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A NOTE ON THE RECURSIVE SEQUENCE $x_{n+1} = p_k x_n + p_{k-1} x_{n-1} + \ldots + p_1 x_{n-k+1}$

ПРО РЕКУРЕНТНУ ПОСЛІДОВНІСТЬ $x_{n+1} = p_k x_n + p_{k-1} x_{n-1} + \ldots + p_1 x_{n-k+1}$

We present some comments on the behaviour of solutions of the difference equation

$$x_{n+1} = p_k x_n + p_{k-1} x_{n-1} + \ldots + p_1 x_{n-k+1}, \quad n = -1, 0, 1, \ldots,$$

where $p_i \geq 0$, $i = 1, \ldots, k$, $k \in \mathbf{N}$ and $x_{-k}, \ldots, x_{-1} \in \mathbf{R}$.

Розглядається поведінка розв'язків різницевого рівняння

$$x_{n+1} = p_k x_n + p_{k-1} x_{n-1} + \ldots + p_1 x_{n-k+1}, \quad n = -1, 0, 1, \ldots,$$

де $p_i \geq 0$, $i = 1, \ldots, k$, $k \in \mathbf{N}$ та $x_{-k}, \ldots, x_{-1} \in \mathbf{R}$.

1. Introduction. In [1] the authors considered the following linear homogeneous difference equation

$$x_{n+1} = p_k x_n + p_{k-1} x_{n-1} + \ldots + p_1 x_{n-k+1}, \quad n = -1, 0, 1 \ldots, \tag{1}$$

where $p_i > 0$, $i = 1, \ldots, k$, $k \in \mathbf{N}$ and $x_{-k}, \ldots, x_{-1} \in \mathbf{R}$. Equation (1) is very important because it presents the linearized equation of a large class of mathematical biology models. For example:

Discrete delay logistic difference equation:

$$x_{n+1} = \frac{\alpha x_n}{1 + \beta x_{n-k}}, \quad \alpha, \beta > 0, \quad \text{and} \quad k \in \mathbf{N}, \tag{2}$$

which was considered in books [2,3] by E. C. Pielou.

Generalized Bedington – Holt stock recruitment model:

$$x_{n+1} = a x_n + \frac{b x_{n-1}}{1 + c x_{n-1} + d x_n}, \quad x_0, x_1 > 0, \quad n = 1, 2, 3, \ldots, \tag{3}$$

where $a \in (0, 1)$, $b \in \mathbf{R}_+$ and $c, d \in \mathbf{R}_+ \cup \{0\}$, with $c + d > 0$.

This equation was considered, for example in [4 – 6]. In [5] it was shown that when $a + b < 1$, or $a + b = 1$ and $c > 0$, then the zero equilibrium is a global attractor of all positive solutions of equation (3).

Mosquito population equations:

$$x_{n+1} = (a x_n + b x_{n-1} e^{-x_{n-1}}) e^{-x_n}, \quad x_0, x_1 > 0, \quad n = 1, 2, 3, \ldots, \tag{4}$$

where $a \in (0, 1)$, $b \in [0, \infty)$ and

$$x_{n+1} = (\alpha x_n + \beta x_{n-1}) e^{-x_n}, \quad x_0, x_1 > 0, \quad n = 1, 2, 3, \ldots, \tag{5}$$

where $\alpha \in [0, 1)$, $\beta \in (0, \infty)$.

Equations (4) describes the growth of a mosquito population. Equations (5) is derived from a two life stage model where the young mature into adults, and adults produce young. The global stability of these equations is studied in [7].

Perenial grass model:

$$x_{n+1} = ax_n + \frac{bx_{n-1}}{e^{x_n}}, \quad x_0, x_1 > 0, \quad n = 1, 2, 3, \ldots, \tag{6}$$

where $a \in (0, 1)$, $b > 0$.

The stability, and the oscillatory character of solutions of somewhat generalized equation have been studied in [5, 8].

Flour beetle population model:

$$x_{n+3} = ax_{n+2} + bx_n e^{-(cx_{n+2}+dx_n)}, \quad n = 0, 1, 2, \ldots, \tag{7}$$

with $a, b, c, d \geq 0$ and $c + d > 0$.

This equation was considered in [9, 10]. In [9] it was shown in particular that when $a + b \leq 1$ and $a,\ b > 0$, then the origin is asymptotically stable relative to the nonnegative octant in $\mathbf{R}^3$ (i.e., all nonnegative solutions are attracted to the origin and the beetles became extinct). The same conclusion holds under the conditions

$$a + b \leq 1 \quad \text{and} \quad b > 0.$$

From the above we see that the most interesting case of these equations are when the sum of the main coefficients of the equations is equal to 1, i.e., when $\alpha = 1$ for Eq. (2), $a + b = 1$ for Eq. (3), (4), (6) and (7) and $\alpha + \beta = 1$ for Eq. (5). In these cases linearized equations of this equations have the form just as in (1).

In [1] the following theorem was established.

Theorem A. *The solutions of Eq.* (1) *are asymptotically stable if and only if* $R_0 = \sum_{i=1}^{k} p_i \leq 1$.

Their proof is not complete and too complicated. They omitted the case when $z = e^{i\theta}$, $\theta \in (0, \pi) \cup (\pi, 2\pi)$, is a zero of the polynomial

$$P_k(z) = z^k - p_k z^{k-1} - p_{k-1} z^{k-2} - \ldots - p_1. \tag{8}$$

Also the formulation of Theorem A is rather awkward, because in the case $R_0 = 1$ the solutions of Eq. (1) are only stable and not asymptotically stable. The purpose of this note is to complete proof of Theorem A and generalize the theorem to the case when $p_i \geq 0$, $i = 1, \ldots, k$.

2. Main results. ***The case*** $R_0 = 1$.

Lemma 1. *Let* $P_k(z) = z^k - p_k z^{k-1} - \ldots - p_1$ *be a polynomial such that* $p_i > 0$, $i = 1, \ldots, k$, *and* $p_1 + \ldots + p_k = 1$. *Then all zeros of the polynomial* $P_k(z)$ *lie in* $|z| < 1$, *except* $z = 1$, *which is a simple.*

Proof. Since, for $|z| > 1$,

$$\left| p_k z^{k-1} + \ldots + p_1 \right| \leq p_k |z|^{k-1} + \ldots + p_1 < |z|^k,$$

we obtain that all zeros of the polynomial $P_k(z)$ lie in $|z| < r$, for every $r > 1$. Hence all zeros of $P_k(z)$ lie in $|z| \leq 1$.

It is easy to see that $P_k(1) = 0$ and $P_k'(1) \neq 0$, which implies that $z = 1$ is a simple zero of $P_k(z)$ (see, Lemma 2 below).

Let $e^{i\theta}$, $\theta \neq 2m\pi$, $m \in \mathbf{Z}$, be a zero of $P_k(z)$. Then

$$e^{ik\theta} = p_k e^{i(k-1)\theta} + \ldots + p_2 e^{i\theta} + p_1$$

i.e.

$$\cos k\theta = p_k \cos(k-1)\theta + \ldots + p_2 \cos\theta + p_1,$$
$$\sin k\theta = p_k \sin(k-1)\theta + \ldots + p_2 \sin\theta.$$

From

$$1 = \left(\sum_{i=1}^{k} p_i \cos(i-1)\theta\right)^2 + \left(\sum_{i=1}^{k} p_i \sin(i-1)\theta\right)^2 =$$

$$= \sum_{i=1}^{k} p_i^2 + 2\sum_{i<j} p_i p_j \cos(i-j)\theta$$

and

$$1 = (p_1 + \ldots + p_m)^2 = \sum_{i=1}^{k} p_i^2 + 2\sum_{i<j} p_i p_j$$

we obtain $\cos(i-j)\theta = 1$ for all $i<j$, i.e., $(i-j)\theta = 2\pi k_{i,j}$, where $k_{i,j} \in \mathbf{Z}$. We may assume $\theta \in (0, 2\pi)$. For $j = i+1$ we obtain $\theta = 2\pi k_{i,i+1}$ for some $k_{i,i+1} \in \mathbf{Z}$, arriving at a contradiction.

Lemma 2. *Let $P_k(z) = z^k - p_k z^{k-1} - \ldots - p_1$ be a polynomial such that $p_i \geq 0$, $i = 1, \ldots, k$, and $p_1 + \ldots + p_k = 1$. Then all zeros of the polynomial $P_k(z)$ which belong to the set $\{z \mid |z| = 1\}$ are simple.*

Proof. Assume that $z = e^{i\theta}$ is a zero of the polynomial $P_k(z)$. If $e^{i\theta}$ is not simple zero, then

$$k = \left|ke^{i(k-1)\theta}\right| = \left|(k-1)p_k e^{i(k-2)\theta} + \ldots + 2p_3 e^{i\theta} + p_2\right| \leq$$

$$\leq (k-1)p_k + \ldots + 2p_3 + p_2 \leq (k-1)\sum_{i=2}^{k} p_i \leq k-1.$$

Hence, $P'(e^{i\theta}) \neq 0$, as desired.

By Lemmas 1 and 2 we obtain the following result.

***Corollary* 1.** *Assume $R_0 = 1$ and $p_i \geq 0$, $i = 1, \ldots, k$. Then the solutions of Eq.* (1) *are stable.*

We present here some comments about convergence of the solutions of Eq. (1).

In [11] the following theorem was established.

Theorem B. *Let $\varphi(y_1, y_2, \ldots, y_k)$ be a continuous real function on $\mathbf{R}^k$ where*

(a) $\varphi(x, x, \ldots, x) \leq x$, *for every* $x \in \mathbf{R}$;

(b) $\varphi \in C(\mathbf{R}^k, \mathbf{R})$ *is nondecreasing in each of its arguments;*

(c) $\varphi(y_1, y_2, \ldots, y_k)$ *is strictly increasing in at least two of its arguments y_i and y_j, where i and j are relatively prime.*

If (x_n) is a bounded sequence which satisfies the inequality

$$x_{n+k} \leq \varphi(x_{n+k-1}, x_{n+k-2}, \ldots, x_n) \quad for \quad n \in \mathbf{N} \cup \{0\},$$

then it must be convergent.

***Corollary* 2.** *Consider Eq.* (1). *Let $R_0 = 1$, $p_{k-i} > 0$ and $p_{k-j} > 0$ where $i+1$ and $j+1$ are relatively prime. Then every solution of Eq.* (1) *converges.*

Proof. From (1) we obtain:

$$|x_{n+1}| \leq \max\{|x_n|, \ldots, |x_{n-k+1}|\}, \qquad n = -1, 0, \ldots,$$

from which the boundedness of (x_n) follows. By Theorem B we obtain the result.

On the other hand, consider the equation

$$x_{n+k} = \frac{x_{n+k-s} + x_{n+k-l} + 0 \cdot x_n}{2}, \tag{9}$$

where $1 \le s < l \le k$, s and l are not relatively prime and $x_0, x_1, \ldots, x_{k-1} \in \mathbf{R}$.

The characteristic polynomial for Eq. (9) is

$$2t^{n+k} - t^{n+k-s} - t^{n+k-l} = t^{n+k-l}(2t^l - t^{l-s} - 1) = 0.$$

Since s and l are not relatively prime there exist $q \in \mathbf{N}\setminus\{1\}$, such that $s = qs_1$ and $l = ql_1$ for some $s_1, l_1 \in \mathbf{N}$. Hence

$$2t^l - t^{l-s} - 1 = 2t^{ql_1} - t^{q(l_1-s_1)} - 1 = t^{q(l_1-s_1)}(t^{qs_1} - 1) + t^{ql_1} - 1. \tag{10}$$

From (10) we see that the polynomial $t^q - 1$ is a factor of characteristic polynomial of Eq. (9). Since $q \ge 2$, this characteristic polynomial has a zero in the set $\{z \mid |z| = 1, z \ne 1\}$. By the well known theorem we obtain that difference equation (2) has a bounded divergent solution, for example $x_n = \cos(2\pi n/q)$, $n \in \mathbf{N}$.

Hence the condition, $i+1$ and $j+1$ are relatively prime, in Corollary 2 is necessary for convergence of all solutions.

The case $R_0 < 1$. The following lemma is a natural generalization of the Theorem A when $R_0 < 1$.

Lemma 3. *Let (x_n) be a sequence of positive numbers which satisfies the inequality*

$$x_{n+k} \le A \max\{x_{n+k-1}, x_{n+k-2}, \ldots, x_n\} \quad \textit{for} \quad n \in \mathbf{N}, \tag{11}$$

where $A \in (0, 1)$ and $k \in \mathbf{N}$ are fixed. Then there exist $L \in \mathbf{R}_+$ such that

$$x_{km+r} \le LA^m \quad \textit{for all} \quad m \in \mathbf{N} \cup \{0\} \quad \textit{and} \quad 1 \le r \le k. \tag{12}$$

Proof. Let $L = \max\{x_1, x_2, \ldots, x_k\}$. We will prove the lemma by induction. For $m = 0$ and $1 \le r \le k$ the result is trivial. Suppose that the result holds for some $m \in \mathbf{N}$ and $1 \le r \le k$. By (11) and the induction hypothesis we have

$$x_{k(m+1)+1} \le A \max\{x_{k(m+1)}, x_{k(m+1)-1}, \ldots, x_{km+1}\} \le A(LA^m) = LA^{m+1}.$$

From that and by the induction hypothesis we get

$$x_{k(m+1)+r} \le A \max\{x_{k(m+1)+r-1}, x_{k(m+1)+r-2}, \ldots, x_{km+r}\} \le A(LA^m) = LA^{m+1},$$

for $2 \le r \le k$, as desired.

Remark. Note that L depends of (x_n).

***Corollary* 3.** *Let (x_n) be the sequence of positive numbers in Lemma 3. Then there exists $M > 0$ such that*

$$x_n \le M\left(\sqrt[k]{A}\right)^n.$$

***Corollary* 4.** *Assume $0 \le R_0 < 1$ and $p_i \ge 0$, $i = 1, \ldots, k$. Then the solutions of Eq.* (1) *are asymptotically stable.*

The case $R_0 > 1$. First, we prove an auxiliary result.

Lemma 4. *Let $R_0 > 1$ then the polynomial* (2) *has a real root $\zeta \in (1, R_0]$.*

Proof. It is clear that

$$P_k(1) = 1 - p_k - \ldots - p_1 = 1 - R_0 < 0. \tag{13}$$

On the other hand

$$P_k(R_0) = R_0^k - p_k R_0^{k-1} - \ldots - p_1 \ge R_0^k - R_0^{k-1}(p_k + \ldots + p_1) = 0. \tag{14}$$

From (13) and (14) the result follows.

By Lemma 4 we obtain the following result.

***Corollary* 5.** *Assume* $R_0 > 1$ *and* $p_i \geq 0$, $i = 1, \ldots, k$. *Then the solutions of Eq.* (1) *are not stable.*

From the above we obtain the following theorem.

Theorem. *Consider Eq.* (1), *where* $p_i \geq 0$, $i = 1, \ldots, k$, $k \in \mathbf{N}$, *and* $x_{-k}, \ldots, x_{-1} \in \mathbf{R}$. *Let* $R_0 = \sum_{i=1}^{k} p_i$. *Then*

(a) *if* $R_0 = 1$ *the solutions of Eq.* (1) *are stable;*

(b) *if* $0 \leq R_0 < 1$ *the solutions of Eq.* (1) *are asymptotically stable;*

(c) *if* $R_0 > 1$ *the solutions of Eq.* (1) *are not stable.*

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