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NONLINEAR-ESTIMATE APPROACH TO THE REGULARITY OF INFINITE-DIMENSIONAL PARABOLIC PROBLEMS*

ПІДХІД ДО РЕГУЛЯРНОСТІ НЕСКІНЧЕННОВИМІРНИХ ПАРАБОЛІЧНИХ ЗАДАЧ, ЩО ҐРУНТУЄТЬСЯ НА НЕЛІНІЙНИХ ОЦІНКАХ

We show how the use of nonlinear symmetries of higher-order derivatives allows one to study the regularity of solutions of nonlinear differential equations in the case where the classical Cauchy – Liouville – Picard scheme is not applicable. In particular, we obtain nonlinear estimates for the boundedness and continuity of variations with respect to initial data and discuss their applications to the dynamics of unbounded lattice Gibbs models.

Показано, яким чином застосування нелінійних симетрій похідних високого порядку дозволяє вивчати регулярність розв'язків нелінійних диференціальних рівнянь у випадку, коли класичну схему Коші – Ліувілля – Пікара неможливо застосувати. Зокрема, отримано нелінійні оцінки на обмеженість та неперервність варіацій за початковими умовами і розглянуто їх застосування до динаміки необмежених граткових гіббсівських систем.

1. Statement of problem: symmetries of variational equations and regularity schemes in the non-Lipschitz case. The problem of correct definition of differential operators is naturally linked with the construction of the functional spaces of their action and study of the associated regularity problems. There is one classical regularity scheme, usually attributed to Cauchy, Liouville, and Picard, that demonstrates how the Lipschitz assumptions on coefficients of equation lead to the regular dependence of solutions with respect to the initial data and parameters of different kinds [1 – 4].

Consider the first order differential equation

$$y_t(x) = x - \int_0^t F\big(y_s(x)\big)\,ds \tag{1}$$

with nonlinear drift F such that

$$\exists\, K_n:\ \sup_{x\in\mathbb{R}^1} \left| F^{(n)}(x) \right| \le K_n,$$

in particular, F is Lipschitz continuous, i.e., $|F(x) - F(y)| \le K_1 |x - y|$. In adaptation to (1), the Cauchy – Liouville – Picard scheme applies the fixed point arguments to the construction of the unique solution to (1) via the following iteration formula:

$$y_t^{n+1}(x) = x - \int_0^t F\big(y_s^n\big)\,ds, \quad n \ge 1.$$

In a similar way, the implicit function arguments

$$\{\Phi(x, y)\}_t = y_t - x + \int_0^t F(y_s)\,ds = 0$$

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and the estimate $\left\| \Phi'_y - 1 \right\| \leq t K_1 \leq \varepsilon$ for small t lead to the first order differentiability with respect to the initial data: $\exists \dfrac{\partial}{\partial x} y_t(x) = -\left[\Phi'_y \right] \Phi'_x$.

The higher-order differentiability of $y_t(x)$ on initial data x for small $t > 0$ is derived from the theorem on the differentiability of implicit function. Finally, due to the semigroup property of flow $y_{t+s}(x) = y_t(y_s(x))$, solution $y_t(x)$ is regular with respect to the initial data for all $t \geq 0$.

In application to the associated semigroup $(P_t f)(x) = f(y_t(x))$, the Cauchy – Liouville – Picard scheme leads to the quasicontractive regularity estimates of any order:

$$\| P_t \|_{L(C_b^n)} \leq \exp\left(t M_{K_1, \ldots, K_n} \right) \tag{2}$$

in the standard spaces of continuously differentiable functions with bounded derivatives C_b^n. This scheme admits natural generalizations to the infinite-dimensional Banach space X and, in fact, inspired to a great degree the development of modern functional analysis. In particular, the concept of full metric space and Browder index theorems were closely related with the fixed point results [1 – 4]. In a similar way, the development of infinite-dimensional analogies of implicit function theorems inspired the study of differentiability in spaces of Frechet differentiable functions and the interpretation of variations as Frechet derivatives $y_t^{(n)}(x) \in B_n = L(X, B_{n-1})$, $B_0 = X$.

However, it is still a question what happens, when the map F is essentially nonlinear, i.e., does not have bounded derivatives and, therefore, is no more globally Lipschitz. In this case, the fixed point arguments and implicit function techniques could not be applied because of unbounded derivatives $F^{(j)}$. One can also construct a counter-example [5] (Chapter 1.2), which demonstrates that spaces C_b^n of continuously differentiable functions with bounded derivatives are not preserved by non-Lipschitz semigroups.

What techniques can be introduced in the essentially nonlinear case and how to work with the C^∞ regularity problems in this case is the main question of this article.

The principal idea lies in the backgrounds of differentiable calculus. Because nonlinearity means nonlinear responses to linear operations, like derivative, let us consider the n th derivative of nonlinear function

$$[F(y)]^{(n)} = \left[F'(y) y^{(1)} \right]^{(n-1)} =$$

$$= F'(y) y^{(n)} + \sum_{j_1 + \ldots + j_s = n,\, s = 2, n-1} F^{(s)}(y) y^{(j_1)} \ldots y^{(j_s)} + F^{(n)}(y) \left[y^{(1)} \right]^n. \tag{3}$$

The main observation is that the right-hand side of (3) contains simultaneously n th order derivative $y^{(n)}$ and first derivative in n th degree $\left[y^{(1)} \right]^n$. Similar symmetry is also reflected in the intermediate terms

$$y^{(j_1)} \ldots y^{(j_s)} \sim \left[y^{(1)} \right]^{j_1} \ldots \left[y^{(1)} \right]^{j_s} \sim \left[y^{(1)} \right]^n \quad \text{because} \quad j_1 + \ldots + j_s = n. \tag{4}$$

Symmetry (4) has immediate consequences for the nonlinear evolutional equations like (1). Consider the higher-order derivative of process y_t^x with respect to the initial data

$$y_t^{(n)} = \frac{\partial^n}{\partial x^n} y_t^x = \text{i. d.} - \int_0^t \frac{\partial^n}{\partial x^n} F\left(y_t^x\right) dt =$$

$$= \text{i. d.} - \int_0^t \sum_{j_1+\ldots+j_s=n,\, s\geq 1} F^{(s)}\left(y_t^x\right)\left[y_t^{(j_1)}, \ldots, y_t^{(j_s)}\right] dt, \qquad (5)$$

where $H^{(a)}(x)[h_1, \ldots, h_a]$ means a th order directional derivative at point $x \in \mathbb{R}^n$.

For equation (5), variation $y^{(n)}$ on the left-hand side is proportional to the first variation $y^{(1)}$ in n th power on the right-hand side, or, after taking the n th root,

$$y^{(1)} \approx \left[y^{(n)}\right]^{1/n}. \qquad (6)$$

This property becomes fundamental. It appears that the knowledge of symmetry (6) is sufficient for the study of regularity in the essentially non-Lipschitz case. Let us introduce the expression which is homogeneous with respect to the symmetry (6):

$$\rho_n(y, t) = \sum_{j=1}^n p_j\left(\left|y_t^x\right|^2\right)\left\| y_t^{(j)} \right\|^{m/j}. \qquad (7)$$

Due to $y_t^{(j)} = \partial_x^{(j)} y_t^x$, this expression reflects the regularity of solution $y_t(x)$ with respect to the initial data x. The main statement is that there is a hierarchy of weights $\{p\}$, determined by nonlinearity parameters of map F, that leads to quasicontractive a priori estimate on regularity.

In the following sections, we consider an example of infinite dimensional model, for which it could be done a sufficiently detail research of regularity properties. In particular, we obtain a set of nonlinear estimates on the a priori boundedness and continuity of variations for dynamics of such models and derive applications to the C^∞-properties of associated parabolic evolutions. More motivational details and peculiarities of research of these models are discussed in [5 – 8].

2. Influence of nonlinearity on the properties of infinite-dimensional systems. Notations and preliminary results. We discuss below an important infinite-dimensional model, that describes the lattice approximations of multidimensional Euclidean $:P(\varphi):_d$-field theory [9, 10]. It corresponds to the formal Gibbs measure

$$d\mu(x) = \frac{1}{Z}\exp\left\{-\frac{1}{2}\sum_{j,k:|j-k|\leq r_0} b_{k-j}\, x_k\, x_j\right\}\prod_{k\in\mathbb{Z}^d} e^{-F(x_k)} dx_k$$

and its kinetic energy operator H, defined by integration by parts via

$$(Hu, u)_{L_2\left(\mathbb{R}^{\mathbb{Z}^d}, \mu\right)} = \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}^{\mathbb{Z}^d}} \left|\frac{\partial}{\partial x_k} u(x)\right|^2 d\mu(x),$$

has the form

$$H = \sum_{k\in\mathbb{Z}^d}\left\{-\Delta_k + \left[F'(x_k) + (Bx)_k\right]\frac{\partial}{\partial x_k}\right\}.$$

The corresponding semigroup $P_t f(x^{(0)}) = \mathbf{E}\; f\left(\xi_t^{(0)}\left(x^{(0)}\right)\right)$ is defined in terms of solutions to the infinite system of stochastic differential equations

$$d\xi_k^{(0)}\big(t, x^{(0)}\big) = \sqrt{2}\, dW_k(t) - \Big\{ F\Big(\xi_k^{(0)}\big(t, x^{(0)}\big)\Big) + \Big(B\xi^{(0)}\big(t, x^{(0)}\big)\Big)_k \Big\} dt,$$
$$\xi_k^{(0)}\big(0, x^{(0)}\big) = x_k^{(0)}, \quad k \in \mathbb{Z}^d, \tag{8}$$

where, for the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\mathcal{F}_t$, the process $W(t) = \{W_k(t)\}_{k\in\mathbb{Z}^d}$, $t \geq 0$, is an $\mathcal{F}_t$-adapted Wiener process defined on Ω with values in $l_2(a) = l_2(a, \mathbb{Z}^d)$, $\sum_{k\in\mathbb{Z}^d} a_k = 1$, and identity covariance operator. The linear finite-diagonal map $B: \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}^{\mathbb{Z}^d}$ is defined by

$$Bx = \left\{ \sum_{j:|j-k|\leq r_0} b(k-j)\, x_j \right\}_{k\in\mathbb{Z}^d}$$

and the nonlinear map $F: \mathbb{R}^{\mathbb{Z}^d} \ni x \to F(x) = \{F(x_k)\}_{k\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$ is generated by the C^∞ monotone function F, $F(0) = 0$, which satisfies the condition of polynomial growth at the infinity:

$$\exists\, \mathbf{k} \geq -1 \quad \forall\, i \geq 1: \left| F^{(i)}(x) - F^{(i)}(y) \right| \leq C_i |x-y| (1 + |x| + |y|)^{\mathbf{k}}. \tag{9}$$

To study the regular properties of semigroup P_t, we need to write the representation for derivatives of semigroup $\partial_\tau P_t f$: for $\tau = \{j_1, \ldots, j_m\}$ and $\partial_\tau = \partial^{|\tau|}/\partial x_{j_1} \ldots \partial x_{j_m}$,

$$\partial_\tau (P_t f)(x^{(0)}) = \sum_{s=1}^{m} \sum_{\gamma_1 \cup \ldots \cup \gamma_s = \tau} \mathbf{E} \left\langle \partial^{(s)} f\big(\xi^{(0)}\big), \xi_{\gamma_1} \otimes \ldots \otimes \xi_{\gamma_s} \right\rangle, \tag{10}$$

where $\partial^{(s)} f = \{\partial_\gamma f\}_{|\gamma|=s}$ denotes the set of sth order partial derivatives of function. We also use the notation

$$\left\langle \partial^{(s)} f\big(\xi^{(0)}\big), \xi_{\gamma_1} \otimes \ldots \otimes \xi_{\gamma_s} \right\rangle = \sum_{j_1,\ldots,j_s \in \mathbb{Z}^d} \Big(\partial_{\{j_1,\ldots,j_s\}} f\Big)\big(\xi^{(0)}\big) \xi_{j_1,\gamma_1} \ldots \xi_{j_s,\gamma_s}.$$

The equation on variational process

$$\xi_\tau = \left\{ \xi_{k,\tau} = \frac{\partial^{|\tau|} \xi_k^{(0)}(t, x^{(0)})}{\partial x_{j_n}^{(0)} \ldots \partial x_{j_1}^{(0)}} \right\}_{k\in\mathbb{Z}^d}$$

is derived by the formal successive differentiation of (8) with respect to $x^{(0)}$:

$$\frac{d\xi_{k,\tau}}{dt} = -F'\big(\xi_k^{(0)}\big)\xi_{k,\tau} - \sum_{j:|j-k|\leq r_0} b(k-j)\,\xi_{j,\tau} - \varphi_{k,\tau},$$
$$\xi_{k,\tau}(0) = \begin{cases} \delta_{k,j}, & |\tau| = 1, \quad \tau = \{j\} \subset \mathbb{Z}^d, \\ 0, & |\tau| > 1, \end{cases} \tag{11}$$

where $\varphi_{k,\tau} = \varphi_{k,\tau}(\xi^{(0)}, \xi_{\cdot,\gamma}, \gamma \subset \tau, \gamma \neq \tau)$ equals to

$$\varphi_{k,\tau} = \sum_{\gamma_1 \cup \ldots \cup \gamma_s = \tau,\ s\geq 2} F^{(s)}\big(\xi^{(0)}\big) \xi_{k,\gamma_1} \ldots \xi_{k,\gamma_s}. \tag{12}$$

In (12), the summation $\sum_{\gamma_1 \cup \ldots \cup \gamma_s = \tau,\, s \geq 2}$ is made over all possible subdivisions of the set $\tau = \{ j_1, \ldots, j_n \}$, $j_i \in \mathbb{Z}^d$, on the nonintersecting subsets $\gamma_1, \ldots, \gamma_s \subset \tau$, with $|\gamma_1| + \ldots + |\gamma_s| = |\tau|$, $s \geq 2$, $|\gamma_i| \geq 1$.

Let us introduce nonlinear expression, which reflects the symmetries of variations for lattice models:

$$\rho_\tau(\xi; t) = \mathbf{E} \sum_{s=1}^{n} \left\{ p_s(z_t) \sum_{\gamma \subset \tau, |\gamma| = s} \| \xi_\gamma \|_{l_{m_\gamma}(c_\gamma)}^{m_\gamma} \right\}, \tag{13}$$

where $\tau = \{ j_1, \ldots, j_n \}$, $j_i \in \mathbb{Z}^d$, p_s are polynomial functions depending on $z_t = \left\| \xi^{(0)}\left(t, x^{(0)}\right) \right\|_{l_2(a)}^2$ and $m_\gamma = m_1 / |\gamma|$, $|\gamma|$ is a number of points in the set $\gamma \in \mathbb{Z}^d$. The set of all vectors $c = \{ c_k \}_{k \in \mathbb{Z}^d}$ such that $\delta_c = \sup_{|k-j|=1} |c_k / c_j| < \infty$ is denoted by $\mathbb{P}$.

Theorem 1. *Let* F *satisfy* (9), $x^{(0)} \in l_{2(\mathbf{k}+1)^2}(a)$, $\sum_{k \in \mathbb{Z}^d} a_k = 1$, $a \in \mathbb{P}$, $x_\gamma \in l_{m_\gamma}(dc_\gamma)$, $\gamma \subset \tau$, $d \geq a^{-(\mathbf{k}+1)m_1/2}$, $m_\gamma = m_1 / |\gamma|$, $m_1 \geq |\tau|$ *and* $\xi^{(0)}$, $\{ \xi_\gamma \}_{\gamma \subset \tau}$ *form the strong solutions to systems* (8), (11). *Suppose that functions* $p_i(z)$, $i = 1, \ldots, n$, *and vectors* $\{ c_\gamma \}_{\gamma \subset \tau} \subset \mathbb{P}$ *in* (13) *satisfy the following conditions*:

1) $\exists\, \varepsilon > 0$, $\exists\, K > 0$ *such that*

$$\forall\, z \in \mathbb{R}_+ : p_i(z) \geq \varepsilon, \quad (1+z)\left(|p_i'(z)| + |p_i''(z)| \right) \leq K p_i(z); \tag{14}$$

2) $\exists\, K_p$ $\forall\, j = 2, \ldots, n$ $\forall\, i_1, \ldots, i_s$, $i_1 + \ldots + i_s = j$, $s \geq 2$:

$$\left[p_j(z) \right]^j (1+z)^{(\mathbf{k}+1)m_1/2} \leq K_p \left[p_{i_1}(z) \right]^{i_1} \ldots \left[p_{i_s}(z) \right]^{i_s}; \tag{15}$$

3) *for any subdivision of the set* $\gamma = \alpha_1 \cup \ldots \cup \alpha_s$, $\gamma \subset \tau$ *on nonempty nonintersecting subsets* $\alpha_1, \ldots, \alpha_s$, $s \geq 2$, *there is* $R_{\gamma, \alpha_1, \ldots, \alpha_s}$ *such that*

$$\forall\, k \in \mathbb{Z}^d : \left[c_{k,\gamma} \right]^{|\gamma|} a_k^{-(\mathbf{k}+1)m_1/2} \leq R_{\gamma, \alpha_1, \ldots, \alpha_s} \left[c_{k,\alpha_1} \right]^{|\alpha_1|} \ldots \left[c_{k,\alpha_s} \right]^{|\alpha_s|}. \tag{16}$$

Then there is a constant $M \in \mathbb{R}^1$ *such that the nonlinear quasicontractive estimate*

$$\rho_\tau(\xi; t) \leq e^{Mt} \rho_\tau(\xi; 0) \tag{17}$$

holds.

Proof of this result may be found in [5, 6]. In Theorem 3, we give a more general result, which implies, in particular, estimate (17).

As a consequence of nonlinear estimate, we can obtain regular properties of semigroup P_t in the spaces of continuously differentiable functions.

Denote by $\mathrm{Lip}_r\left(l_2(a)\right)$ the space of continuous functions on the space $l_2(a) \subset \mathbb{R}^{\mathbb{Z}^d}$, equipped with norm

$$\| f \|_{\mathrm{Lip}_r} = \sup_{x \in l_2(a)} \frac{|f(x)|}{\left(1 + \|x\|_{l_2(a)}\right)^{r+1}} + \sup_{x, y \in l_2(a)} \frac{|f(x) - f(y)|}{\|x - y\|_{l_2(a)} \left(1 + \|x\|_{l_2(a)} + \|y\|_{l_2(a)}\right)^r} < \infty. \tag{18}$$

For some $m \in \mathbb{N}$, denote by Θ^m the array of pairs $\{(p,\mathcal{G}):(p,\mathcal{G})\in\Theta^m\}$, where $\mathcal{G} = G^1 \otimes \ldots \otimes G^m$ is m-tensor constructed by vectors $G^i \in \mathbb{P}$, $i = 1,\ldots,m$, and p is a smooth function of polynomial behavior (14).

Definition 1. *The array* $\Theta = \Theta^1 \cup \ldots \cup \Theta^n$, $n \in \mathbb{N}$, *is quasicontractive with parameter* $\mathbf{k}$ *iff for any* $m = 2,\ldots,n$, *for any* $(p,\mathcal{G}) \in \Theta^m$, *and all* $i, j \in \{1,\ldots,m\}$, $i \neq j$, *there exists a pair* $(\tilde{p},\tilde{\mathcal{G}}) \in \Theta^{m-1}$ *such that*

$$\exists K \quad \forall z \in \mathbb{R}^1_+ : (1+z)^{(\mathbf{k}+1)/2}\,\tilde{p}(z) \le Kp(z),$$
$$\left(\hat{\mathcal{G}}^{\{i,j\}}\right)^l \le K\tilde{G}^l, \quad l = 1,\ldots,m-1. \tag{19}$$

Above $(m-1)$-tensor $\hat{\mathcal{G}}^{\{i,j\}}$ is constructed from m-tensor $\mathcal{G}$ by the rule

$$\hat{\mathcal{G}}^{\{i,j\}} = G^1 \otimes \underset{\hat{i}}{\ldots} \otimes \underset{\uparrow j}{(A)^{-(\mathbf{k}+1)}G^i G^j} \otimes \ldots \otimes G^m \mathrm{x}.$$

The notation $G^1 \otimes \underset{\hat{i}}{\ldots} \otimes G^s$ means that the ith vector is omitted in tensor product and $G^1 \otimes \ldots \otimes \underset{\uparrow j}{B} \otimes \ldots \otimes G^s$ means that the vector B is inserted on jth place in tensor product. Inequality (19) is understood as a coordinate inequality between two vectors.

For multifunction of m th order $u^{(m)}(x) = \{u_\tau(x),\ \tau = \{k_1,\ldots,k_m\},\ k_i \in \mathbb{Z}^d\}$, $x \in l_2(a)$, we introduce the seminorm

$$\left\| u^{(m)} \right\|_{\Theta^m} = \sup_{x \in l_2(a)} \sup_{(p_m,\mathcal{G}^m)\in\Theta^m} \frac{\left| u^{(m)}(x) \right|_{\mathcal{G}^m}}{p_m\left(\|x\|^2_{l_2(a)}\right)} \tag{20}$$

with $\left|u^{(m)}(x)\right|^2_{\mathcal{G}^m} = \sum_{\tau=\{j_1,\ldots,j_m\}\subset\mathbb{Z}^d} G^1_{j_1}\ldots G^m_{j_m}\left|u_\tau(x)\right|^2$ for $\mathcal{G}^m = G^1 \otimes \ldots \otimes G^m$.

Let $r \ge 0$, $n \ge 1$, and let $\Theta = \Theta^1 \cup \ldots \cup \Theta^n$ be a quasicontractive array with parameter $\mathbf{k}$. We say that a function f belongs to the space $C_{\Theta,r}(l_2(a))$ iff $f \in \mathrm{Lip}_r(l_2(a))$ and the following statement are true:

1) there is a set of partial derivatives $\{\partial^{(1)}f,\ldots,\partial^{(n)}f\}$ such that for any $m \in \{1,\ldots,n\}$ the coordinates of multifunctions $\{\partial^{(m)}f(x)\}_{j_1\ldots j_m} = \partial_\tau f(x)$, $\tau = \{j_1,\ldots,j_m\}$ are continuous: $\partial_\tau f \in C(l_2(a),\mathbb{R}^1)$, and for all $x^{(0)} \in l_2(a)$, $h \in X_\infty([a,b])$ the equalities

$$f\left(x^{(0)} + h(\cdot)\right)\Big|_a^b = \int_a^b ds \sum_{k\in\mathbb{Z}^d} \partial_k f\left(x^{(0)} + h(s)\right) h_k'(s) \tag{21}$$

and for $\tau = \{j_1,\ldots,j_l\}$, $|\tau| = l \le n-1$:

$$\partial_\tau f\left(x^{(0)} + h(\cdot)\right)\Big|_a^b = \int_a^b ds \sum_{k\in\mathbb{Z}^d} \partial_{\tau\cup\{k\}} f\left(x^{(0)} + h(s)\right) h_k'(s), \tag{22}$$

hold;

2) the norm is finite:

$$\|f\|_{C_{\Theta,r}} = \|f\|_{\mathrm{Lip}_r} + \max_{m=1,n} \left\| \partial^{(m)} f \right\|_{\Theta^m} < \infty. \tag{23}$$

The space $X_\infty([a, b])$ is defined as

$$X_\infty([a, b]) = \bigcap_{p \ge 1,\, c \in \mathbb{P}} AC_\infty\big([a, b], l_p(c)\big) \tag{24}$$

and $AC_\infty([a, b], X) = \{h \in C([a, b], X) : \exists h' \in L_\infty([a, b], X)\}$ for the Banach space X.

The following theorem states the regular properties of semigroup P_t in the scale $C_{\Theta, r}$.

Theorem 2. *Let* F *satisfy* (9) *and let* Θ *be a quasicontractive array with parameter* $\mathbf{k}$. *Then, for any* $t \ge 0$, $P_t : C_{\Theta, r} \to C_{\Theta, r}$ *and there exist constants* $K_{\Theta, r}$, $M_{\Theta, r}$ *such that*

$$\forall f \in C_{\Theta, r} : \| P_t f \|_{C_{\Theta, r}} \le K_{\Theta, r}\, e^{M_{\Theta, r} t} \| f \|_{C_{\Theta, r}}.$$

Proof of this result is quite complicated and may be found in [5, 6].

The relation between variations and derivatives of a semigroup

$$\partial^{(n)} P_t f(x) = \sum_{j_1 + \ldots + j_s = n,\, s \ge 1} \mathbf{E} f^{(s)}\big(\xi^0_{t,x}\big)\xi^{(j_1)}_{t,x} \ldots \xi^{(j_s)}_{t,x} \tag{25}$$

is applied as a main tool to obtain the quasicontractive estimates on the regularity of semigroup from estimates (17).

3. Main result: non-Lipschitz gap between boundedness and continuity. In this section, we discuss a principally infinite-dimensional effect inherent for the infinite-dimensional nonlinear evolutions.

First, recall that the classical Cauchy – Liouville – Picard scheme for equations with coefficients with bounded derivatives implies C^∞-properties of semigroups in the spaces of smooth functions with *the same topology of boundedness and continuity of derivatives*: for a function f such that

$$\forall\, i = 1, \ldots, n \quad \forall\, x, y \in \mathcal{B}_0 : \left\| \partial^{(i)} f(x) \right\|_{\mathcal{B}_i} \le K,$$

$$\left\| \partial^{(i)} f(x) - \partial^{(i)} f(y) \right\|_{\mathcal{B}_i} \le K \| x - y \|_{\mathcal{B}_0},$$

the consideration of the semigroup $(P_t f)(x) = f(y_t(x))$, generated by Lipschitz differential flow $y_t(x) = x - \int_0^t F(y_s(x))\,ds$, gives no more that exponential growth of constants in the recurrently defined sequence of spaces $\mathcal{B}_i = \mathcal{L}(\mathcal{B}_0, \mathcal{B}_{i-1})$ over the Banach space $\mathcal{B}_0$:

$$\exists\, M \quad \forall\, t \ge 0 \quad \forall\, i = 1, \ldots, n \quad \forall\, x, y \in \mathcal{B}_0 : \left\| \partial^{(i)} (P_t f)(x) \right\|_{\mathcal{B}_i} \le K e^{Mt},$$

$$\left\| \partial^{(i)} (P_t f)(x) - \partial^{(i)} (P_t f)(y) \right\|_{\mathcal{B}_i} \le K e^{Mt} \| x - y \|_{\mathcal{B}_0}.$$

In other words, in terms of the norm

$$\| f \|_{C^n} = \max_{i = 0, \ldots, n} \left(\sup_{x \in \mathcal{B}_0} \left\| \partial^{(i)} f(x) \right\|_{\mathcal{B}_i},\ \sup_{x, y \in \mathcal{B}_0} \frac{\left\| \partial^{(i)} f(x) - \partial^{(i)} f(y) \right\|_{\mathcal{B}_i}}{\| x - y \|_{\mathcal{B}_0}} \right),$$

there is a quasicontractive property:

$$\exists\, M \quad \forall f \in C^n : \| P_t f \|_{C^n} \le e^{Mt} \| f \|_{C^n}.$$

Below we show that the non-Lipschitz coefficients in equation cause *the depending on the nonlinearity parameters gap* between the topologies of the *boundedness and continuity* of semigroup derivatives. We study the action of semigroup P_t in spaces equipped with topology

$$\| f \|_{\mathcal{E}^n} = \max_{j=0,\ldots,n} \left[\sup_x \frac{\left\| \partial^{(j)} f(x) \right\|_{\mathcal{B}_j}}{q_j(\| x \|)}, \sup_{x,y} \frac{\left\| \partial^{(j)} f(x) - \partial^{(j)} f(y) \right\|_{\tilde{\mathcal{B}}_j}}{\| x - y \| p_j(\| x \| + \| y \|)} \right]. \tag{26}$$

We show that the study of smooth properties of non-Lipschitz semigroup, in particular, the quasicontractive estimates

$$\exists\, M = M_{\mathcal{E}^n} \quad \forall f \in \mathcal{E}^n : \| P_t f \|_{\mathcal{E}^n} \le e^{Mt} \| f \|_{\mathcal{E}^n}$$

requires to introduce a gap between the topologies of boundedness and continuity

$$p_j(z) = \mathrm{Pol}_{\mathbf{k}}(z) \cdot q_j(z), \qquad \tilde{\mathcal{B}}_j = C_{\mathbf{k}} \mathcal{B}_j$$

depending on the non-Lipschitz order $\mathbf{k}$ of F. Note that the operator gap $C_{\mathbf{k}}$ displays an essentially infinite-dimensional effect, because the finite-dimensional norms are equivalent.

Let us shortly discuss the key idea. Due to representation (25), the continuity of variations, i.e., estimates on $\xi_{t,x}^{(j)} - \xi_{t,y}^{(j)}$ for solutions of (11), is necessary in order to control the continuity of semigroup derivatives. The principal part of equation in $\xi_{t,x}^{(j)} - \xi_{t,y}^{(j)}$:

$$\frac{d}{dt}\left(\xi_{t,x}^{(j)} - \xi_{t,y}^{(j)} \right) = - \frac{\left[F'\left(\xi_{t,x}^{0}\right) + F'\left(\xi_{t,y}^{0}\right) \right]}{2} \left(\xi_{t,x}^{(j)} - \xi_{t,y}^{(j)} \right) -$$
$$- \frac{\left[F'\left(\xi_{t,x}^{0}\right) - F'\left(\xi_{t,y}^{0}\right) \right]}{2} \left(\xi_{t,x}^{(j)} + \xi_{t,y}^{(j)} \right) + \ldots$$

points on the similarity of behaviour,

$$\frac{\left\| \xi_{t,x}^{(j)} - \xi_{t,y}^{(j)} \right\|_{Y_j}}{\left\| \xi_{t,x}^{0} - \xi_{t,y}^{0} \right\|} \sim \left\| \xi_{t,x}^{(j)} \right\|_{X_j} + \left\| \xi_{t,y}^{(j)} \right\|_{X_j}, \tag{27}$$

due to the relation $\left(F'\left(\xi_{t,x}^{0}\right) - F'\left(\xi_{t,y}^{0}\right) \right) \sim \left(\xi_{t,x}^{0} - \xi_{t,y}^{0} \right)$ up to accuracy of some polynomial $\left(\xi_{t,x}^{0}, \xi_{t,y}^{0} \right)$ factor.

Let $p_\gamma, q_\gamma \in C^\infty(\mathbb{R}_+)$, $\gamma \subset \tau$, be positive monotone functions of polynomial behaviour, i.e., such that

$$\exists\, \varepsilon > 0 \quad \forall z \in \mathbb{R}_+ : p_\gamma(z) \ge \varepsilon, \qquad p'_\gamma(z) \ge \varepsilon,$$

$$\exists\, C > 0 : (1+z) \left| p''_\gamma(z) \right| \le C p'_\gamma(z), \qquad (1+z) p'_\gamma(z) \le C p_\gamma(z). \tag{28}$$

Let us introduce a generalization of expression (13), reflecting the higher-order symmetries of variations

$$\rho_\tau^b(\xi^x, \xi^y; t) = \sum_{\gamma \subset \tau} \mathbf{E} \left\| \xi_\varnothing^x - \xi_\varnothing^y \right\|_{l_2(a)}^{\delta} p_\gamma\left(n_t^{x,y} \right) \left\{ \left\| \xi_\gamma^x \right\|_{l_{m_\gamma}(c_\gamma)}^{m_\gamma} + \left\| \xi_\gamma^y \right\|_{l_{m_\gamma}(c_\gamma)}^{m_\gamma} \right\}, \tag{29}$$

where we use the notation $n_t^{x,y}$ for the sum of norms,

$$n_t^{x,y} \;=\; \left\|\xi_\varnothing^x(t)\right\|_{l_2(a)}^2 \;+\; \left\|\xi_\varnothing^y(t)\right\|_{l_2(a)}^2,$$

with corresponding sense of $n_0^{x,y} = \|x\|_{l_2(a)}^2 + \|y\|_{l_2(a)}^2$.

Then we write the continuity part

$$\rho_\tau^c(\xi^x,\xi^y;t) \;=\; \sum_{\gamma\subset\tau} \mathbf{E}\left\|\xi_\varnothing^x-\xi_\varnothing^y\right\|_{l_2(a)}^{\delta-m_\gamma} q_\gamma\!\left(n_t^{x,y}\right)\left\|\xi_\gamma^x-\xi_\gamma^y\right\|_{l_{m_\gamma}\left(a^{(\mathbf{k}+1)m_\gamma/2}c_\gamma\right)}^{m_\gamma} \tag{30}$$

with the multiple $\left\|\xi_\varnothing^x-\xi_\varnothing^y\right\|^{\delta-m_\gamma}$, that reflects the similarity of behavior (27).

Introduce nonlinear expression

$$\rho_\tau(\xi^x,\xi^y;t) \;=\; \rho_\tau^b(\xi^x,\xi^y;t) \;+\; \rho_\tau^c(\xi^x,\xi^y;t)$$

for the joint boundedness and continuity of variations.

In the following theorem, we obtain the estimates on the continuous dependence of solutions ξ_γ^x of variational equations (11) with respect to the initial data x in terms of quasicontractive behavior of expression $\rho_\tau(\xi^x,\xi^y;t)$.

Theorem 3. *Let* F *satisfy* (9), $\delta\ge m_1\ge|\tau|$, *and let* $\xi_\varnothing^x$, $\xi_\varnothing^y$, ξ_γ^x, ξ_γ^y, $\gamma\subset\tau$, *be generalized solutions to* (8) *and* (11) *with initial data* $x, y\in l_2(a)$ and x_γ, $y_\gamma\in l_{m_\gamma}(dc_\gamma)$, $d_k\ge a_k^{-(\mathbf{k}+1)m_1/2}$, *correspondingly.*

Suppose that

$\{c_\gamma,\ \gamma\subset\tau\}$ *satisfy hierarchy* (16);

functions $\{p_\gamma,\ \gamma\subset\tau\}$ *of polynomial behaviour satisfy the relations*: $\forall\alpha_1\cup\ldots\ldots\cup\alpha_s=\gamma\subset\tau,\ s\ge2\ \ \exists K_p$:

$$\left[p_\gamma(z)\right]^{|\gamma|}(1+z)^{(\mathbf{k}+1)m_1/2} \;\le\; K_p\left[p_{\alpha_1}(z)\right]^{|\alpha_1|}\ldots\left[p_{\alpha_s}(z)\right]^{|\alpha_s|}, \qquad z\in\mathbb{R}_+; \tag{31}$$

functions of polynomial behaviour $\{q_\gamma,\ \gamma\subset\tau\}$ *are such that* $q_\gamma(z)(1+z)^{\mathbf{k}m_\gamma/2}=p_\gamma(z)$.

Then there exists M_γ *such that*

$$\rho_\tau(\xi^x,\xi^y;t) \;=\; e^{M_\tau t}\rho_\tau(\xi^x,\xi^y;0). \tag{32}$$

Proof. This theorem is proved by induction on the number of points in the set τ. First, we suppose that the initial data $x, y\in l_{2(\mathbf{k}+1)^2}(a)$, i.e., $\xi_\varnothing^x$, $\xi_\varnothing^y$ and ξ_γ^x, ξ_γ^y form strong solutions to (8) and (11).

Introduce notation for $i=1,\ldots,|\tau|$:

$$h_\tau^i(\xi^x,\xi^y;t) \;=\; \begin{cases} 0, & i=0,\\ \displaystyle\sum_{\gamma\subset\tau,\,|\gamma|\le i}\left[g_\gamma^b(\xi^x,\xi^y;t)+g_\gamma^c(\xi^x,\xi^y;t)\right], & i\ge1,\end{cases}$$

where

$$g_\gamma^b(t) \;=\; \mathbf{E}\left\|\xi_\varnothing^x-\xi_\varnothing^y\right\|_{l_2(a)}^{\delta} p_\gamma\!\left(n_t^{x,y}\right)\left\{\left\|\xi_\gamma^x\right\|_{l_{m_\gamma}(c_\gamma)}^{m_\gamma}+\left\|\xi_\gamma^y\right\|_{l_{m_\gamma}(c_\gamma)}^{m_\gamma}\right\},$$

$$g_\gamma^c(t) \;=\; \mathbf{E}\left\|\xi_\varnothing^x-\xi_\varnothing^y\right\|_{l_2(a)}^{\delta-m_\gamma} q_\gamma\!\left(n_t^{x,y}\right)\left\|\xi_\gamma^x-\xi_\gamma^y\right\|_{l_{m_\gamma}\left(a^{(\mathbf{k}+1)m_\gamma/2}c_\gamma\right)}^{m_\gamma}.$$

Note that $h^n_\tau(\xi^x, \xi^y; t) = \rho_\tau(\xi^x, \xi^y; t)$ at $n = |\tau|$ and

$$h^i_\tau(\xi^x, \xi^y; t) = h^{i-1}_\tau(\xi^x, \xi^y; t) + \sum_{\gamma \subset \tau, |\gamma| = i} g_\gamma(t) \tag{33}$$

with $g_\gamma(t) = g^b_\gamma(t) + g^c_\gamma(t)$.

If the prove by induction that

$$\forall\, i \le n \quad \exists\, M_i : \; h^i_\tau(\xi^x, \xi^y; t) \le e^{M_i t} h^i_\tau(\xi^x, \xi^y; 0), \tag{34}$$

then we obtain the statement of theorem at $i = n$.

The inductive assumption at $i = 0$ is trivial. Suppose that inequality

$$g_\gamma(t) \le e^{C_1 t} g_\gamma(0) + C_2 \int_0^t e^{C_1(t-s)} h^{i-1}_\tau(\xi^x, \xi^y; s)\, ds \tag{35}$$

is already proved for $i = \{0, \ldots, n_0 - 1\}$.

Then representation (33) and inductive assumption (34) imply estimate (32) for $x, y \in l_{2(\mathbf{k}+1)^2}(a)$:

$$h^i_\tau(\xi^x, \xi^y; t) \le e^{M_{i-1} t} h^{i-1}_\tau(\xi^x, \xi^y; 0) +$$

$$+ \sum_{\gamma \subset \tau, |\gamma| = i} \left\{ e^{C_1 t} g_\gamma(0) + C_2 \int_0^t e^{C_1(t-s)} e^{M_{i-1} s} h^{i-1}_\tau(\xi^x, \xi^y; 0)\, ds \right\} \le$$

$$\le e^{(M_{i-1} + C_1)t} \left(1 + 2^{|\tau|} C_2\, t\right) h^i_\tau(\xi^x, \xi^y; 0) \le e^{\left(M_{i-1} + C_1 + 2^{|\tau|} C_2\right)t} h^i_\tau(\xi^x, \xi^y; 0).$$

We now prove inequality (35). First, we derive the estimate

$$g^b_\gamma(t) \le g^b_\gamma(0) + A_1 \int_0^t g^b_\gamma(s)\, ds + A_2 \int_0^t h^{i-1}_\tau(s)\, ds \tag{36}$$

for boundedness part g^b_γ of nonlinear expression and then, using the special symmetry (27) between g^b_γ and g^c_γ, obtain inequality

$$g^c_\gamma(t) \le g^c_\gamma(0) + B_1 \int_0^t g^c_\gamma\, ds + B_2 \int_0^t g^b_\gamma\, ds + B_3 \int_0^t h^{i-1}_\tau\, ds \tag{37}$$

for continuity part g^c_γ. Together with (36), this finally implies (35).

Estimate (36). The Ito formula for $g^b_\gamma = \mathbf{E} I^\delta_1 I_2\, p_\gamma\left(n^{x,y}_t\right)$ with finite variation processes

$$I_1 = \left\| \xi^x_\varnothing - \xi^y_\varnothing \right\|_{l_2(a)}, \qquad I_2 = \left\| \xi^x_\gamma \right\|^{m_\gamma}_{l_{m_\gamma}(c_\gamma)} + \left\| \xi^y_\gamma \right\|^{m_\gamma}_{l_{m_\gamma}(c_\gamma)} \tag{38}$$

implies

$$g^b_\gamma(t) \le g^b_\gamma(0) + \mathbf{E} \int_0^t I^\delta_1 I_2\, dp_\gamma\left(n^{x,y}_t\right) + p_\gamma\left(n^{x,y}_t\right) I_2\, dI^\delta_1 + I^\delta_1\, p_\gamma\left(n^{x,y}_t\right) dI_2. \tag{39}$$

To estimate first integral in (39), we write the stochastic differential of $p_\gamma\left(n^{x,y}_t\right)$

$$d\,p_\gamma\left(n_t^{x,y}\right) \;=\; -\,L^{x,y}\,p_\gamma\left(n_t^{x,y}\right)dt \;+\; 2p_\gamma'\left(n_t^{x,y}\right)\left\langle\,\xi_\varnothing^x + \xi_\varnothing^y,\,dW_t\,\right\rangle_{l_2(a)}, \qquad (40)$$

where the second order differential operator $L^{x,y}$ acts by the rule

$$L^{x,y}\,p\left(n_0^{x,y}\right) \;=\; -\,2p'\left(n_0^{x,y}\right)\sum_{k\in\mathbb{Z}^d} a_k \;-\; 2p''\left(n_0^{x,y}\right)\sum_{k\in\mathbb{Z}^d} a_k^2(x_k+y_k)^2 \;+$$

$$+\;\; 2p'\left(n_0^{x,y}\right)\Big\{\langle\, x,\,F(x)+Bx\,\rangle_{l_2(a)} + \langle\, y,\,F(y)+By\,\rangle_{l_2(a)}\Big\}. \qquad (41)$$

Using (40) and the estimate

$$L^{x,y}\,p\left(n_0^{x,y}\right) \;\geq\; -\,M_p\,p\left(n_0^{x,y}\right) \qquad (42)$$

which is analogous to [7] (Hint 9), we have

$$\mathbf{E}\int_0^t I_1^\delta\,I_2\,dp_\gamma \;=\; -\int_0^t \mathbf{E}\,I_1^\delta\,I_2\,L^{x,y}\,p_\gamma\left(n_t^{x,y}\right)dt \;\leq\; M_{p_\gamma}\int_0^t g_\gamma^b\,dt. \qquad (43)$$

The monotonicity of map $F:\left\langle\,\xi_\varnothing^x-\xi_\varnothing^y,\,F\left(\xi_\varnothing^x\right)-F\left(\xi_\varnothing^y\right)\right\rangle_{l_2(a)}\geq 0$ and the Ito formula for the stochastic differential of I_1^δ

$$d\,I_1^\delta \;=\; \delta\left\|\,\xi_\varnothing^x-\xi_\varnothing^y\,\right\|_{l_2(a)}^{\delta-2}\left\langle\,\xi_\varnothing^x-\xi_\varnothing^y,\,d\xi_\varnothing^x-d\xi_\varnothing^y\,\right\rangle_{l_2(a)}$$

applied to the second integral in (39) give

$$\mathbf{E}\int_0^t I_2\,p_\gamma\left(n_t^{x,y}\right)d\,I_1^\delta \;=$$

$$=-F\delta\int_0^t \mathbf{E}I_2 p_\gamma\left(n_t^{x,y}\right)\left\|\,\xi_\varnothing^x-\xi_\varnothing^y\,\right\|_{l_2(a)}^{\delta-2}\left\langle\,\xi_\varnothing^x-\xi_\varnothing^y,\,F\left(\xi_\varnothing^x\right)-F\left(\xi_\varnothing^y\right)+B\left(\,\xi_\varnothing^x-\xi_\varnothing^y\right)\right\rangle_{l_2(a)}dt\;\leq$$

$$\leq\;\; \delta\|\,B\,\|_{\mathcal{L}(l_2(a))}\int_0^t g_\gamma^b(t)\,dt. \qquad (44)$$

To estimate the third term in (39), we use the Ito formula for stochastic differential of I_2

$$dI_2 \;=\; m_\gamma\left\{\left\langle\left[\,\xi_\gamma^x\,\right]^\#,\,d\xi_\gamma^x\right\rangle_{l_{m_\gamma}(c_\gamma)} \;+\;\left\langle\left[\,\xi_\gamma^y\,\right]^\#,\,d\xi_\gamma^y\right\rangle_{l_{m_\gamma}(c_\gamma)}\right\}dt$$

for

$$\left\langle\,\xi^\#,\,y\,\right\rangle_{l_p(c)} \;=\; \sum_{k\in\mathbb{Z}^d} c_k\,\xi_k\,|\,\xi_k\,|^{p-2}\,y_k$$

and inequality

$$\left|\left\langle\,\xi^\#,\,\varphi\,\right\rangle_{l_m}\right| \;\leq\; \frac{1}{m}\|\,\varphi\,\|_{l_m}^m \;+\; \frac{m-1}{m}\|\,\xi\,\|_{l_m}^m. \qquad (45)$$

Therefore, by $F'\geq 0$, we have

$$\mathbf{E}\int_0^t I_1^\delta\,p_\gamma\left(n_t^{x,y}\right)dI_2 \;=\; -m_\gamma\int_0^t \mathbf{E}\,I_1^\delta\,p_\gamma\left(n_t^{x,y}\right)\times$$

$$\times \left\{ \left\langle \left[\xi_\gamma^x \right]^\#, F'\left(\xi_\varnothing^x\right)\xi_\gamma^x + B\xi_\gamma^x + \varphi_\gamma^x \right\rangle_{l_{m_\gamma}(c_\gamma)} + \left\langle \left[\xi_\gamma^y \right]^\#, F'\left(\xi_\varnothing^y\right)\xi_\gamma^y + B\xi_\gamma^y + \varphi_\gamma^y \right\rangle_{l_{m_\gamma}(c_\gamma)} \right\} dt \le$$

$$\le \left(m_\gamma \| B \|_{\mathcal{L}\left(l_{m_\gamma}(c_\gamma)\right)} + (m_\gamma - 1) K_\gamma \right) \int_0^t g_\gamma^b(t)\, dt \; + \; \mathbf{E} \int_0^t I_1^\delta \, p_\gamma\left(n_t^{x,y}\right) \times$$

$$\times \sum_{\alpha_1 \cup \ldots \cup \alpha_s = \gamma,\, s \ge 2} \left\{ \left\| F^{(s)}\left(\xi_\varnothing^x\right) \xi_{\alpha_1}^x \ldots \xi_{\alpha_s}^x \right\|_{l_{m_\gamma}(c_\gamma)}^{m_\gamma} + \left\| F^{(s)}\left(\xi_\varnothing^y\right) \xi_{\alpha_1}^y \ldots \xi_{\alpha_s}^y \right\|_{l_{m_\gamma}(c_\gamma)}^{m_\gamma} \right\} dt, \quad (46)$$

where K_γ is a number of all possible subdivisions of the set γ on sets $\alpha_1, \ldots, \alpha_s$, $s \ge 2$.

Both terms in (46) will be estimated analogously. Using condition (9) on map F

$$\left| F^{(s)}\left(\xi_{k,\varnothing}^x\right) \right|^{m_\gamma} \le \left(C_n^F\right)^{m_\gamma} a_k^{-(\mathbf{k}+1)m_\gamma/2} \left(a_k + a_k \left| \xi_{k,\varnothing}^x \right|^2 + a_k \left| \xi_{k,\varnothing}^y \right|^2 \right)^{(\mathbf{k}+1)m_\gamma/2} \le$$

$$\le \left(C_n^F\right)^{m_\gamma} a_k^{-(\mathbf{k}+1)m_\gamma/2} \left(1 + n_t^{x,y}\right)^{(\mathbf{k}+1)m_\gamma/2}$$

and applying hierarchy (16) of weights $\{c_\gamma\}$ and representation $|\xi_\alpha|^{m_\gamma} = \left(|\xi_\alpha|^{m_\alpha}\right)^{|\alpha|/|\gamma|}$, we estimate first term in (46) by

$$\mathbf{E} \int_0^t I_1^\delta \, p_\gamma\left(n_t^{x,y}\right) \sum_{\alpha_1 \cup \ldots \cup \alpha_s = \gamma,\, s \ge 2} \left\| F^{(s)}\left(\xi_\varnothing^x\right) \xi_{\alpha_1}^x \ldots \xi_{\alpha_s}^x \right\|_{l_{m_\gamma}(c_\gamma)}^{m_\gamma} =$$

$$= \sum_{\alpha_1 \cup \ldots \cup \alpha_s = \gamma,\, s \ge 2} \int_0^t \mathbf{E} I_1^\delta \, p_\gamma\left(n_t^{x,y}\right) \sum_{k \in \mathbb{Z}^d} c_{k,\gamma} \left| F^{(s)}\left(\xi_{k,\varnothing}^x\right) \right|^{m_\gamma} \left| \xi_{k,\alpha_1}^x \right|^{m_\gamma} \ldots \left| \xi_{k,\alpha_s}^x \right|^{m_\gamma} dt \le$$

$$\le \left(C_n^F\right)^{m_\gamma} \sum_{\alpha_1 \cup \ldots \cup \alpha_s = \gamma,\, s \ge 2} R_{\gamma,\alpha}^{1/|\gamma|} \times$$

$$\times \int_0^t \mathbf{E} I_1^\delta \, p_\gamma\left(n_t^{x,y}\right) \left(1 + n_t^{x,y}\right)^{(\mathbf{k}+1)m_\gamma/2} \sum_{k \in \mathbb{Z}^d} \prod_{l=1}^s \left(c_{k,\alpha_l} \left| \xi_{k,\alpha_l}^x \right|^{m_{\alpha_l}} \right)^{|\alpha_l|/|\gamma|} dt. \quad (47)$$

Using hierarchy (31) applied to polynomials $\{p_\gamma\}$ and inequality $|x_1 \ldots x_n| \le$ $\le |x_1|^{q_1}/q_1 + \ldots + |x_n|^{q_n}/q_n$ with $q_l = |\gamma|/|\alpha_l|$ we continue (47) by

$$\left(C_n^F\right)^{m_\gamma} \sum_{\alpha_1 \cup \ldots \cup \alpha_s = \gamma,\, s \ge 2} R_{\gamma,\alpha}^{1/|\gamma|} \times$$

$$\times \int_0^t \mathbf{E} I_1^\delta \, p_\gamma\left(n_t^{x,y}\right) \left(1 + n_t^{x,y}\right)^{(\mathbf{k}+1)m_\gamma/2} \sum_{k \in \mathbb{Z}^d} \prod_{l=1}^s \left(c_{k,\alpha_l} \left| \xi_{k,\alpha_l}^x \right|^{m_{\alpha_l}} \right)^{|\alpha_l|/|\gamma|} dt \le$$

$$\le K_p^{1/|\gamma|} \left(C_n^F\right)^{m_\gamma} \sum_{\alpha_1 \cup \ldots \cup \alpha_s = \gamma,\, s \ge 2} R_{\gamma,\alpha}^{1/|\gamma|} \times$$

$$\times \sum_{l=1}^s \frac{|\alpha_l|}{|\gamma|} \int_0^t \mathbf{E} I_1^\delta \, p_{\alpha_l}\left(n_t^{x,y}\right) \left\{ \left\| \xi_{\alpha_l}^x \right\|_{l_{m_{\alpha_l}}(c_{\alpha_l})}^{m_{\alpha_l}} + \left\| \xi_{\alpha_l}^y \right\|_{l_{m_{\alpha_l}}(c_{\alpha_l})}^{m_{\alpha_l}} \right\} dt \le$$

$$\le K_p^{1/|\gamma|} \left(C_n^F\right)^{m_\gamma} \sum_{\alpha_1 \cup \ldots \cup \alpha_s = \gamma,\, s \ge 2} R_{\gamma,\alpha}^{1/|\gamma|} \int_0^t h_\tau^{i-1}(s)\, ds. \quad (48)$$

Finally, (44), (48), and analogous estimates for second term in (46) prove inequality (36).

Estimate (37). Similar to (36), $g_\gamma^c(t) = \mathbf{E} I_1^{\delta - m_\gamma} I_3 q_\gamma\left(n_t^{x,y}\right)$ with I_1 introduced in (38) and

$$I_3 = \left\| \xi_\gamma^x - \xi_\gamma^y \right\|_{l_{m_\gamma}\left(a_k^{(\mathbf{k}+1)m_\gamma/2} c_\gamma\right)}^{m_\gamma}.$$

Because $I_1^{\delta - m_\gamma}$ and I_3 are finite variation processes, by applying the Ito formula to $g_\gamma^c(t)$, we obtain

$$g_\gamma^c(t) = g_\gamma^c(0) +$$

$$+ \mathbf{E} \int_0^t \left\{ I_1^{\delta - m_\gamma} I_3 \, dq_\gamma\left(n_t^{x,y}\right) + q_\gamma\left(n_t^{x,y}\right) I_3 \, dI_1^{\delta - m_\gamma} + I_1^{\delta - m_\gamma} q_\gamma\left(n_t^{x,y}\right) dI_3 \right\}. \tag{49}$$

Representation (40) of stochastic differential of $q_\gamma\left(n_t^{x,y}\right)$, inequality (42), and the monotonicity of map F imply the following estimates for the first and second terms in (49):

$$\mathbf{E} \int_0^t I_1^{\delta - m_\gamma} I_3 \, dq_\gamma\left(n_t^{x,y}\right) = -\int_0^t \mathbf{E} I_1^{\delta - m_\gamma} I_3 L^{x,y} q_\gamma\left(n_t^{x,y}\right) dt \le M_{q_\gamma} \int_0^t g_\gamma^c(t) \, dt, \tag{50}$$

$$\mathbf{E} \int_0^t q_\gamma\left(n_t^{x,y}\right) I_3 \, dI_1^{\delta - m_\gamma} = -(\delta - m_\gamma) \int_0^t \mathbf{E} q_\gamma\left(n_t^{x,y}\right) I_3 \left\| \xi_\varnothing^x - \xi_\varnothing^y \right\|_{l_2(a)}^{\delta - m_\gamma - 2} \times$$

$$\times \left\langle \xi_\varnothing^x - \xi_\varnothing^y, F\left(\xi_\varnothing^x\right) - F\left(\xi_\varnothing^y\right) + B\left(\xi_\varnothing^x - \xi_\varnothing^y\right) \right\rangle_{l_2(a)} dt \le$$

$$\le (\delta - m_\gamma) \| B \|_{\mathcal{L}(l_2(a))} \int_0^t q_\gamma^c(s) \, ds. \tag{51}$$

The estimation of the third term in (49) reflects the similarity of behaviour (27). Using the Ito formula for I_3 with $\tilde{c}_{k,\gamma} = a_k^{(\mathbf{k}+1)m_\gamma/2} c_{k,\gamma}$,

$$dI_3 = -m_\gamma \left\langle \left[\xi_\gamma^x - \xi_\gamma^y \right]^\#, F'\left(\xi_\varnothing^x\right)\xi_\gamma^x - F'\left(\xi_\varnothing^y\right)\xi_\gamma^y + B\left(\xi_\gamma^x - \xi_\gamma^y\right) + \varphi_\gamma^x - \varphi_\gamma^y \right\rangle_{l_{m_\gamma}(\tilde{c}_\gamma)} dt,$$

representation (12) of φ_γ, the monotonicity of map F $(F' \ge 0)$ and, where necessary, adding and subtracting new terms, we have

$$\mathbf{E} \int_0^t I_1^{\delta - m_\gamma} q_\gamma\left(n_t^{x,y}\right) dI_3 = -m_\gamma \int_0^t \mathbf{E} I_1^{\delta - m_\gamma} q_\gamma\left(n_t^{x,y}\right) \times$$

$$\times \left\langle \left[\xi_\gamma^x - \xi_\gamma^y \right]^\#, F'\left(\xi_\varnothing^x\right)\xi_\gamma^x - F'\left(\xi_\varnothing^y\right)\xi_\gamma^y + B\left(\xi_\gamma^x - \xi_\gamma^y\right) + \varphi_\gamma^x - \varphi_\gamma^y \right\rangle_{l_{m_\gamma}(\tilde{c}_\gamma)} dt \le$$

$$\le \left(m_\gamma \| B \|_{\mathcal{L}\left(l_{m_\gamma}(\tilde{c}_\gamma)\right)} + (m_\gamma - 1)(|\gamma| + 1) K_\gamma \right) \int_0^t q_\gamma^c \, dt + \tag{52}$$

$$+ \int_0^t \mathbf{E} I_1^{\delta - m_\gamma} q_\gamma\left(n_t^{x,y}\right) \left\| \left(F'\left(\xi_\varnothing^x\right) - F'\left(\xi_\varnothing^y\right)\right) \xi_\gamma^y \right\|_{l_{m_\gamma}(\tilde{c}_\gamma)}^{m_\gamma} dt + \tag{53}$$

$$+ \sum_{\alpha_1 \cup \ldots \cup \alpha_s = \gamma,\, s \geq 2} \int_0^t \mathbf{E}\, I_1^{\delta - m_\gamma}\, q_\gamma\left(n_t^{x,y}\right) \times$$

$$\times \left\| \left[F^{(s)}\left(\xi_\varnothing^x\right) - F^{(s)}\left(\xi_\varnothing^y\right) \right] \xi_{\alpha_1}^y \ldots \xi_{\alpha_s}^y \right\|_{l_{m_\gamma}(\tilde{c}_\gamma)}^{m_\gamma} dt \; + \tag{54}$$

$$+ \sum_{\alpha_1 \cup \ldots \cup \alpha_s = \gamma,\, s \geq 2} \sum_{j=1}^{s} \int_0^t \mathbf{E}\, I_1^{\delta - m_\gamma}\, q_\gamma\left(n_t^{x,y}\right) \times$$

$$\times \left\| F^{(s)}\left(\xi_\varnothing^x\right) \xi_{\alpha_1}^y \ldots \xi_{\alpha_{j-1}}^y \left(\xi_{\alpha_j}^x - \xi_{\alpha_j}^y\right) \xi_{\alpha_{j+1}}^x \ldots \xi_{\alpha_s}^x \right\|_{l_{m_\gamma}(\tilde{c}_\gamma)}^{m_\gamma} dt. \tag{55}$$

Above inequality (45) and the boundedness of finite diagonal map B in any space $l_p(c)$, $c \in \mathbb{P}$, were applied.

We use the relation $q_\gamma\left(n_t^{x,y}\right)\left(1 + n_t^{x,y}\right)^{\mathbf{k} m_\gamma / 2} = p_\gamma\left(n_t^{x,y}\right)$ and assumption (9) on map F,

$$\left\| \left[F'\left(\xi_\varnothing^x\right) - F'\left(\xi_\varnothing^y\right) \right] \xi_\gamma^y \right\|_{l_{m_\gamma}\left(a^{(\mathbf{k}+1) m_\gamma / 2} c_\gamma\right)}^{m_\gamma} \leq$$

$$\leq \left(C_n^F\right)^{m_\gamma} \sum_{k \in \mathbb{Z}^d} c_{k,\gamma} \left(a_k \left|\xi_{k,\varnothing}^x - \xi_{k,\varnothing}^y\right|^2\right)^{m_\gamma / 2} \times$$

$$\times \left(a_k + a_k \left|\xi_{k,\varnothing}^x\right|^2 + a_k \left|\xi_{k,\varnothing}^y\right|^2\right)^{\mathbf{k} m_\gamma / 2} \left|\xi_{k,\gamma}^y\right|^{m_\gamma} \leq$$

$$\leq \left(C_n^F\right)^{m_\gamma} \left\| \xi_\varnothing^x - \xi_\varnothing^y \right\|_{l_2(a)}^{m_\gamma} \left(1 + n_t^{x,y}\right)^{\mathbf{k} m_\gamma / 2} \left\| \xi_\gamma^y \right\|_{l_{m_\gamma}(c_\gamma)}^{m_\gamma}, \tag{56}$$

to estimate term (53):

$$\int_0^t \mathbf{E}\, I_1^{\delta - m_\gamma}\, q_\gamma\left(n_t^{x,y}\right) \left\| \left(F'\left(\xi_\varnothing^x\right) - F'\left(\xi_\varnothing^y\right)\right) \xi_\gamma^y \right\|_{l_{m_\gamma}(\tilde{c}_\gamma)}^{m_\gamma} dt \leq \left(C_n^F\right)^{m_\gamma} \int_0^t g_\gamma^b(t)\, dt. \tag{57}$$

To estimate term (54), we first use $1 \leq a_k^{-(\mathbf{k}+1) m_\gamma / 2}$ and hierarchy (16) to get the following estimate:

$$\left\| \left[F^{(s)}\left(\xi_\varnothing^x\right) - F^{(s)}\left(\xi_\varnothing^y\right) \right] \xi_{\alpha_1}^y \ldots \xi_{\alpha_s}^y \right\|_{l_{m_\gamma}(\tilde{c}_\gamma)}^{m_\gamma} \leq$$

$$\leq \left(C_n^F\right)^{m_\gamma} \sum_{k \in \mathbb{Z}^d} c_{k,\gamma} a_k^{(\mathbf{k}+1) m_\gamma / 2} \left|\xi_{k,\varnothing}^x - \xi_{k,\varnothing}^y\right|^{m_\gamma} \left(1 + \left|\xi_{k,\varnothing}^x\right|^2 + \left|\xi_{k,\varnothing}^y\right|^2\right)^{\mathbf{k} m_\gamma / 2} \left|\xi_{k,\alpha_1}^y \ldots \xi_{k,\alpha_s}^y\right|^{m_\gamma} \leq$$

$$\leq \left(C_n^F\right)^{m_\gamma} \left\| \xi_\varnothing^x - \xi_\varnothing^y \right\|_{l_2(a)}^{m_\gamma} \left(1 + n_t^{x,y}\right)^{\mathbf{k} m_\gamma / 2} \sum_{k \in \mathbb{Z}^d} c_{k,\gamma} \left|\xi_{k,\alpha_1}^y \ldots \xi_{k,\alpha_s}^y\right|^{m_\gamma} \leq$$

$$\leq \left(C_n^F\right)^{m_\gamma} \left\| \xi_\varnothing^x - \xi_\varnothing^y \right\|_{l_2(a)}^{m_\gamma} \left(1 + n_t^{x,y}\right)^{\mathbf{k} m_\gamma / 2} R_{\gamma,\alpha}^{1/|\gamma|} \sum_{k \in \mathbb{Z}^d} \prod_{j=1}^{s} \left(c_{k,\alpha_j} \left|\xi_{k,\alpha_j}^y\right|^{m_{\alpha_j}}\right)^{|\alpha_j|/|\gamma|} \leq$$

$$\leq \left(C_n^F\right)^{m_\gamma} \left\| \xi_\varnothing^x - \xi_\varnothing^y \right\|_{l_2(a)}^{m_\gamma} \left(1 + n_t^{x,y}\right)^{\mathbf{k} m_\gamma / 2} R_{\gamma,\alpha}^{1/|\gamma|} \prod_{j=1}^{s} \left[\left\| \xi_{\alpha_j}^y \right\|_{l_{m_{\alpha_j}}(c_{\alpha_j})}^{m_{\alpha_j}} \right]^{|\alpha_j|/|\gamma|}. \tag{58}$$

Hence, by using the inequality $\left|x_1 \ldots x_n\right| \leq |x_1|^{q_1}/q_1 + \ldots + |x_n|^{q_n}/q_n$ with $q_j = |\gamma|/|\alpha_j|$, the relation $q_\gamma(z)(1+z)^{\mathbf{k}m_\gamma/2} = p_\gamma(z)$, and hierarchy (31) of polynomials $\{p_\gamma\}$, we obtain the estimate for term (54):

$$\sum_{\alpha_1\cup\ldots\cup\alpha_s=\gamma,\, s\geq 2} \int_0^t \mathbf{E}\, I_1^{\delta-m_\gamma} q_\gamma\left(n_t^{x,y}\right) \left\| \left[F^{(s)}\left(\xi_\varnothing^x\right) - F^{(s)}\left(\xi_\varnothing^y\right) \right] \xi_{\alpha_1}^y \ldots \xi_{\alpha_s}^y \right\|_{l_{m_\gamma}(\tilde c_\gamma)}^{m_\gamma} dt \leq$$

$$\leq \left(C_n^F\right)^{m_\gamma} \sum_{\alpha_1\cup\ldots\cup\alpha_s=\gamma,\, s\geq 2} R_{\gamma,\alpha}^{1/|\gamma|} \int_0^t \mathbf{E}\, I_1^{\delta-m_\gamma} q_\gamma\left(n_t^{x,y}\right) I_1^{m_\gamma} \left(1+n_t^{x,y}\right)^{\mathbf{k}m_\gamma/2} \times$$

$$\times \prod_{j=1}^{s} \left[\left\| \xi_{\alpha_j}^y \right\|_{l_{m_{\alpha_j}}(c_{\alpha_j})}^{m_{\alpha_j}} \right]^{|\alpha_j|/|\gamma|} \leq$$

$$\leq \left(C_n^F\right)^{m_\gamma} K_p^{1/|\gamma|} \sum_{\alpha_1\cup\ldots\cup\alpha_s=\gamma,\, s\geq 2} R_{\gamma,\alpha}^{1/|\gamma|} \int_0^t \mathbf{E}\, I_1^{\delta} \prod_{j=1}^{s} \left[p_{\alpha_j}\left(n_t^{x,y}\right) \left\| \xi_{\alpha_j}^y \right\|_{l_{m_{\alpha_j}}(c_{\alpha_j})}^{m_{\alpha_j}} \right]^{|\alpha_j|/|\gamma|} \leq$$

$$\leq \left(C_n^F\right)^{m_\gamma} K_p^{1/|\gamma|} \sum_{\alpha_1\cup\ldots\cup\alpha_s=\gamma,\, s\geq 2} R_{\gamma,\alpha}^{1/|\gamma|} \sum_{j=1}^{s} \frac{|\alpha_j|}{|\gamma|} \int_0^t g_{\alpha_j}^b(t)\, dt. \tag{59}$$

It remains to estimate (55). Assumption (9) on map F gives

$$\left| F^{(s)}(x) \right| \leq C_n^F \left(a_k + |x|^2 + |y|^2\right)^{(\mathbf{k}+1)/2}$$

and, by taking $\tilde c_{k,\gamma} = a_k^{(\mathbf{k}+1)m_\gamma/2} c_{k,\gamma}$, we arrive at the estimate

$$\left\| F^{(s)}\left(\xi_\varnothing^x\right) \xi_{\alpha_1}^y \ldots \xi_{\alpha_{j-1}}^y \left(\xi_{\alpha_j}^x - \xi_{\alpha_j}^y\right) \xi_{\alpha_{j+1}}^x \ldots \xi_{\alpha_s}^x \right\|_{l_{m_\gamma}(\tilde c_\gamma)}^{m_\gamma} \leq$$

$$\leq \left(C_n^F\right)^{m_\gamma} \sum_{k\in\mathbb{Z}^d} a_k^{(\mathbf{k}+1)m_\gamma/2} c_{k,\gamma} \left(1 + \left|\xi_{k,\varnothing}^x\right|^2 + \left|\xi_{k,\varnothing}^y\right|^2\right)^{(\mathbf{k}+1)m_\gamma/2} \times$$

$$\times \left|\xi_{k,\alpha_1}^y\right|^{m_\gamma} \ldots \left|\xi_{k,\alpha_j}^x - \xi_{k,\alpha_j}^y\right|^{m_\gamma} \ldots \left|\xi_{k,\alpha_i}^x\right|^{m_\gamma} \leq$$

$$\leq \left(C_n^F\right)^{m_\gamma} \left(1+n_t^{x,y}\right)^{(\mathbf{k}+1)m_\gamma/2} R_{\gamma,\alpha}^{1/|\gamma|} \left\| \xi_{\alpha_j}^x - \xi_{\alpha_j}^y \right\|_{l_{m_{\alpha_j}}(\tilde c_{\alpha_j})}^{m_{\alpha_j}|\alpha_j|/|\gamma|} \times$$

$$\times \prod_{l=1,\, l\neq j}^{s} \left(1 + \left\| \xi_{\alpha_l}^x \right\|_{l_{m_{\alpha_l}}(c_{\alpha_l})}^{m_{\alpha_l}} + \left\| \xi_{\alpha_l}^y \right\|_{l_{m_{\alpha_l}}(c_{\alpha_l})}^{m_{\alpha_l}} \right)^{|\alpha_l|/|\gamma|}, \tag{60}$$

where, on the last step, we use

$$c_{k,\gamma} = c_{k,\gamma} a_k^{-(\mathbf{k}+1)m_\gamma/2} a_k^{(\mathbf{k}+1)m_\gamma/2} \leq$$

$$\leq R_{\gamma,\alpha}^{1/|\gamma|} \left[c_{k,\alpha_1}\right]^{|\alpha_1|/|\gamma|} \ldots \left[c_{k,\alpha_s}\right]^{|\alpha_s|/|\gamma|} a_k^{(\mathbf{k}+1)m_\gamma/2} =$$

$$= R_{\gamma,\alpha}^{1/|\gamma|} \left[c_{k,\alpha_j} a_k^{(\mathbf{k}+1)m_{\alpha_j}/2} \right]^{|\alpha_j|/|\gamma|} \prod_{l=1,\, l\neq j}^{s} \left[c_{k,\alpha_l}\right]^{|\alpha_l|/|\gamma|}.$$

To estimate (55), we note that hierarchy (31) of weights p_γ and the relation $q_\gamma(z)(1+z)^{\mathbf{k}m_\gamma/2} = p_\gamma(z)$ imply

$$q_\gamma(z)(1+z)^{(\mathbf{k}+1)m_\gamma/2} \le (1+z)^{-\mathbf{k}m_\gamma/2} K_p^{1/|\gamma|} p_{\alpha_1}^{|\alpha_1|/|\gamma|} \ldots p_{\alpha_s}^{|\alpha_s|/|\gamma|} =$$

$$= K_p^{1/p} q_{\alpha_j}^{|\alpha_j|/|\gamma|} \prod_{l=1,\, l\neq j}^{s} p_{\alpha_l}^{|\alpha_l|/|\gamma|}.$$

Finally, substituting (60) in (55) and representing $\delta - m_\gamma$ as

$$\delta - m_\gamma = \left(\delta - m_{\alpha_j}\right)\frac{|\alpha_j|}{|\gamma|} + \sum_{l=1,\, l\neq j}^{s} \delta\frac{|\alpha_l|}{|\gamma|},$$

we have the estimate for term (55):

$$\sum_{\alpha_1\cup\ldots\cup\alpha_s=\gamma,\, s\ge 2} \sum_{j=1}^{s} \int_0^t \mathbf{E}\, I_1^{\delta-m_\gamma} q_\gamma\left(n_t^{x,y}\right) \times$$

$$\times \left\| F^{(s)}\left(\xi_\varnothing^x\right)\xi_{\alpha_1}^y \ldots \xi_{\alpha_{j-1}}^y\left(\xi_{\alpha_j}^x - \xi_{\alpha_j}^y\right)\xi_{\alpha_{j+1}}^x \ldots \xi_{\alpha_s}^x \right\|_{l_{m_\gamma}(\tilde c_\gamma)}^{m_\gamma} dt \le$$

$$\le \left(C_n^F\right)^{m_\gamma} \sum_{\alpha_1\cup\ldots\cup\alpha_s=\gamma,\, s\ge 2} R_{\gamma,\alpha}^{1/|\gamma|} K_p^{1/|\gamma|} \times$$

$$\times \sum_{j=1}^{s} \int_0^t \mathbf{E}\left(\left\|\xi_\varnothing^x - \xi_\varnothing^y\right\|_{l_2(a)}^{\delta - m_{\alpha_j}} q_{\alpha_j}\left(n_t^{x,y}\right) \left\|\xi_{\alpha_j}^x - \xi_{\alpha_j}^y\right\|_{l_{m_{\alpha_j}}\left(\tilde c_{\alpha_j}\right)}^{m_{\alpha_j}} \right)^{|\alpha_j|/|\gamma|} \times$$

$$\times \prod_{l=1,\, l\neq j}^{s} \left(\left\|\xi_\varnothing^x - \xi_\varnothing^y\right\|_{l_2(a)}^{\delta} p_{\alpha_l}\left(n_t^{x,y}\right) \left(1 + \left\|\xi_{\alpha_l}^x\right\|_{l_{m_{\alpha_l}}(c_{\alpha_l})}^{m_{\alpha_l}} + \left\|\xi_{\alpha_l}^y\right\|_{l_{m_{\alpha_l}}(c_{\alpha_l})}^{m_{\alpha_l}}\right)\right)^{|\alpha_l|/|\gamma|} dt \le$$

$$\le \left(C_n^F\right)^{m_\gamma} \sum_{\alpha_1\cup\ldots\cup\alpha_s=\gamma,\, s\ge 2} R_{\gamma,\alpha}^{1/|\gamma|} K_p^{1/|\gamma|} \sum_{j=1}^{s} \int_0^t \left\{ \frac{|\alpha_j|}{|\gamma|} g_{\alpha_j}^c + \sum_{l=1,\, l\neq j}^{s} \frac{|\alpha_l|}{|\gamma|} g_{\alpha_l}^b \right\} dt \le$$

$$\le \left(C_n^F\right)^{m_\gamma} \sum_{\alpha_1\cup\ldots\cup\alpha_s=\gamma,\, s\ge 2} R_{\gamma,\alpha}^{1/|\gamma|} K_p^{1/|\gamma|} \int_0^t h_\tau^{i-1}\, dt. \tag{61}$$

Collecting together (50) – (52), (57), (59), and (61), we conclude that inequality (37) is proved.

The possibility to close nonlinear estimate (32) from $x, y \in l_{2(\mathbf{k}+1)^2}(a)$ to x, $y \in l_2(a)$ follows from [6] (Theorems 3.4 and 3.11).

4. Conclusion: application to the regularity of heat semigroups. As a consequence of the nonlinear estimate for the joint boundedness and continuity of variations (32), one can estimate the gap between the boundedness and continuity of semigroup derivatives in the corresponding scales of functional spaces. Next theorem announces this result.

The Banach space $\mathcal{E}_{\Theta,r}(l_2(a))$, $\Theta = \Theta_b \cup \Theta_c$, consists of functions $f \in \mathrm{Lip}_r(l_2(a))$ which have partial derivatives up to n th order $\left\{\partial^{(1)} f, \ldots, \partial^{(n)} f\right\}$, $\left\{\partial^{(m)} f\right\}_{k_1,\ldots,k_m} = \partial_{\{k_1,\ldots,k_m\}} f(x)$ and whose norm is finite:

$$\|f\|_{\mathcal{E}_{\Theta,r}} = \|f\|_{\mathrm{Lip}_r} + \max_{m=1,\ldots,n}\left(\left\|\partial^{(m)} f\right\|_{\Theta_b^m}, \left\|\partial^{(m)} f\right\|_{\Theta_c^m} \right) < \infty,$$

where

$$\left\|\partial^{(m)} f\right\|_{\Theta_b^m} = \max_{(q_m,\mathcal{G}^m)\in\Theta_b^m} \sup_{x\in l_2(a)} \frac{\left|\partial^{(m)} f(x)\right|_{\mathcal{G}^m}}{q_m\left(\|x\|^2_{l_2(a)}\right)}, \tag{62}$$

$$\left\|\partial^{(m)} f\right\|_{\Theta_c^m} =$$

$$= \max_{(q_m,\mathcal{H}^m)\in\Theta_c^m} \sup_{x,y\in l_2(a)} \frac{\left|\partial^{(m)} f(x) - \partial^{(m)} f(y)\right|_{\mathcal{H}^m}}{\|x-y\|_{l_2(a)}\, q_m\left(\|x\|^2_{l_2(a)} + \|y\|^2_{l_2(a)}\right)\left(1+\|x\|^2_{l_2(a)} + \|y\|^2_{l_2(a)}\right)^{\mathbf{k}/2}} \tag{63}$$

with

$$\left|\partial^{(m)} f\right|^2_{\mathcal{G}^m} = \sum_{\tau=\{j_1,\ldots,j_m\}\subset\mathbb{Z}^d} G^1_{j_1}\ldots G^m_{j_m}\,|\partial_\tau f(x)|^2$$

for $\mathcal{G}^m = G^1\otimes\ldots\otimes G^m$, $G^i\in\mathbb{P}$, and similar expression for $\mathcal{H}^m = H^1\otimes\ldots\otimes H^m$.

The partial derivatives $\left\{\partial^{(1)} f,\ldots,\partial^{(n)} f\right\}$ of function $f\in\mathcal{E}_{\Theta,r}$ are understood in the sense of identities (21), (22).

Henceforth, we demand that array Θ_b in (62) be generated by the array Θ_c (63) by the law

$$\begin{aligned} \forall\, m = 1,\ldots,n:\ \Theta_b^m = \Big\{ &\left(q_m, \mathcal{G}^m_j\right)_{j=1}^m \text{ such that} \\ &\mathcal{G}^m_j = H^1\otimes\ldots\otimes A^{-(\mathbf{k}+1)}H^j\otimes\ldots\otimes H^m \\ &\text{for } \left(q_m, \mathcal{H}^m = H^1\otimes\ldots\otimes H^m\right)\in\Theta_c^m\Big\}. \end{aligned} \tag{64}$$

Note that for quasicontractive array Θ_c with parameter $\mathbf{k}$, the array $\Theta_b = \Theta_b^1\cup\ldots\cup\Theta_b^n$ generated by (64) is also a quasicontractive one, which could be directly checked.

Theorem 4. *Let* F *satisfy* (9), *let* $\Theta = \Theta_c\cup\Theta_b$ *for* Θ_c *be a quasicontractive array with parameter* $\mathbf{k}$ (*Definition* 2), *and* Θ_b *be generated by* Θ_c *by rule* (64).

Then, for all $t\ge 0$, $P_t: \mathcal{E}_{\Theta,r}\to\mathcal{E}_{\Theta,r}$ *and there exist* $K_{\Theta,r}$, $M_{\Theta,r}$ *such that*

$$\forall f\in\mathcal{E}_{\Theta,r}:\ \|P_t f\|_{\mathcal{E}_{\Theta,r}} \le K_{\Theta,r}\, e^{M_{\Theta,r}t}\|f\|_{\mathcal{E}_{\Theta,r}}. \tag{65}$$

Proof. The detailed proof will appear in [8], here we only sketch its main steps.

First, one can derive estimates

$$\begin{aligned} &\mathbf{E}\, q_n\left(\left\|\xi^0_{t,x}\right\| + \left\|\xi^0_{t,y}\right\|\right)\left\|\xi^{(n)}_{t,x} - \xi^{(n)}_{t,y}\right\|^{m/n}_{Y_n} \le \\ &\le e^{\tilde{M}_n t}\|x-y\|^{m/n} p_1(\|x\|+\|y\|)\left\{\left\|\xi^{(1)}_{0,x}\right\|^m_{X_1} + \left\|\xi^{(1)}_{0,y}\right\|^m_{X_1}\right\}, \end{aligned} \tag{66}$$

which follow by formal appeal to (30), (32) at $\delta = m/n < m$, if n th term of ρ^c_n is remained on the left-hand side of (32) and the special initial data from (11), are substituted therefore $\rho^c_n|_{t=0} = 0$ and summands on $j\ge 2$ in $\rho^b_n|_{t=0}$ disappear on the right-hand side of (32).

However, estimate (32) holds only in the domain $\delta\ge m$, otherwise one would face singular terms like $1/\left\|\xi^0_{t,x} - \xi^0_{t,y}\right\|^{m-\delta}$ in expression (30). Having guessed in

Theorem 3 the precise form of weights $\{p_j, q_j\}$ and topologies $\{X_j, Y_j\}$ for boundedness and continuity, we apply in [8] (Theorem 5) the evolutionary equation techniques to the nonautonomous inhomogeneous equation (11) with its special initial data to reach the value $\delta = m / n$ and obtain estimate (66), important in the proof of Theorem 4.

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