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FIXED POINT THEOREMS FOR MULTIVALUED GENERALIZED NONLINEAR CONTRACTIVE MAPS IN PARTIAL METRIC SPACES

ТЕОРЕМИ ПРО НЕРУХОМУ ТОЧКУ ДЛЯ БАГАТОЗНАЧНИХ УЗАГАЛЬНЕНИХ НЕЛІНІЙНИХ СТИСКАЮЧИХ ВІДОБРАЖЕНЬ В ЧАСТКОВО МЕТРИЧНИХ ПРОСТОРАХ

We prove some fixed point results for multivalued generalized nonlinear contractive mappings in partial metric spaces which generalize and improve the corresponding recent fixed point results due to Ćirić [*Ćirić L. B.* Multivalued nonlinear contraction mappings // Nonlinear Anal. – 2009. – **71**. – P. 2716 – 2723], and Klim and Wardowski [*Klim D., Wardowski D.* Fixed point theorems for set-valued contractions in complete metric spaces // J. Math. Anal. and Appl. – 2007. – **334**. – P. 132 – 139].

Доведено деякі теореми про нерухому точку в частково метричних просторах, що узагальнюють та покращують відповідні нові результати про нерухому точку, отримані Чірічем (*Ćirić L. B.* Multivalued nonlinear contraction mappings // Nonlinear Anal. – 2009. – **71**. – P. 2716 – 2723) та Клімом і Вардовським (*Klim D., Wardowski D.* Fixed point theorems for set-valued contractions in complete metric spaces // J. Math. Anal. and Appl. – 2007. – **334**. – P. 132 – 139).

1. Introduction and preliminaries. Let (X, d) be a metric space, 2^X the collection of nonempty subsets of X, $\mathrm{Cl}(X)$ be the subcollection of nonempty closed subsets of X. Investigations on the existence of fixed points of multivalued contractions in metric spaces were initiated by Nadler [15] and has been extended in many different directions by many authors, in particular, by Reich [19], Mizoguchi, Takahashi [14] and Feng, Liu [8]. In this context Klim and Wardowski proved the following nice result:

Theorem 1.1 [12]**.** *Let (X, d) be a complete metric space and let $T\colon X \longrightarrow \mathrm{Cl}(X)$. Assume that the following conditions hold*:

(i) *there exist a number $b \in (0, 1)$ and a function $k\colon [0,\infty) \longrightarrow [0, b)$ such that for each $t \in [0,\infty)$,*

$$\limsup_{r\longrightarrow t^+} k(r) < b,$$

(ii) *for any $x \in X$ there is $y \in T(x)$ satisfying*

$$bd(x, y) \leq d(x, T(x))$$

and

$$d(y, T(y)) \leq k(d(x, y))d(x, y).$$

Then $\mathrm{Fix}(T) \neq \varnothing$ provided the real-valued function g on X, $g(x) = d(x, T(x))$ is lower semicontinuous.

Recently, Ćirić [6] extended the above result and proved the following two interesting theorems:

Theorem 1.2. *Let* (X,d) *be a complete metric space and let* $T\colon X \longrightarrow \mathrm{Cl}(X)$. *Assume that the following conditions hold*:

(i) *the map* $f\colon X \longrightarrow R$, *defined by* $f(x) = d(x, T(x))$ *is lower semicontinuous,*

(ii) *there exists a function* $\varphi\colon [0,\infty) \longrightarrow [b,1)$, $0 < b < 1$, *satisfying*

$$\limsup_{r\longrightarrow t^+} \varphi(r) < 1 \quad for\ each \quad t \in [0,\infty),$$

(iii) *for any* $x \in X$, *there is* $y \in T(x)$ *satisfying*

$$\sqrt{\varphi(f(x))}\, d(x,y) \leq f(x)$$

such that

$$f(y) \leq \varphi(f(x))d(x,y).$$

Then, there exists $z \in X$ *such that* $z \in T(z)$.

Theorem 1.3. *Let* (X,d) *be a complete metric space and let* $T\colon X \longrightarrow \mathrm{Cl}(X)$. *Assume that the following conditions hold*:

(i) *the map* $f\colon X \longrightarrow R$, *defined by* $f(x) = d(x, T(x))$ *is lower semicontinuous,*

(ii) *there exists a function* $\varphi\colon [0,\infty) \longrightarrow [b,1)$, $0 < b < 1$, *satisfying*

$$\limsup_{r\longrightarrow t^+} \varphi(r) < 1 \quad for\ each \quad t \in [0,\infty),$$

(iii) *for any* $x \in X$, *there is* $y \in T(x)$ *satisfying*

$$\sqrt{\varphi(d(x,y)}\, d(x,y) \leq f(x)$$

such that

$$f(y) \leq \varphi(d(x,y))d(x,y).$$

Then, there exists $z \in X$ *such that* $z \in T(z)$.

Note that Theorem 1.3 is a generalization of Theorem 1.1.

The aim of this paper is to prove generalized corresponding theorems in the setting of partial metric space which includes as a special case the standard metric space. Also we present an example to show that our results improve on the above results.

Here we present some definitions and properties of partial metric spaces, for more detail see [4, 7, 9, 13, 16, 17].

A partial metric space is a pair (X,p), where X is a nonempty set and p is a partial metric on X. A partial metric is a function $p\colon X \times X \longrightarrow \mathbb{R}^+$ such that for all x, y, $z \in X$ we have:

(p_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,

(p_2) $p(x,x) \leq p(x,y)$,

(p_3) $p(x,y) = p(y,x)$,

(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

It is clear that, if $p(x,y) = 0$, then from (p_1) and (p_2), $x = y$. But if $x = y$, $p(x,y)$ may not be 0. A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$ for all x, $y \in \mathbb{R}^+$. Additional references, mainly from the computational point of view, may be found in [1–3, 5, 7, 9–11, 17, 18, 20].

Each partial metric p on X induces a T_0 topology τ_p on X which has as a base the family open p-balls $\{B_p(x,\epsilon) \colon x \in X, \epsilon > 0\}$, where $B_p(x,\epsilon) = \{y \in X \colon p(x,y) < p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$ (see [13]).

Let (X,p) be a partial metric space, then we have the following:

(i) A sequence $\{x_n\}$ in a partial metric space (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n \longrightarrow \infty} p(x, x_n)$.

(ii) A sequence $\{x_n\}$ in a partial metric space (X,p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m \longrightarrow \infty} p(x_m, x_n)$ ([13], Definition 5.2).

(iii) A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m \longrightarrow \infty} p(x_m, x_n)$ ([13], Definition 5.3).

(iv) A map $f \colon X \longrightarrow X$ is τ_p-continuous if $\lim_{n \longrightarrow \infty} x_n = x$ with respect to τ_p implies $\lim_{n \longrightarrow \infty} fx_n = fx$ with respect to τ_p.

It is easy to see that, every closed subset of a complete partial metric space is complete. The following lemma is crucial for the proofs of our results.

Lemma 1.1 [4]. *Let (X,p) be a partial metric space, $A \subset X$ and $x_0 \in X$. Define $p(x_0, A) = \inf\{p(x_0, x) \colon x \in A\}$. Then $a \in \overline{A}$ if and only if $p(a, A) = p(a,a)$.*

2. Main results. In this section, we prove some fixed point results for multivalued generalized nonlinear contractive maps in partial metric space which generalize and improve the mentioned corresponding results of Ćirić [6], Klim and Wardowski [12] and several authors.

Theorem 2.1. *Let (X,p) be a complete partial metric space. Let $T \colon X \longrightarrow \mathrm{Cl}(X)$. Assume that the following conditions hold*:

(i) *there exist a constant $b \in (0,1)$ and two functions $\alpha \colon [0,\infty) \longrightarrow [b,\infty)$ and $\beta \colon [0,\infty) \longrightarrow (0,\infty)$ such that for each $t \in [0,\infty)$, $\beta(t) \leq \alpha(t)$ and*

$$\limsup_{r \longrightarrow t^+} \frac{\beta(r)}{\alpha(r)} < 1,$$

(ii) *the map $f \colon X \longrightarrow \mathbb{R}$, defined by $f(x) = p(x, T(x))$ is lower semicontinuous,*

(iii) *for any $x \in X$, there exists $y \in T(x)$ satisfying*

$$\alpha(f(x))p(x,y) \leq f(x)$$

and

$$f(y) \leq \beta(f(x))p(x,y).$$

Then there exists $v_0 \in X$ such that $f(v_0) = 0$. Further, if $p(v_0, v_0) = 0$, then $v_0 \in T(v_0)$.

***Remark* 2.1.** Notice that a complete metric space (X, d) is a complete partial metric space (since every metric d is a partial metric).

Therefore, taking $\beta(t) = \varphi(t)$, $\alpha(t) = \sqrt{\varphi(t)}$ and $p(x, y) = d(x, y)$ in Theorem 2.1, we get Theorem 1.2.

***Proof of Theorem* 2.1.** Let $x_0 \in X$. From condition (iii) there exists $x_1 \in T(x_0)$ such that

$$\alpha(f(x_0))p(x_0, x_1) \leq f(x_0) \tag{1}$$

and

$$f(x_1) \leq \beta(f(x_0))p(x_0, x_1). \tag{2}$$

From (1) and (2), we get

$$f(x_1) \leq \frac{\beta(f(x_0))}{\alpha(f(x_0))} f(x_0). \tag{3}$$

Similarly, there exists $x_2 \in T(x_1)$ such that

$$\alpha(f(x_1))p(x_1, x_2) \leq f(x_1)$$

and

$$f(x_2) \leq \beta(f(x_1))p(x_1, x_2),$$

which imply $f(x_2) \leq \dfrac{\beta(f(x_1))}{\alpha(f(x_1))} f(x_1)$.

By induction we get an orbit $\{x_n\}$ of T in X such that for all integers $n \geq 0$

$$\alpha(f(x_n))p(x_n, x_{n+1}) \leq f(x_n) \tag{4}$$

and

$$f(x_{n+1}) \leq \frac{\beta(f(x_n))}{\alpha(f(x_n))} f(x_n). \tag{5}$$

Since by condition (i) we have

$$\frac{\beta(f(x))}{\alpha(f(x))} \leq 1 \qquad \text{for all} \qquad x \in X.$$

From (5), we get $f(x_{n+1}) \leq f(x_n)$.

Thus $\{f(x_n)\}$ is a nonincreasing sequence of real numbers which is bounded from below by 0, so is convergent to a $L \geq 0$. We show that $L = 0$. Suppose to the contrary that $L > 0$. From (5) and taking the limit when $n \longrightarrow \infty$ we obtain

$$L \leq \limsup_{f(x_n) \longrightarrow L^+} \frac{\beta(f(x_n))}{\alpha(f(x_n))} L < L,$$

which is a contradiction, so $L = 0$.

Now, from condition (i), we can choose q such that

$$\limsup_{f(x_n)\longrightarrow 0^+} \frac{\beta(f(x_n))}{\alpha(f(x_n))} < q < 1.$$

So $\dfrac{\beta(f(x_n))}{\alpha(f(x_n))} < q$ for all $n \geq n_0$, for some $n_0 \in N$.

Again using (5), we get $f(x_{n+1}) \leq q f(x_n)$ for all $n \geq n_0$. Hence, by induction, we have

$$f(x_{n+1}) \leq q^{n+1-n_0} f(x_{n_0}) \qquad \text{for all} \qquad n \geq n_0. \tag{6}$$

From (4), (6) and the fact that $\alpha(t) \geq b > 0$ for all $t \geq 0$, we obtain

$$p(x_n, x_{n+1}) \leq \frac{1}{b} q^{n-n_0} f(x_{n_0}) \qquad \text{for all} \qquad n \geq n_0. \tag{7}$$

In this step of the proof, we show that $\{x_n\}$ is a Cauchy sequence. For any $m, n \in \mathbb{N}, m > n \geq n_0$, and using property (p_4) of the partial metric p we get

$$p(x_n, x_m) \leq \sum_{k=n}^{m-1} p(x_k, x_{k+1}) \leq \frac{1}{b} \sum_{k=n}^{m-1} q^{k-n_0} f(x_{n_0}) \leq \frac{1}{b} \left(\frac{q^{n-n_0}}{1-q} \right) f(x_{n_0}).$$

Since $q < 1$, we conclude that the sequence $\{x_n\}$ is a Cauchy sequence. Due to the completeness of X, there exists $v_0 \in X$ such that $\lim_{n\longrightarrow\infty} x_n = v_0$, with respect to τ_p. Since f is lower semicontinuous and have in mind that $\{f(x_n)\}$ is convergent to 0, we have

$$0 \leq f(v_0) \leq \liminf_{n\longrightarrow\infty} f(x_n) = 0,$$

and thus, $f(v_0) = p(v_0, T(v_0)) = 0$. Now if $p(v_0, v_0) = 0$, since $T(v_0)$ is closed, it follows from Lemma 1.4 that $v_0 \in T(v_0)$.

Theorem 2.1 is proved.

Theorem 2.2. *Let (X, p) be a complete partial metric space. Let $T\colon X \longrightarrow \mathrm{Cl}(X)$. Assume that the following conditions hold*:

(i) *there exist a constant $b \in (0, 1)$ and two functions $\alpha\colon [0, \infty) \longrightarrow [b, \infty)$ and $\beta\colon [0, \infty) \longrightarrow \longrightarrow (0, \infty)$ such that for each $t \in [0, \infty)$, $\beta(t) \leq \alpha(t)$ and*

$$\limsup_{r\longrightarrow t^+} \frac{\beta(r)}{\alpha(r)} < 1,$$

(ii) *assume that* $\inf\{p(x, v) + p(x, T(x))\colon x \in X\} > 0$, *for every $v \in X$ with $v \notin T(v)$*,

(iii) *for any $x \in X$, there exists $y \in T(x)$ satisfying*

$$\alpha(f(x))p(x, y) \leq f(x)$$

and

$$f(y) \leq \beta(f(x))p(x, y).$$

Then $\mathrm{Fix}(T) \neq \varnothing$.

Proof. As in the proof of Theorem 2.1, we can obtain a Cauchy sequence $\{x_n\}$ with $x_n \in \in T(x_{n-1})$. Since (X,p) is a complete partial metric space, there exists some $v_0 \in X$ such that $\lim_{n\longrightarrow\infty} x_n = v_0$, with respect to τ_p. Also by property (p_4) of partial metric p on X

$$p(x_n, v_0) \le p(x_n, x_m) + p(x_m, v_0) - p(x_m, x_m).$$

By taking the limit when $m \longrightarrow \infty$ we get

$$p(x_n, v_0) \le \lim_{m\longrightarrow\infty} p(x_n, x_m) + \lim_{m\longrightarrow\infty} p(x_m, v_0) - \lim_{m\longrightarrow\infty} p(x_m, x_m).$$

Now, since $x_m \longrightarrow v_0 \in X$, with respect to τ_p, we have

$$\lim_{m\longrightarrow\infty} p(x_m, v_0) = \lim_{m\longrightarrow\infty} p(x_m, x_m) = p(v_0, v_0).$$

Therefore, it follows that for all $n \ge n_0$

$$p(x_n, v_0) \le \lim_{m\longrightarrow\infty} p(x_n, x_m) \le \frac{1}{b}\left(\frac{q^{n-n_0}}{1-q}\right) f(x_{n_0})$$

and

$$p(x_n, T(x_n)) \le p(x_n, x_{n+1}) \le \frac{q^{n-n_0}}{b} f(x_{n_0}).$$

Assume that $v_0 \notin T(v_0)$. Then, we obtain

$$0 < \inf\{p(x, v_0) + p(x, T(x))\colon x \in X\} \le$$

$$\le \inf\{p(x_n, v_0) + p(x_n, T(x_n))\colon n \ge n_0\} \le$$

$$\le \inf\left\{\frac{1}{b}\left(\frac{q^{n-n_0}}{1-q}\right) f(x_{n_0}) + \frac{1}{b} q^{n-n_0} f(x_{n_0})\colon n \ge n_0\right\} =$$

$$= \left\{\frac{1}{b}\left(\frac{2-q}{1-q}\right) f(x_{n_0})\right\} \inf\{q^{n-n_0} : n \ge n_0\} = 0,$$

which is impossible and hence $v_0 \in \mathrm{Fix}(T)$.

Theorem 2.2 is proved.

Theorem 2.3. *Let (X,p) be a complete partial metric space. Let $T\colon X \longrightarrow \mathrm{Cl}(X)$. Assume that the following conditions hold*:

(i) *there exist a constant $b \in (0,1)$ and two functions $\alpha\colon [0,\infty) \longrightarrow [b,\infty)$ and $\beta\colon [0,\infty) \longrightarrow \longrightarrow (0,\infty)$ such that for each $t \in [0,\infty)$, $\beta(t) \le \min\{\alpha(t), \alpha^2(t)\}$ and*

$$\limsup_{r\longrightarrow t^+} \frac{\beta(r)}{\alpha(r)} < 1,$$

(ii) *the map $f\colon X \longrightarrow \mathbb{R}$, defined by $f(x) = p(x, T(x))$ is lower semicontinuous,*

(iii) *for any* $x \in X$, *there exists* $y \in T(x)$ *satisfying*

$$\alpha(P(x,y))p(x,y) \le p(x,T(x))$$

and

$$p(y,T(y)) \le \beta(p(x,y))p(x,y).$$

Then there exists $v_0 \in X$ *such that* $f(v_0) = 0$. *Further, if* $p(v_0, v_0) = 0$, *then* $v_0 \in T(v_0)$.

Proof. Let $x_0 \in X$. From condition (iii) there exists $x_1 \in T(x_0)$ such that

$$\alpha(p(x_0,x_1))p(x_0,x_1) \le p(x_0,T(x_0)) \tag{8}$$

and

$$p(x_1,T(x_1)) \le \beta(p(x_0,x_1))p(x_0,x_1). \tag{9}$$

From (8) and (9), we get

$$p(x_1,T(x_1)) \le \frac{\beta(p(x_0,x_1))}{\alpha(p(x_0,x_1))}p(x_0,T(x_0)).$$

Similarly, there exists $x_2 \in T(x_1)$ such that

$$\alpha(p(x_1,x_2))p(x_1,x_2) \le p(x_1,T(x_1))$$

and

$$p(x_2,T(x_2)) \le \beta(p(x_1,x_2))p(x_1,x_2),$$

which imply

$$p(x_2,T(x_2)) \le \frac{\beta(p(x_1,x_2))}{\alpha(p(x_1,x_2))}p(x_1,T(x_1)).$$

By induction we get an orbit $\{x_n\}$ of T in X such that for all integers $n \ge 0$

$$\alpha(p(x_n,x_{n+1}))p(x_n,x_{n+1}) \le p(x_n,T(x_n)) \tag{10}$$

and

$$p(x_{n+1},T(x_{n+1})) \le \frac{\beta(p(x_n,x_{n+1}))}{\alpha(p(x_n,x_{n+1}))}p(x_n,T(x_n)). \tag{11}$$

Since by condition (i) we have

$$\frac{\beta(t)}{\alpha(t)} \le 1 \qquad \text{for all} \qquad t \in [0,\infty),$$

from (11), we get

$$p(x_{n+1},T(x_{n+1})) \le p(x_n,T(x_n)). \tag{12}$$

Thus $\{p(x_n, T(x_n))\}$ is a nonincreasing sequence of real numbers which is bounded from below by 0, so is convergent to an $L \geq 0$. From (10) and the fact that $\alpha(t) \geq b > 0$ for all $t \geq 0$, we obtain

$$p(x_n, x_{n+1}) \leq \frac{1}{b} p(x_n, T(x_n)). \tag{13}$$

By using (13) and the convergence of $\{p(x_n, T(x_n))\}$, we have that $\{p(x_n, x_{n+1})\}$ is bounded. Therefore, there exists $K \geq 0$ such that

$$\liminf_{n \longrightarrow \infty} p(x_n, x_{n+1}) = K. \tag{14}$$

Since $x_{n+1} \in T(x_n)$, we have $p(x_n, x_{n+1}) \geq p(x_n, T(x_n))$ for each $n \geq 0$, which implies that $K \geq L$. We claim that $K = L$. First suppose that $L = 0$. Again using (13) and the convergence of $\{p(x_n, T(x_n))\}$ we get $\lim_{n \longrightarrow \infty} p(x_n, x_{n+1}) = 0$.

Hence, if $L = 0$, then $K = L$. Now, suppose that $L > 0$ and assume to the contrary that $K > L$. Then $K - L > 0$. So from (14) and the fact that $\{p(x_n, T(x_n))\}$ is convergent to an $L \geq 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$L < p(x_n, T(x_n)) < L + \frac{K - L}{4} \qquad \text{for all} \qquad n \geq n_0 \tag{15}$$

and

$$K - \frac{K - L}{4} < p(x_n, x_{n+1}) \qquad \text{for all} \qquad n \geq n_0. \tag{16}$$

From (10), (15) and (16) we have

$$\alpha(p(x_n, x_{n+1})) \left(K - \frac{K - L}{4} \right) < \alpha(p(x_n, x_{n+1})) p(x_n, x_{n+1}) \leq$$

$$\leq p(x_n, T(x_n)) < L + \frac{K - L}{4}.$$

This implies that

$$\alpha(p(x_n, x_{n+1})) \leq \frac{K + 3L}{3K + L} \qquad \text{for all} \qquad n \geq n_0. \tag{17}$$

Take $h = \dfrac{K + 3L}{3K + L}$. Now, From (11), (17) and condition (i) we get

$$p(x_{n+1}, T(x_{n+1})) \leq \frac{\beta(p(x_n, x_{n+1}))}{\alpha^2(p(x_n, x_{n+1}))} \alpha(p(x_n, x_{n+1})) p(x_n, T(x_n)) \leq$$

$$\leq \frac{\beta(p(x_n, x_{n+1}))}{\min\{\alpha^2(p(x_n, x_{n+1})), \alpha(p(x_n, x_{n+1}))\}} \alpha(p(x_n, x_{n+1})) p(x_n, T(x_n)) \leq$$

$$\leq \alpha(p(x_n, x_{n+1})) p(x_n, T(x_n)) \leq h(p(x_n, T(x_n))) \qquad \text{for all} \qquad n \geq n_0. \tag{18}$$

By using (15) and (18), we have for any $\delta \geq 1$

$$L < p(x_{n_0+\delta}, T(x_{n_0+\delta})) \leq h^{\delta} p(x_{n_0}, T(x_{n_0})). \tag{19}$$

Since $L > 0$ and $h = \dfrac{K+3L}{3K+L} = \left(1 - \dfrac{2(K-L)}{3K+L}\right) < 1$, there exists a $\delta_0 \in N$ such that $h^{\delta_0} p(x_{n_0}, T(x_{n_0})) < L$. From (19), we get

$$L < p(x_{n_0+\delta}, T(x_{n_0+\delta})) \leq h^{\delta_0} p(x_{n_0}, T(x_{n_0})) < L,$$

which is a contradiction. Hence, assumption $K > L$ is wrong, so $L = K$. Now, we claim that $K = 0$. Since $K = L \leq p(x_n, T(x_n)) \leq p(x_n, x_{n+1})$, then from (14) we have

$$\liminf_{n \longrightarrow \infty} p(x_n, x_{n+1}) = K^+.$$

Hence, we can choose a subsequence $\{p(x_{n_\delta}, x_{n_\delta+1})\}$ of $\{p(x_n, x_{n+1})\}$ such that

$$\lim_{\delta \longrightarrow \infty} p(x_{n_\delta}, x_{n_\delta+1}) = K^+.$$

By condition (i), we get

$$\limsup_{p(x_{n_\delta}, x_{n_\delta+1}) \longrightarrow K^+} \frac{\beta(p(x_{n_\delta}, x_{n_\delta+1}))}{\alpha(p(x_{n_\delta}, x_{n_\delta+1}))} < 1 \tag{20}$$

and from (11), we obtain

$$p(x_{n_\delta+1}, T(x_{n_\delta+1})) \leq \frac{\beta(p(x_{n_\delta}, x_{n_\delta+1}))}{\alpha(p(x_{n_\delta}, x_{n_\delta+1}))} p(x_{n_\delta}, T(x_{n_\delta})).$$

Taking the limit as $\delta \longrightarrow \infty$ and have in mind that $\{p(x_n, T(x_n))\}$ is convergent to $L \geq 0$, thus, we get

$$L = \limsup_{\delta \longrightarrow \infty} p(x_{n_\delta+1}, T(x_{n_\delta+1})) \leq$$

$$\leq \left(\limsup_{\delta \longrightarrow \infty} \frac{\beta(p(x_{n_\delta}, x_{n_\delta+1}))}{\alpha(p(x_{n_\delta}, x_{n_\delta+1}))}\right) \limsup_{\delta \longrightarrow \infty} p(x_{n_\delta}, T(x_{n_\delta})) =$$

$$= \left(\limsup_{p(x_{n_\delta}, x_{n_\delta+1}) \longrightarrow K^+} \frac{\beta(p(x_{n_\delta}, x_{n_\delta+1}))}{\alpha(p(x_{n_\delta}, x_{n_\delta+1}))}\right) L.$$

If we suppose that $L > 0$, then from high inequality, we have

$$\limsup_{p(x_{n_\delta}, x_{n_\delta+1}) \longrightarrow K^+} \frac{\beta(p(x_{n_\delta}, x_{n_\delta+1}))}{\alpha(p(x_{n_\delta}, x_{n_\delta+1}))} \geq 1,$$

which contradicts (20). Thus $L = 0$. Obviously, $\{p(x_n, T(x_n))\}$ is convergent to $L \geq 0$, and since $p(x_n, x_{n+1}) \leq \dfrac{1}{b} p(x_n, T(x_n))$, then we obtain

$$\lim_{n \longrightarrow \infty} p(x_n, T(x_n)) = 0^+ \tag{21}$$

and

$$\lim_{n\longrightarrow\infty} p(x_n, x_{n+1}) = 0^+.$$

Now, from condition (i), we can choose q such that

$$\limsup_{p(x_{n_\delta}, x_{n_\delta+1})\longrightarrow 0^+} \frac{\beta(p(x_{n_\delta}, x_{n_\delta+1}))}{\alpha(p(x_{n_\delta}, x_{n_\delta+1}))} < q < 1.$$

So $\dfrac{\beta(p(x_n, x_{n+1}))}{\alpha(p(x_n, x_{n+1}))} < q$ for all $n \geq n_0$, for some $n_0 \in \mathbb{N}$.

Again using (11), we get

$$p(x_{n+1}, T(x_{n+1})) \leq q\, p(x_n, T(x_n)) \qquad \text{for all} \qquad n \geq n_0.$$

Hence, by induction, we have

$$p(x_{n+1}, T(x_{n+1}) \leq q^{n+1-n_0} p(x_{n_0}, T(x_{n_0})) \qquad \text{for all} \qquad n \geq n_0. \tag{22}$$

From (13), (22) and the fact that $x_{n+1} \in T(x_n)$, we obtain

$$p(x_n, x_{n+1}) \leq \frac{1}{b} q^{n-n_0} p(x_{n_0}, T(x_{n_0})) \qquad \text{for all} \qquad n \geq n_0.$$

In this step of the proof, we show that $\{x_n\}$ is a Cauchy sequence.

For any $m, n \in \mathbb{N}, m > n \geq n_0$, and using property (p_4) of the partial metric p we have

$$p(x_n, x_m) \leq \sum_{\delta=n}^{m-1} p(x_\delta, x_{\delta+1}) \leq \frac{1}{b} \sum_{\delta=n}^{m-1} q^{\delta-n_0} f(x_{n_0}) \leq \frac{1}{b} \left(\frac{q^{n-n_0}}{1-q} \right) f(x_{n_0}).$$

Since $q < 1$, we conclude that the sequence $\{x_n\}$ is a Cauchy sequence. Due to the completeness of X, there exists $v_0 \in X$ such that $\lim_{n\longrightarrow\infty} x_n = v_0$, with respect to τ_p. Since f is lower semicontinuous and from (21), we get

$$0 \leq f(v_0) \leq \liminf_{n\longrightarrow\infty} f(x_n) = \liminf_{n\longrightarrow\infty} p(x_n, T(x_n)) = 0,$$

and thus, $f(v_0) = p(v_0, T(v_0)) = 0$. Now if $p(v_0, v_0) = 0$, since $T(v_0)$ is closed, it follows from Lemma 1.1 that $v_0 \in T(v_0)$.

Theorem 2.4. *Suppose that all the hypotheses of Theorem* 2.3 *except* (ii) *hold. Assume that*

$$\inf\{p(x, v) + p(x, T(x))\colon x \in X\} > 0, \quad for\ every\ \ v \in X\ \ with\ \ v \notin T(v).$$

Then $\mathrm{Fix}(T) \neq \varnothing$.

Proof. Since the proof of this theorem can be completed essentially on the line of the proof of Theorem 2.2, hence details are omitted.

3. Examples.

***Example* 3.1.** Let $X = [0,1]$ and $p(x,y) = \max\{x,y\}$, then it is clear that (X,p) is a complete partial metric space. Note that function p is not a metric. Let $T\colon X \longrightarrow \mathrm{Cl}(X)$ be defined by the following formula:

$$T(x) = \begin{cases} \left\{\dfrac{1}{6}x^2\right\} & \text{for } x \in \left[0, \dfrac{1}{2}\right) \bigcup \left(\dfrac{1}{2}, 1\right], \\[2ex] \left\{\dfrac{9}{120}, \dfrac{9}{20}\right\} & \text{for } x = \dfrac{1}{2}. \end{cases}$$

Define $\alpha\colon [0,\infty) \longrightarrow (0,\infty)$ by

$$\alpha(t) = \begin{cases} \dfrac{1}{2} & \text{for } t \in [0,1], \\[2ex] \dfrac{1}{2}t & \text{for } t \in (1,\infty), \end{cases}$$

and, $\beta\colon [0,\infty) \longrightarrow (0,\infty)$ by $\beta(t) = \dfrac{1}{3}\alpha(t)$.

Since $T(x) \le x$, we have $f(x) = \max\{x, T(x)\} = x$ for all $x \in [0,1]$ and f is lower semicontinuous. Moreover, for each $x \in \left[0, \dfrac{1}{2}\right) \bigcup \left(\dfrac{1}{2}, 1\right]$, we have $T(x) = \left\{\dfrac{1}{6}x^2\right\}$, so $y \in T(x)$ implies $y = \dfrac{1}{6}x^2$. Hence

$$p(x,y) = p\left(x, \frac{1}{6}x^2\right) = \max\left\{x, \frac{1}{6}x^2\right\} = x \in [0,1].$$

Now, for all $x \in X - \left\{\dfrac{1}{2}\right\}$ and $y = \dfrac{1}{6}x^2$

$$\alpha(p(x,y))p(x,y) = \frac{1}{2}x \le x = p(x,T(x))$$

and

$$p(y,T(y)) = \left(\frac{1}{6}x^2\right) \le \frac{1}{6}x = \beta(p(x,y))p(x,y).$$

If $x = \dfrac{1}{2}$ then we have $T(x) = \left\{\dfrac{9}{120}, \dfrac{9}{20}\right\}$, and $p(x,T(x)) = \dfrac{1}{2}$. Thus, for $x = \dfrac{1}{2}$ we can choose $y = \dfrac{9}{120} \in T(x)$ such that

$$\alpha(p(x,y))p(x,y) = \alpha\left(\frac{1}{2}\right)\frac{1}{2} = \frac{1}{4} < p(x,T(x))$$

and

$$p(y,T(y)) = \frac{9}{120} < \beta\left(\frac{1}{2}\right)\frac{1}{2} = \frac{1}{12} = \beta(p(x,y))p(x,y).$$

Hence, T satisfies all the conditions of Theorem 2.3 and note that $\mathrm{Fix}(T) = \{0\}$.

***Example* 3.2.** Consider $x_n = 1 - \dfrac{1}{n}$ for $n \in N$ and $x_0 = 1$. Suppose $\mathbb{N}$, $\mathbb{N}_e$, $\mathbb{N}_o$ denote the sets of all positive integers, even positive integers and odd positive integers, respectively. Let $X = \{x_0, x_1, x_2, x_3, x_4, \ldots\}$ that is a bounded complete subset of $\mathbb{R}$. Let $p(x,y) = d(x,y)$, for all $x, y \in X$. Define a mapping T from X into $\mathrm{Cl}(X)$ by

$$T(x_n) = \begin{cases} x_0 & \text{if } n = 0, \\ \{x_{n-1}, x_{n^2}\} & \text{if } n \in \{n \in \mathbb{N}_e : n \geq 2\}, \\ x_1 & \text{if } n \in \mathbb{N}_o. \end{cases}$$

It is easy to verify that

$$f(x_n) = p(x_n, T(x_n)) = \begin{cases} 0 & \text{if } n \in \{0,1\}, \\ \dfrac{1}{n-1} - \dfrac{1}{n} & \text{if } n \in \{n \in \mathbb{N}_e \colon n \geq 2\}, \\ 1 - \dfrac{1}{n} & \text{if } n \in \{n \in \mathbb{N}_o \colon n > 2\}. \end{cases}$$

So, f is lower semicontinuous in X.

Suppose $\alpha(t) \colon [0,\infty) \longrightarrow (0,1)$ and $\beta(t) \colon [0,\infty) \longrightarrow (0,1)$ are defined by

$$\alpha(t) = \begin{cases} \dfrac{1}{4} & \text{if } t \in \{0\} \cup [1,\infty), \\ \sqrt{t} & \text{if } t \in (0,1), \end{cases}$$

and

$$\beta(t) = \begin{cases} t & \text{if } t \in [0,1), \\ \dfrac{1}{8} & \text{if } t \in [1,\infty). \end{cases}$$

Since

$$\frac{\beta(t)}{\alpha(t)} = \begin{cases} \sqrt{t} & \text{if } t \in (0,1), \\ 0 & \text{if } t \in \{0\}, \\ \dfrac{1}{2} & \text{if } t \in [1,\infty). \end{cases}$$

Then, we have for each $t \in [0,\infty)$, $\beta(t) \leq \alpha(t)$ and

$$\limsup_{r \longrightarrow t^+} \frac{\beta(r)}{\alpha(r)} < 1.$$

For $x = x_1,\, x_0$ let $y = x \in T(x)$, so

$$\alpha(f(x))p(x,y) = 0 = f(x), \qquad f(y) = 0 = \beta(f(x))p(x,y).$$

For $x = x_n$, $n \geq 2$, and $n \in \mathbb{N}_e$ if we let $y = x_{n^2} \in T(x)$ then

$$\alpha(f(x))p(x,y) = \alpha\left(\frac{1}{n-1} - \frac{1}{n}\right)\left(\frac{1}{n} - \frac{1}{n^2}\right) =$$

$$= \left(\sqrt{\frac{1}{n-1} - \frac{1}{n}}\right)\left(\frac{1}{n} - \frac{1}{n^2}\right) < \left(\frac{1}{n-1} - \frac{1}{n}\right) = f(x),$$

$$f(y) = \left(\frac{1}{n^2-1} - \frac{1}{n^2}\right) < \left(\frac{1}{n-1} - \frac{1}{n}\right)\left(\frac{1}{n} - \frac{1}{n^2}\right) = \beta(f(x))p(x,y).$$

For $x = x_n$, $n > 2$, and $n \in \mathbb{N}_o$, let $y = x_1 \in T(x)$, so

$$\alpha(f(x))p(x,y) = \alpha\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n}\right) = \left(\sqrt{1 - \frac{1}{n}}\right)\left(1 - \frac{1}{n}\right) < \left(1 - \frac{1}{n}\right) = f(x),$$

$$f(y) = 0 < \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n}\right) = \beta(f(x))p(x,y).$$

Therefore, all the assumptions of Theorem 2.1 are satisfied and x_1, x_0 are two fixed points of T. Let us observe that T does not satisfy the assumptions of Theorem 1.2 provided $p(x,y) = d(x,y)$, for all $x, y \in X$. Indeed, for any function $\varphi : [0,\infty) \longrightarrow [b,1)$, $b \in (0,1)$, there exists $n > 2$, $n \in \mathbb{N}_e$, such that for $x = x_n$, if $y = x_{n^2} \in T(x)$, we have

$$f(x) = \frac{1}{n-1} - \frac{1}{n} < \sqrt{b}\left(\frac{1}{n} - \frac{1}{n^2}\right) \leq \sqrt{\varphi(f(x))}d(x,y),$$

and if $y = x_{n-1} \in T(x)$, we have

$$f(y) = d(y, T(y)) = 1 - \frac{1}{n-1} > \frac{1}{n(n-1)} \geq \varphi(f(x))d(x,y),$$

which does not match the assumptions of Theorem 1.2.

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