired when calculating $\sum_\rho \frac{1}{\rho}$ and $\sum_\rho\left(1-\left(1-\frac{1}{1-\rho}\right)^n\right)$ for $n=1$.) Here the first term is the contribution of the $n+1$ order pole of $g(z)$ at $z = 1$, second term is the contribution of the second order pole arising from the term $-\frac{n}{(z-1)^2}$ occurring in (3), the third and fourth terms are the integrals $-\int_{-\infty+iT_i}^{\rho_i} g(z)\,dz$. (Clearly,

$$\frac{n}{(z-1)^2}\left(\frac{z}{z-1}\right)^{n-1} = \frac{d}{dz}\left(1-\left(1-\frac{1}{1-z}\right)^n\right)$$

which explains while function $g(z)$ in form (3) is used; the term $-\frac{n}{(z-1)^2}$ is added just to ensure the asymptotic $g(z) \cong O(1/z^3)$ necessary to bring the contour integral value to zero. Note also evident

$$\sum_\rho\left(1-\left(1-\frac{1}{1-\rho}\right)^n\right) = \sum_\rho\left(1-\left(1-\frac{1}{\rho}\right)^n\right).\Bigg)$$

We know that $\frac{\xi'}{\xi}(1) = -\sum_\rho \frac{1}{\rho}$ [1], so that we have

$$n\frac{1}{(n-1)!}\frac{d^n}{dz^n}\left(z^{n-1}\ln(\xi(z))\right)\Big|_{z=1} = \sum_\rho\left(1-\left(1-\frac{1}{\rho}\right)^n\right)$$

which is a relation we have searched for.

Quite similar consideration is applied to our case, where now we introduce the function

$$\tilde{g}(z) = -\frac{n(2a-1)(z-a)^{n-1}}{(z+a-1)^{n+1}} + \frac{n(2a-1)}{(z+a-1)^2} \tag{5}$$

and consider contour integral $\int_C \tilde{g}(z)\ln(\xi(z))\,dz$ taken round the same contour as above. Application of Theorem 4 (generalized Littlewood theorem) gives

$$-\frac{n(2a-1)}{(n-1)!}\frac{d^n}{dz^n}\left((z-a)^{n-1}\ln(\xi(z))\right)\Big|_{z=1-a} + n(2a-1)\frac{\xi'}{\xi}(z)\Big|_{z=1-a} -$$

$$-\sum_\rho\left(1-\left(\frac{\rho-a}{\rho+a-1}\right)^n\right) + n(2a-1)\sum_\rho\frac{1}{\rho+a-1} = 0. \tag{6}$$

(Again, complex conjugate zeroes are to be paired whenever necessary.) Using well known $\frac{\xi'}{\xi}(z)\Big|_{z=1-a} = -\sum_\rho\frac{1}{\rho+a-1}$ [1] and reminding our Theorem 1 we have the following theorem.

Theorem 5. *Riemann hypothesis is equivalent to the nonnegativity of all derivatives* $\frac{1}{(n-1)!}\frac{d^n}{dz^n}((z-a)^{n-1}\ln(\xi(z)))\Big|_{z=1-a}$ *for all nonnegative integers* n *and any real* $a<1/2$; *correspondingly, it is equivalent also to the nonpositivity of all derivatives* $\frac{1}{(n-1)!}\frac{d^n}{dz^n}((z-a)^{n-1}\ln(\xi(z)))\Big|_{z=1-a}$ *for all nonnegative integers* n *and any real* $a>1/2$.

***Remark* 3.** Another possibility to arrive to the same conclusions is to see the formula $\left|\frac{\rho-a}{\rho+a-1}\right|=\left|\frac{-a+1/2+iT}{a-1/2+iT}\right|=1$ as a precursor for the conformal mapping $s=\frac{z-a}{z+a-1}$. For $a<1/2$ and $\operatorname{Re} z\le 1/2$, module of s is always less or equal to 1; this equality is realized only on the line $z=1/2+it$. Correspondingly, on RH the function $\ln\xi\left(\frac{z-a}{z+a-1}\right)$ is analytic in the interior of the discus $|s|<1$. We will not pursue this line of researches here, see again [2, 3] and our paper [8] where similar idea was used to generalize Balazard – Saias – Yor's criterion equivalent to the Riemann hypothesis [9]. Similarly, for more general case $s=\frac{z-a}{z+a-2\sigma}$, if $a<\sigma$ and $\operatorname{Re} z\le\sigma$, module of s is always less or equal to 1, and if $a>\sigma$ and $\operatorname{Re} z\ge\sigma$, this module also is always less or equal to 1. This illustrates again our Theorem 2 (the generalized Bombieri – Lagarias' theorem).

***Remark* 4.** Along similar lines, analogous formulae connecting generalized Li's sums and certain derivatives of the logarithm, can be established for numerous other zeta-functions. We will not pursue this line of researches here.

Now we want to prove the following minor theorem.

Theorem 6. *The statement that there are no nontrivial Riemann function zeroes with* $\operatorname{Re}\rho>\sigma>1/2$ *is equivalent to the statement that for any* $a<\sigma$ *all derivatives* $\frac{1}{(n-1)!}\frac{d^n}{dz^n}((z-a)^{n-1}\ln(\xi(z)))\Big|_{z=2\sigma-a}$ *are nonnegative and for any* $a>1-\sigma$ *all derivatives* $\frac{1}{(n-1)!}\frac{d^n}{dz^n}((z-a)^{n-1}\ln(\xi(z)))\Big|_{z=2-2\sigma-a}$ *are nonpositive.*

Proof. From our Theorem 2 (generalized Bombieri – Lagarias' theorem), we know that the condition that there are no nontrivial Riemann function zeroes with $\operatorname{Re}\rho=\sigma>1/2$ is equivalent to the statement that for any $a<\sigma$ all $\sum_\rho\left(1-\left(\frac{\rho-a}{\rho-2\sigma+a}\right)^n\right)\ge 0$ for $n=1,2,3,\ldots$. These sums are calculated using our Theorem 4 (the generalized Littlewood theorem) exactly as above, with the only difference that now the function $\tilde{\tilde{g}}(z)=\frac{n(2\sigma-2a)(z-a)^{n-1}}{(z+a-2\sigma)^{n+1}}-\frac{n(2\sigma-2a)}{(z+a-2\sigma)^2}$ instead of the function $\tilde{g}(z)$ given by (5) is exploited. Such change brings the equality between the sums

$\sum_{\rho}\left(1-\left(\frac{\rho-a}{\rho-2\sigma+a}\right)^n\right)\geq 0$ and derivatives $\frac{n(2\sigma-2a)}{(n-1)!}\frac{d^n}{dz^n}\left((z-a)^{n-1}\ln(\xi(z))\right)\Big|_{z=2\sigma-a}$ which, consequently, also should be nonnegative.

If there are no zeroes with $\operatorname{Re}\rho>\sigma>1/2$, there are also no zeroes with $\operatorname{Re}\rho<1-\sigma$ and we are able to apply Theorem 2 with $a>1-\sigma$; all corresponding sums

$$\sum_{\rho}\left(1-\left(\frac{\rho-a}{\rho-2+2\sigma+a}\right)^n\right)\geq 0$$

and are given by $\frac{n(2(1-\sigma)-2a)}{(n-1)!}\frac{d^n}{dz^n}\left((z-a)^{n-1}\ln(\xi(z))\right)\Big|_{z=2-2\sigma-a}$ whence the derivatives $\frac{1}{(n-1)!}\frac{d^n}{dz^n}\left((z-a)^{n-1}\ln(\xi(z))\right)\Big|_{z=2-2\sigma-a}$ should be nonpositive.

Theorem 6 is proved.

3. An arithmetic interpretation of the generalized Li's criterion. In Ref. [3], Bombieri and Lagarias demonstrated the relation of Li's criterion with the so called Weil's explicit formula in the theory of prime numbers and Weil's criterion of the truth of the Riemann hypothesis (see [10, 11]) and gave an arithmetic interpretation of the Li's criterion. Lately, such an interpretation has been given for some other Zeta-functions (see, e.g., [12]). For completeness, we would like to conclude this paper establishing an arithmetic interpretation of the generalized Li's criterion. We closely follow the lines used in [3] here.

For suitable function f, Mellin transform is defined as $\hat{f}(s)=\int_0^\infty f(x)x^{s-1}\,dx$ while inverse Mellin transform formula is $f(x)=\frac{1}{2\pi i}\int_{\operatorname{Re}s=c}\hat{f}(s)x^{-s}\,ds$ with an appropriate value of c. The following is more or less a repetition of Lemma 2 from [3], which is a particular case corresponding to $a=1$.

Lemma 1. *For* $n=1,2,3,\ldots$, *and an arbitrary complex number* a *the inverse Mellin transform of the function* $k_{n,a}(s)=1-\left(1-\frac{2a-1}{s+a-1}\right)^n$ *is*

$$g_{n,a}(x)=P_{n,a}(x) \quad \text{if} \quad 0<x<1,$$

$$g_{n,a}(x)=\frac{n}{2}(2a-1) \quad \text{if} \quad x=1, \tag{7}$$

$$g_{n,a}(x)=0 \quad \text{if} \quad x>1,$$

where $P_{n,a}(x)=x^{a-1}\sum_{j=1}^{n}C_n^j\frac{(2a-1)^j\ln^{j-1}x}{(j-1)!}$; $C_n^j=\frac{n!}{j!(n-j)!}$ *is a binomial coefficient.*

Proof. We have for $\operatorname{Re}(s+a)>1$:

$$\sum_{j=1}^{n} C_n^j \frac{(2a-1)^j}{(j-1)!} \int_0^1 (\ln^{j-1} x) x^{s+a-2}\, dx = \sum_{j=1}^{n} C_n^j \frac{(2a-1)^j}{(j-1)!} \frac{d^{j-1}}{ds^{j-1}} \int_0^1 x^{s+a-2}\, dx =$$

$$= \sum_{j=1}^{n} C_n^j \frac{(2a-1)^j (-1)^{j-1}}{(s+a-1)^j} = 1 - \left(1 - \frac{2a-1}{s+a-1}\right)^n.$$

If a is an arbitrary complex number with $\operatorname{Re} a > 1$, for the function $g_n(x)$ we can apply the so called Explicit Formula of Weil (see [3, 10, 11]), which is, as given in [3]:

$$\sum_{\rho} \hat{f}(\rho) = \int_0^{\infty} f(x)\,dx + \int_0^{\infty} \tilde{f}(x)\,dx - \sum_{n=1}^{\infty} \Lambda(n)(f(n)+\tilde{f}(n)) -$$

$$- (\ln \pi + \gamma) f(1) - \int_1^{\infty} \left\{ f(x) + \tilde{f}(x) - \frac{2}{x^2} f(1) \right\} \frac{x dx}{x^2-1}. \tag{8}$$

Here $\Lambda(n)$ is a van Mangoldt function $\left(\text{let us remind that for } \operatorname{Re} s > 1 \ \ \frac{\varsigma'(s)}{\varsigma(s)} = -\sum_{m=1}^{\infty} \frac{\Lambda(m)}{m^s}\right.$ $\left.[1]\right)$, $\gamma = 0.572\ldots$ is Euler – Mascheroni constant and $\tilde{f}(x) := \frac{1}{x} f\left(\frac{1}{x}\right)$, thus in our case the function $\tilde{P}_{n,a}(x) = x^{-a} \sum_{j=1}^{n} C_n^j \frac{(-1)^{j-1}(2a-1)^j \ln^{j-1} x}{(j-1)!}$ should be used whenever appropriate. Surely, $\tilde{P}_{n,a}(x)$ is inverse Mellin transform of $k_{n,a}(1-s) = 1 - \left(1 - \frac{2a-1}{a-s}\right)^n$.

Such an application is justified because this is easy to check that for $\operatorname{Re} a > 1$ the functions $g_{n,a}(x)$ do possess the following necessary properties for eq. (8) can be used for a function $f(x)$ [3, 10, 11]:

(A) $f(x)$ is continuous and continuously differentiable everywhere except at finitely many points a_i, in which both $f(x)$ and $f'(x)$ have at most a discontinuity of the first kind, and in which one sets $f(a_i) = \frac{1}{2}[f(a_i+0) + f(a_i-0)]$;

(B) there is $\delta > 0$ such that $f(x) = O(x^{\delta})$ as $x \to 0+$ and $f(x) = O(x^{-1-\delta})$ as $x \to +\infty$.

The use of eq. (8) gives

$$\sum_{\rho} \left(1 - \left(\frac{\rho - a}{\rho + a - 1}\right)^n\right) = \sum_{\rho} \left(1 - \left(\frac{\rho + a - 1}{\rho - a}\right)^n\right) =$$

$$= \sum_{j=1}^{n} C_n^j \frac{(2a-1)^j}{(j-1)!} \left\{ \int_0^1 x^{a-1} \ln^{j-1} x dx + (-1)^{j-1} \int_1^{\infty} x^{-a} \ln^{j-1} x \, dx - (-1)^{j-1} \sum_{m=1}^{\infty} \frac{\Lambda(m) \ln^{j-1} m}{m^a} \right\} -$$

$$- \frac{n}{2}(2a-1)(\ln \pi + \gamma) - \int_1^{\infty} \left\{ \sum_{j=1}^{n} C_n^j \frac{(-1)^{j-1}(2a-1)^{j-1}}{(j-1)!} x^{-a} \ln^{j-1} x - \frac{n}{x^2}(2a-1) \right\} \frac{xdx}{x^2-1}. \qquad (9)$$

Now, in the second and third integrals in the right-hand side of (9) we make a variable transform x to $1/x$, after what these integrals take the forms $I_2 = \int_0^1 x^{a-2} \ln^{j-1}(x) dx$ and

$$I_3 = \int_0^1 \left\{ \sum_{j=2}^{n} C_n^j \frac{\ln^{j-1} x}{(j-1)!} (2a-1)^j x^{a-1} + n(2a-1)(x^{a-1} - x) \right\} \frac{dx}{1-x^2}.$$

(Note, that when writing I_3 we move the summation term corresponding to $j = 1$ from the sum to the second term under the integral sign.) The first two integrals are handled by virtue of an Example 4.272.6 of GR book [13]:

$$\int_0^1 \ln^{\mu-1}(1/x) x^{\nu-1} dx = \frac{1}{\nu^{\mu}} \Gamma(\mu); \qquad \operatorname{Re} \mu > 0 \quad \text{and} \quad \operatorname{Re} \nu > 0.$$

Adopting for our case, we get

$$\int_0^1 \ln^{j-1}(x) x^{a-1} dx = \frac{(-1)^{j-1}}{a^j}(j-1)!, \qquad \int_0^1 \ln^{j-1}(x) x^{a-2} dx = \frac{(-1)^{j-1}}{(a-1)^j}(j-1)!.$$

The "second part" of the third integral I_3 is, by virtue of an Example 3.244.3 of GR book [13], equal to

$$I_{32} = n(2a-1) \int_0^1 \frac{x^{a-1} - x}{1-x^2} dx = -\frac{n}{2}(2a-1)(\gamma + \psi(a/2));$$

ψ is a digamma function. In the first part of this integral we make the variable change $x = \exp(-t)$:

$$I_{31} = \int_0^1 \sum_{j=2}^{n} C_n^j \frac{\ln^{j-1} x}{(j-1)!} (2a-1)^j x^{a-1} \frac{dx}{1-x^2} = \sum_{j=2}^{n} C_n^j (-1)^{j-1} \frac{(2a-1)^j}{(j-1)!} \int_0^{\infty} t^{j-1} \frac{e^{-at}}{1-e^{-2t}} dt.$$

Applying Taylor expansion $(1-e^{-2t})^{-1} = 1 + e^{-2t} + e^{-4t} + e^{-6t} + \ldots$ we get further

$$I_{31} = \sum_{j=2}^{n} C_n^j (-1)^{j-1} \frac{(2a-1)^j}{(j-1)!} \sum_{m=0}^{\infty} \frac{(j-1)!}{(2m+a)^j} = \sum_{j=2}^{n} C_n^j (-1)^{j-1} 2^{-j} (2a-1)^j \varsigma(j, a/2),$$

where $\varsigma(s,a) := \sum_{m=0}^{\infty} \frac{1}{(m+a)^s}$ is Hurwitz zeta-function.

Using the relations

$$\sum_{j=1}^{n} C_n^j (-1)^{j-1}(2a-1)^j a^{-j} = 1 - \sum_{j=0}^{n} C_n^j (-1)^j \left(\frac{2a-1}{a} \right)^j = 1 - \left(-1 + \frac{1}{a} \right)^n ,$$

$$\sum_{j=1}^{n} C_n^j (-1)^{j-1}(2a-1)^j (a-1)^{-j} = -1 - \left(-1 - \frac{1}{a-1} \right)^n ,$$

and collecting everything together we have proven the following theorem.

Theorem 7. *For* $n = 1, 2, 3, \ldots$ *and an arbitrary complex a with* $\operatorname{Re} a > 1$ *we have*

$$\sum_{\rho} \left(1 - \left(\frac{\rho - a}{\rho + a - 1} \right)^n \right) = \sum_{\rho} \left(1 - \left(\frac{\rho + a - 1}{\rho - a} \right)^n \right) = 2 - \left(-1 + \frac{1}{a} \right)^n - \left(-1 - \frac{1}{a-1} \right)^n +$$

$$+ \sum_{j=1}^{n} C_n^j (2a-1)^j \frac{(-1)^j}{(j-1)!} \sum_{m=1}^{\infty} \frac{\Lambda(m) \ln^{j-1} m}{m^a} +$$

$$+ \frac{n}{2}(2a-1)(\psi(a/2) - \ln \pi) + \sum_{j=2}^{n} C_n^j (-1)^j 2^{-j} (2a-1)^j \varsigma(j, a/2) . \tag{10}$$

***Remark* 5.** The case $n = 1$ of the Theorem 1 gives well known equality

$$\sum_{\rho} \frac{1}{a - \rho} = \frac{1}{a} + \frac{1}{a-1} - \sum_{m=1}^{\infty} \frac{\Lambda(m)}{m^a} + \frac{1}{2}(\psi(a/2) - \ln \pi)$$

(see, e.g., [1]).

The same connection of Li's criterion with Weil's criterion of the truth of the Riemann hypothesis that has been discussed in [3], takes place also for the generalized Li's criterion. This can be seen as follows.

The multiplicative convolution of functions $f(x)$, $g(x)$ satisfying conditions (A), (B) given above, is defined as $(f * g)(x) = \int_0^{\infty} f(x/y) g(y) \frac{dy}{y}$, and Mellin transform of such a convolution is $\hat{f}(s) \cdot \hat{g}(s)$. For the multiplicative convolution $f * \tilde{\bar{f}}$ $\Big($signs of complex conjugation and definition $\tilde{f}(x) := \frac{1}{x} f\left(\frac{1}{x} \right)$ are used$\Big)$, Mellin transform is given by $\hat{f}(s) \cdot \hat{\bar{f}}(1-s)$ — an expression which clearly is real and positive for $\operatorname{Re} s = 1/2$. Correspondingly, for any function expressible as $f * \tilde{\bar{f}}$, on RH the sum over the nontrivial Riemann zeroes should be positive. Weil showed that this is also a sufficient condition for RH to be true.

Now let us remind that if $h(s) = \dfrac{s-a}{s+a-1}$ then $h(1-s) = 1/h(s)$ and thus

$$k_{n,a}(s) \cdot k_{n,a}(1-s) = k_{n,a}(s) + k_{n,a}(1-s), \tag{11}$$

where $k_{n,a}(s) = 1 - \left(\dfrac{s-a}{s+a-1} \right)^n$. By construction, $k_{n,a}(s) = \hat{g}_{n,a}(s)$ and, due to general properties of the Mellin transform, $\hat{\tilde{g}}_{n,a}(s) = \hat{g}_{n,a}(1-s)$ thus (11) can be rewritten as $\hat{g}_{n,a}(s) \cdot \hat{\tilde{g}}_{n,a}(s) = \hat{g}_{n,a}(s) + \hat{\tilde{g}}_{n,a}(s)$. Whence, applying inverse Mellin transform, $g_{n,a}(x) + \tilde{g}_{n,a}(x) = (g_{n,a} * \tilde{g}_{n,a})(x)$, and this establishes the aforementioned connection: right-hand side of eq. (8) is invariant with respect to the change of $f(x)$ into $\tilde{f}(x)$.

4. Conclusion. Thus we see that to judge the truth of the Riemann hypothesis, evaluation of certain derivatives of the Riemann xi-function can be effected at any point of the real axis apart from the point $z = 1/2$. In particular, such a point can lie arbitrary far to the right from the critical strip: however large the number $b > -1/2$ is, all derivatives $\dfrac{1}{(n-1)!} \dfrac{d^n}{dz^n} \left((z+b)^{n-1} \ln(\xi(z))\right)\Big|_{z=b+1}$ should be nonnegative for RH is true, and vice versa. The present author sincerely hopes that this, and other related interesting possibilities might be useful for Riemann researches. Finally, we are also sure that there is a room to use the approach presented in the paper to study other than Riemann function analytic functions.

1. *Titchmarsh E. C., Heath-Brown E. R.* The theory of the Riemann Zeta-function. – Oxford: Clarendon Press, 1988.
2. *Li X.-E.* The positivity of a sequence of numbers and the Riemann hypothesis // J. Number Theory. – 1997. – **65**. – P. 325 – 333.
3. *Bombieri E., Lagarias J. C.* Complements to Li's criterion for the Riemann hypothesis // J. Number Theory. – 1999. – **77**. – P. 274 – 287.
4. *Sekatskii S. K., Beltraminelli S., Merlini D.* On equalities involving integrals of the logarithm of the Riemann ς-function and equivalent to the Riemann hypothesis // Ukr. Math. J. – 2012. – **64**, № 2. – P. 218 – 228.
5. *Titchmarsh E. C.* The theory of functions. – Oxford: Oxford Univ. Press, 1939.
6. *Wang F. T.* A note on the Riemann Zeta-function // Bull. Amer. Math. Soc. – 1946. – **52**. – P. 319 – 321.
7. *Sekatskii S. K., Beltraminelli S., Merlini D.* A few equalities involving integrals of the logarithm of the Riemann ς-function and equivalent to the Riemann hypothesis III. Exponential weight functions // arXiv: 1006.0323v2.
8. *Sekatskii S. K., Beltraminelli S., Merlini D.* A few equalities involving integrals of the logarithm of the Riemann ς-function and equivalent to the Riemann hypothesis I // arXiv: 0806.1596v1.
9. *Balazard M., Saias E., Yor M.* Notes sur la fonction de Riemann // Adv. Math. – 1999. – **143**. – P. 284 – 287.
10. *Weil A.* Sur les "formules explicites" de la théorie des nombres premiers // Meddelanden Fran Lunds Univ. Math. Sem. (dedié a M. Riesz). – 1952. – P. 252 – 265 (see also *Weil A.* OEvres scientifique. – Collected papers. – New York: Springer, 1980. – Vol. 2. – P. 48 – 61).
11. *Bombieri E.* Remarks on Weil's quadratic functional in the theory of prime numbers I // Rend. Mat. Accad. Lincei. – 2000. – **9**. – P. 183-233.
12. *Smajlovic L.* On Li's criterion for the Riemann hypothesis for the Selberg class // J. Number Theory. – 2010. – **130**. – P. 828 – 851.
13. *Gradshtein I. S., Ryzhik I. M.* Tables of integrals, series and products. – New York: Academic, 1990.

Received 22.07.13